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On the elementary solutions for one operator-differential equation of the second order

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Abstract. Sufficient conditions of the completeness of decreasing elementary solutions for one class elliptic type operator-differential equations of the second order, considered on the semi-axis with an operator in the boundary condition, are obtained in this paper.

Key Words and Phrases: operator pencil, operator-differential equation, regular solution, eigen and adjoint vectors, elementary solutions.
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Let H be a separable Hilbert space, A be a self-adjoint positive-defined operator in H, i.e. $A = A^* \geq cE$, $c > 0$, E be an identity operator. We denote by H_γ, $\gamma \geq 0$, a scale of Hilbert spaces, generated by the operator A, i.e. $H_\gamma = D\left(A^\gamma\right)$, $(x,y)_\gamma = \left(A^\gamma x,\ A^\gamma y\right)$, $x, y \in D\left(A^\gamma\right)$.

We suppose

$$L_2\left(R_+;H\right) = \left\{u\left(t\right) : \|u\|^2_{L_2(R_+;H)} = \int_0^{+\infty} \|u\left(t\right)\|^2_H \, dt < +\infty\right\},$$

$$W_2^2\left(R_+;H\right) = \left\{u\left(t\right) : \|u\|^2_{W_2^2(R_+;H)} = \int_0^{+\infty} \left(\left\|u''\left(t\right)\right\|^2_H + \left\|A^2 u\left(t\right)\right\|^2_H\right) dt < +\infty\right\}$$

(see [1, ch. 1]), where $R_+ = (0, +\infty)$. Here and further the derivatives are understood in sense of distributions theory.

Let $L(X,Y)$ be a set of the linear bounded operators, acting from Hilbert space X to another Hilbert space Y; $\sigma_\infty(H)$ is an ideal of absolutely continuous operators in $L(H,H)$; $\sigma_p(H)$ is Shatten-von Neumann ideal (see [2, ch. 3]); $\sigma(\cdot)$ is the spectrum of the operator $(\cdot)$.

We consider the polynomial operator pencil of the second order in H:

$$P\left(\lambda\right) = -\left(\lambda E - \omega_1 A\right)\left(\lambda E - \omega_2 A\right) + \lambda B A, \tag{1}$$

and connect with it the operator-differential equation

$$P\,(d/dt)\,u\,(t)=0, t\in R_{+}, \tag{2}$$

with the boundary condition at zero

$$u(0)-Tu'(0)=\varphi, \tag{3}$$

where

$$\omega_1<0,\ \omega_2>0,\ A=A^{*}\geq cE,\ c>0,$$
$$B\in L(H,H),\ T\in L(H_{1/2},H_{3/2}),$$
$$\varphi\in H_{3/2},\ u\,(t)\in W_2^2\,(R_{+};H)\,.$$

We suppose

$$W_{2,T}^{2}\,(R_{+};H)=\left\{u\,(t):u\,(t)\in W_2^2\,(R_{+};H)\,,\,u(0)=Tu'(0)\right\}.$$

From the theorem on traces [1, ch. 1] and from the condition $T\in L(H_{1/2},H_{3/2})$ it follows that $W_{2,T}^{2}\,(R_{+};H)$ is the closed subspace of the space $W_2^2\,(R_{+};H)$.

Definition. *If the vector-function $u\,(t)\in W_2^2\,(R_{+};H)$ satisfies the equation* (2) *almost everywhere in R_{+}, and the boundary condition* (3) *is satisfied in sense of*

$$\lim_{t\to 0}\left\|u\,(t)-Tu'\,(t)-\varphi\right\|_{H_{3/2}}=0,$$

then $u(t)$ is called the regular solution of the boundary value problem (2), (3).

The conditions, for which the boundary value problem (2), (3) has unique regular solution, are shown in the paper [3]. Namely, the following theorem is true.

Theorem 1. *Let $\frac{1}{\omega_1}\notin\sigma(C)$, $C=A^{3/2}TA^{-1/2}$ and $\|B\|<N_T^{-1}$. Then the boundary value problem* (2), (3) *has the unique regular solution for any $\varphi\in H_{3/2}$. Here*

$$N_T=(\omega_2-\omega_1)^{-1},$$

if $Q(\xi)\geq 0$ and

$$N_T=(\omega_2-\omