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Limit at infinity of potential type integrals on abstract spaces

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Abstract. If is non-negative and compactly supported, then classical Riesz potential has the order at infinity. D. Siegel and E. Talvila [9] found necessary and sufficient conditions on f for the validity of as even when is not compactly supported. In the present paper we consider the limit at infinity of convolution type integrals on quasi-metric measure spaces with a monotone decreasing kernel satisfying the so-called doubling condition and find a necessary and sufficient condition for the order of this limit. The limit at infinity of convolution type integrals, on normal homogeneous spaces, which are generalizations of classic Riesz potentials is also studied.

Key Words and Phrases: Riesz potential, quasi-metric, convolution type integrals, normal homogeneous space.
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1. Introduction

Let $0 < \alpha < n$. The operator

$$I_\alpha f(x) = \int_{R^n} |x-y|^{\alpha-n} f(y)\, dy$$

is known as the classical Riesz potential. We refer to the monographs [1], [8] for various properties of the Riesz potentials. Their behavior at infinity was investigated in [6], [7], [9].

It is easy to see that if f is non-negative and compactly supported, then $I_\alpha f(x)$ has the order $|x|^{\alpha-n}$ at infinity. D. Siegel and E. Talvila [9] found necessary and sufficient conditions on f for the validity of $I_\alpha f(x) = O\left(|x|^{\alpha-n}\right)$ as $|x| \to \infty$ even when f is not compactly supported.

Theorem A. ([9]) If $f \geq 0$, then a necessary and sufficient condition for $I_\alpha f(x)$ to exist on R^n and be $O\left(|x|^{\alpha-n}\right)$ as $|x| \to \infty$ is that

$$\int_{R^n} |x-y|^{\alpha-n} f(y) (1+|y|)^{n-\alpha}\, dy$$

is bounded on R^n.

We generalize this fact for convolution type integrals on abstract spaces with a monotone decreasing kernel satisfying the so-called "doubling" condition. The limit at infinity of convolution type integrals, on normal homogeneous spaces, which are generalizations of classic Riesz potentials is also studied.

2. The necessary and sufficient condition

Definition 1. *Let X be a set. A function $\rho : X \times X \to [0, \infty)$ is called quasi-metric if*

1. $\rho(x, y) = 0 \Leftrightarrow x = y$;
2. $\rho(x, y) = \rho(y, x)$;
3. *there exists a constant $c \geq 1$ such that for every $x, y, z \in X$*

$$\rho(x, y) \leq c(\rho(x, z) + \rho(z, y)).$$

If (X, ρ) is a set endowed with a quasi-metric, then the balls $B(x, r) = \{y \in X : \rho(x, y) < r\}$, where $x \in X$ and $r > 0$, satisfy the axioms of a complete system of neighborhoods in X, and therefore induce a (separated) topology. With respect to this topology, the balls $B(x, r)$ need not be open.

We denote $diam\, X = \sup\{\rho(x, y) : x \in X, y \in X\}$.

Lemma 1. *Let (X, ρ) be a set with a quasi-metric, $diamX = \infty$, $\xi \in X$ and $m > c$. Then $B(x, m\rho(\xi, x)) \to X$ as $\rho(\xi, x) \to \infty$.*

Proof. Assume the contrary. Suppose that there is an $y \in X$ such that for all $\delta > 0$ there exists an $x \in X$ such that the inequality $\rho(\xi, x) > \delta$ implies $\rho(x, y) \geq m\rho(\xi, x)$. Then by Definition 1 we have

$$m\rho(\xi, x) \leq \rho(x, y) \leq c(\rho(\xi, x) + \rho(\xi, y)).$$

Hence $\rho(\xi, x) \leq \frac{c}{m-c}\rho(\xi, y)$. Choosing $\delta > \frac{c}{m-c}\rho(\xi, y)$, we arrive at contradiction. Lemma is proved.

Let X be a set with a quasi-metric ρ and a nonnegative measure μ and $diam\, X = \infty$. Consider the integral

$$K_\mu(x) = \int_X K(\rho(x, y))\, d\mu(y) \tag{1}$$

where $K : (0, \infty) \to [0, \infty)$ is a monotone decreasing function and there exists a constant $C \geq 1$ such that $K(r) \leq CK(2r)$ for $r > 0$.

Lemma 2. *Let $K_\mu(x) = O(K(\rho(\xi, x)))$ as $\rho(\xi, x) \to \infty$. Then $\int_X d\mu(y) < \infty$.*

Proof. Let $m > c$. Then

$$\begin{aligned} K_\mu(x) &\geq \int\limits_{B(x,m\rho(\xi,x))} K(\rho(x,y))\, d\mu(y) \\ &\geq K(m\rho(\xi,x)) \int\limits_{B(x,m\rho(\xi,x))} d\mu(y) \\ &\geq C_1 K(\rho(\xi,x)) \int\limits_{B(x,m\rho(\xi,x))} d\mu(y). \end{aligned}$$

Hence $\int\limits_{B(x,m\rho(\xi,x))} d\mu(y) < \infty$. By Lemma 1, $B(x,m\rho(\xi,x)) \to X$ as $\rho(\xi,x) \to \infty$. Then $\int\limits_X d\mu(y) < \infty$. Lemma is proved.

Theorem 1. *Let $\xi \in X$. A necessary and sufficient condition for integral (1) to exist on X and be $O(K(\rho(0,x)))$, as $\rho(\xi,x) \to \infty$, is that*

$$\int\limits_X \frac{K(\rho(x,y))}{K(1+\rho(\xi,y))} d\mu(y) \tag{2}$$

is bounded on X.

Proof. Let integral (1) exist on X and $K_\mu(x) = O(K(\rho(\xi,x)))$ as $\rho(\xi,x) \to \infty$. Fix any $z \in X$. To prove that $\int\limits_X \frac{K(\rho(z,y))}{K(1+\rho(\xi,y))} d\mu(y) < \infty$, take $m > c$ such that $m\rho(\xi,z) > 1$. Then

$$\begin{aligned} \int\limits_X \frac{K(\rho(z,y))}{K(1+\rho(\xi,y))} d\mu(y) &= \int\limits_{B(\xi,1)} \frac{K(\rho(z,y))}{K(1+\rho(\xi,y))} d\mu(y) \\ &\quad + \int\limits_{B(\xi,m\rho(\xi,z))\setminus B(\xi,1)} \frac{K(\rho(z,y))}{K(1+\rho(\xi,y))} d\mu(y) \\ &\quad + \int\limits_{X\setminus B(\xi,m\rho(\xi,z))} \frac{K(\rho(z,y))}{K(1+\rho(\xi,y))} d\mu(y) \\ &= I_1(z) + I_2(z) + I_3(z). \end{aligned}$$

It is clear that

$$I_1(z) \leq \frac{1}{K(1+\rho(\xi,1))} \int\limits_{B(\xi,1)} K(\rho(z,y))\, d\mu(y) < \infty.$$

If $1 \leq \rho(z,y) < m\rho(\xi,z)$, then

$$\begin{aligned} 1+\rho(\xi,y) &\leq 1+c(\rho(\xi,z)+\rho(z,y)) \\ &< 1+c(1+m)\rho(\xi,z) \\ &< d\rho(\xi,z), \end{aligned}$$

where $d = m + c(1+m)$. Hence

$$I_2(z) \leq \frac{1}{K(d\rho(\xi,z))} \int\limits_{B(\xi,m\rho(\xi,z))\setminus B(\xi,1)} K(\rho(z,y))\, d\mu(y) < \infty.$$

Consider $I_3(z)$. If $1 < m\rho(\xi,z) \leq \rho(z,y)$, then there exists $C_1 \geq 1$ such that

$$\begin{aligned} \frac{K(\rho(z,y))}{K(1+\rho(\xi,y))} &\leq \frac{K(\rho(z,y))}{K(1+c(\rho(\xi,z)+\rho(z,y)))} \\ &\leq \frac{K(\rho(z,y))}{K\left(1+c\left(1+\frac{1}{m}\right)\rho(z,y)\right)} \\ &\leq \frac{K(\rho(z,y))}{K\left(\left(1+c\left(1+\frac{1}{m}\right)\right)\rho(z,y)\right)} \leq C_1 \end{aligned}$$

Then $I_3(z) \leq C_1 \int\limits_X d\mu(y)$. By Lemma 2, we have $I_3(z) < \infty$. Therefore

$$\int\limits_X \frac{K(\rho(z,y))}{K(1+\rho(\xi,y))} d\mu(y) < \infty.$$

The necessity part of the theorem has been proved.

Now let $\int\limits_X \frac{K(\rho(x,y))}{K(1+\rho(\xi,y))} d\mu(y) < \infty$ for any $x \in X$. To prove that integral (1) exists on X and is $O(K(\rho(\xi,x)))$ as $\rho(\xi,x) \to \infty$, take $a \in (\xi, c^{-1})$. Then

$$\begin{aligned} K_\mu(x) &= \int\limits_{X\setminus B(x,a\rho(\xi,x))} K(\rho(x,y))\, d\mu(y) \\ &\quad + \int\limits_{B(x,a\rho(\xi,x))} K(\rho(x,y))\, d\mu(y) \\ &= J_1(x) + J_2(x). \end{aligned}$$

It is clear that

$$\int\limits_X d\mu(y) \leq \int\limits_X \frac{K(\rho(\xi,y))}{K(1+\rho(\xi,y))} d\mu(y) < \infty.$$

Then

$$J_1(x) \le K(a\rho(\xi,x)) \int\limits_{X\setminus B(x,a\rho(\xi,x))} d\mu(y) \le C_2 K(\rho(\xi,x)).$$

Consider $J_2(x)$. If $\rho(x,y) < a\rho(\xi,x)$, then

$$1+\rho(\xi,y) > c^{-1}\rho(\xi,x) - \rho(x,y) > \left(c^{-1}-a\right)\rho(\xi,x).$$

Hence

$$J_2(x) \le K\left(\left(c^{-1}-a\right)\rho(\xi,x)\right) \int\limits_{B(x,a\rho(\xi,x))} \frac{K(\rho(x,y))}{K(1+\rho(\xi,y))} d\mu(y) = C_3 K(\rho(\xi,x)).$$

From the estimates of $J_1(x)$ and $J_2(x)$ the proof of the sufficiency of the condition follows. Theorem is proved.

3. Limit at infinity

Lemmas 3 and 4, were formulated for classical Riesz potentials in [7], for anisotropic Riesz potentials in [2], for Riesz potentials on homogeneous groups in [3]. The variants of these lemmas which we will use were given in [4].

Lemma 3. *([4]) Let X be a set with a quasi-metric ρ and a nonnegative Borel measure μ on X with $\operatorname{supp}\mu = X$, $\operatorname{diam} X = \infty$ and f be a nonnegative μ-locally integrable function on X. Suppose that a function $K : (0,\infty) \to [0,\infty)$ satisfies the following conditions:*

(K_1) *$K(t)$ is an almost decreasing function, i.e., there exists a constant $D > 1$ such that*

$$K(s_2) \le DK(s_1) \quad \text{for} \quad 0 < s_1 < s_2 < \infty;$$

(K_2) *there exists a constant $M \ge 1$ such that $K(r) \le MK(2r)$ for $r > 0$;*

(K_3)

$$\int\limits_{B(x,1)} K(\rho(x,y))\, d\mu(y) < \infty.$$

Then for the existence of

$$U_K f(x) = \int\limits_X K(\rho(x,y)) f(y)\, d\mu(y)$$

μ-almost everywhere on X, it is necessary and sufficient that one of the following equivalent conditions be fulfilled:

1) *there exists $x_0 \in X$ such that*

$$\int\limits_{X\setminus B(x_0,1)} K(\rho(x_0,y)) f(y)\, d\mu(y) < \infty;$$

2) *for arbitrary* $x \in X$

$$\int\limits_{X\backslash B(x,1)} K\left(\rho\left(x,y\right)\right) f\left(y\right) d\mu\left(y\right) < \infty;$$

3)

$$\int\limits_{X} K\left(1+\rho\left(0,y\right)\right) f\left(y\right) d\mu\left(y\right) < \infty. \tag{3}$$

Definition 2. *Let* $\beta > 0$. *A space* $(X, \rho, \mu)_\beta$ *is a set* X *with a quasi-metric* ρ *and a nonnegative Borel measure* μ *on* X *with* $\operatorname{supp}\mu = X$, $\operatorname{diam} X = \infty$ *such that*

$$C^{-1} r^\beta \le \mu\left(B\left(x,r\right)\right) \le C r^\beta$$

for all $r > 0$ *and all* $x \in X$, *where the constant* $C \ge 1$ *does not depend on* x *and* r.

Lemma 4. *([4]) Let* $K : (0,\infty) \to [0,\infty)$ *be a continuous function satisfying conditions* (K_1), (K_2) *and*

(K_4) *there exist a constant* $F > 0$ *and* $0 < \sigma < \beta$ *such that*

$$\int\limits_{B(x,r)} K\left(\rho\left(x,y\right)\right) d\mu\left(y\right) < F r^\sigma \ \text{ for any } \ r > 0.$$

Let f *is a nonnegative* μ*- locally integrable function on* X *satisfying the condition*

$$\int\limits_{X} f\left(y\right)^p w\left(f\left(y\right)\right) d\mu\left(y\right) < \infty,$$

where $p = \dfrac{\beta}{\sigma}$ *and the following conditions are fulfilled*

(w_1) w *is a positive, monotone increasing function on the interval* $(0,\infty)$;

(w_2) $\int\limits_1^\infty w\left(r\right)^{-\frac{1}{p-1}} r^{-1} dr < \infty;$

(w_3) *there exists a constant* $A > 0$ *such that*

$$w\left(2r\right) < A w\left(r\right) \ \text{ for } \ \text{ any } \ r > 0.$$

Then there exists a positive constant L *such that*

$$\int\limits_{\{y\in X: f(y)\ge a\}} K\left(\rho\left(x,y\right)\right) f\left(y\right) d\mu\left(y\right)$$

$$< L \left(\int\limits_{\{y\in X: |f(y)|\ge a\}} f\left(y\right)^p w\left(f\left(y\right)\right) d\mu\left(y\right) \right)^{\frac{1}{p}} \left(\int\limits_a^\infty w\left(t\right)^{-\frac{1}{p-1}} t^{-1} dt \right)^{\frac{1}{p'}},$$

for any $a > 0$, *where* $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

Lemma 5. *Let (X,ρ) be a set with a quasi-metric, $diam X=\infty$, $\xi\in X$ and $m<c^{-1}$. Then*

$$X\backslash B\left(x,m\rho\left(\xi,x\right)\right)\to X,\ as\,\rho\left(\xi,x\right)\to\infty.$$

Proof. Assume the contrary. Suppose that there is an $y\in X$ such that for all $\delta>0$ there exists an $x\in X$ such that $\rho\left(\xi,x\right)>\delta$ yields $\rho\left(x,y\right)<m\rho\left(\xi,x\right)$. Then by Definition 1 we have.

$$\rho\left(\xi,x\right)\leq c\left(\rho\left(x,y\right)+\rho\left(\xi,y\right)\right)\leq c\left(m\rho\left(\xi,x\right)+\rho\left(\xi,y\right)\right).$$

Hence

$$\rho\left(\xi,x\right)\leq\frac{c}{1-mc}\rho\left(\xi,y\right),$$

which is impossible under the choice $\delta>\dfrac{c}{1-mc}\rho\left(\xi,y\right)$. Lemma is proved.

The following theorem generalizes the corresponding theorem in [7].

Theorem 2. *Let the assumptions of Lemma 4 and condition (3) be fulfilled and let also K and w satisfy the conditions*

(K_5) $\quad\lim\limits_{r\to\infty}K\left(r\right)=0$

(w_4) $\quad w\left(r^2\right)\leq A_1w\left(r\right),$ *for* $r\in\left(1,\infty\right)$. *Then*

$$w^*\left(\rho\left(\xi,x\right)^{-1}\right)^{\frac{1}{p}}U_Kf\left(x\right)\to0\quad as\quad\rho\left(\xi,x\right)\to\infty,$$

where $w^*\left(r\right)=\left(\int\limits_r^\infty w\left(t\right)^{-\frac{1}{p-1}}t^{-1}dt\right)^{1-p}$.

Proof. Let $m<c^{-1}$. For $x\in X\backslash\left\{\xi\right\}$, we write

$$\begin{aligned}U_Kf\left(x\right)=&\int\limits_{X\backslash X\left(x,m\rho\left(\xi,x\right)\right)}K\left(\rho\left(x,y\right)\right)f\left(y\right)dy\\&+\int\limits_{B\left(x,m\rho\left(\xi,x\right)\right)}K\left(\rho\left(x,y\right)\right)f\left(y\right)dy\\&=J_1\left(x\right)+J_2\left(x\right).\end{aligned}$$

If $y\in X\backslash B\left(x,m\rho\left(\xi,x\right)\right)$, then

$$\begin{aligned}\rho\left(\xi,x\right)+\rho\left(\xi,y\right)&\leq\rho\left(\xi,x\right)+c\left(\rho\left(\xi,x\right)+\rho\left(x,y\right)\right)\\&\leq\left(\left(c+1\right)m^{-1}+1\right)\rho\left(x,y\right).\end{aligned}$$

Then one have by $\left(K_2\right)$,

$$J_1\left(x\right)\leq\int\limits_{X\backslash B\left(x,m\rho\left(\xi,x\right)\right)}K\left(\frac{1}{\left(c+1\right)m^{-1}+1}\left(\rho\left(\xi,x\right)+\rho\left(\xi,y\right)\right)\right)f\left(y\right)dy$$

$$\leq C_1 \int\limits_X K\left(\rho\left(\xi,x\right)+\rho\left(\xi,y\right)\right) f\left(y\right) dy.$$

By conditions (3), (K_5) and Lebesgue's dominated convergence theorem,

$$J_1\left(x\right)\to 0, \text{ as } \rho\left(\xi,x\right)\to\infty.$$

Consider $J_2\left(x\right)$. Let $l>\sigma$. It is clear that

$$J_2\left(x\right)=\int\limits_{\left\{y;\rho(x,y)<m\rho(\xi,x),f(y)<\rho(\xi,x)^{-l}\right\}} K\left(\rho\left(x,y\right)\right) f\left(y\right) dy$$
$$+\int\limits_{\left\{y;\rho(x,y)<m\rho(\xi,x),f(y)\geq\rho(\xi,x)^{-l}\right\}} K\left(\rho\left(x,y\right)\right) f\left(y\right) dy$$
$$=J_{21}\left(x\right)+J_{22}\left(x\right).$$

By (K_4), we have

$$J_{21}\left(x\right)\leq\rho\left(\xi,x\right)^{-l}\int\limits_{B(x,m\rho(\xi,x))} K\left(\rho\left(x,y\right)\right) dy$$
$$\leq Fm^{\sigma}\rho\left(\xi,x\right)^{\sigma-l}\to 0, \text{ as } \rho\left(\xi,x\right)\to\infty.$$

By Lemma 4 and the assumptions of the theorem,

$$J_{22}\left(x\right)<L\left(\int\limits_{B(x,\rho(\xi,x))} f\left(y\right)^p w\left(f\left(y\right)\right) d\mu\left(y\right)\right)^{\frac{1}{p}}\left(\int\limits_{\rho(\xi,x)^{-l}}^{\infty} w\left(t\right)^{-\frac{1}{p-1}} t^{-1} dt\right)^{\frac{1}{p'}}$$
$$\leq L\left(\int\limits_{B(x,\rho(\xi,x))} f\left(y\right)^p w\left(f\left(y\right)\right) d\mu\left(y\right)\right)^{\frac{1}{p}} w^*\left(\rho\left(\xi,x\right)^{-1}\right).$$

Using Lemma 5, we have

$$w^*\left(\rho\left(\xi,x\right)^{-1}\right) J_{22}\left(x\right)\to 0, \text{ as } \rho\left(\xi,x\right)\to\infty.$$

So that

$$w^*\left(\rho\left(\xi,x\right)^{-1}\right)^{\frac{1}{p}} U_K f\left(x\right)\to 0, \text{ as } \rho\left(\xi,x\right)\to\infty.$$

Theorem is proved.

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References

[1] D. Adams and L. Hedberg, *Function spaces and potential theory.* Springer Verlag, Berlin, Heidelberg, New York, 1996.

[2] A. Gadjiev and O. Doğru, On combination of Riesz potentials with non-isotropic kernels. *Indian J. Pure Appl. Math.*, 30(6):545–556, 1999. 545-556.

[3] M. Hajibayov, The potential type operators on homogeneous groups. *Trans. of NAS of Azerbaijan, series of physical-tech. and math. sci.*, 26(1):81–88, 2006.

[4] M. Hajibayov, Continuity properties of potentials on spaces of homogeneous type. *International Centre for Theoretical Physics*, IC/2006/076, 1–13. *Int. J. Math. Anal.*, 2(5-8):315–328, 2008.

[5] M. Hajibayov, Behavior at infinity of convolution type integrals. *Acta Math. Univ. Comenian.* 78(1):75–85, 2009.

[6] T. Kurokowa and Y. Mizuta, On the order at infinity of Riesz potentials. *Hiroshima Math. J.*, 9:533–545, 1979.

[7] Y. Mizuta, Continuity properties of Riesz potentials and boundary limits of Beppo-Levi functions. *Math. Scand.,* 63:238–260, 1988.

[8] Y. Mizuta, *Potential theory in Euclidean spaces.* Gakuto International Series, Tokyo, 1996.

[9] D. Siegel and E. Talvila, Pointwise growth estimates of the Riesz potential. *Dynamics of Continuous, Discrete and Impulsive System,* 5:185–194, 1999.

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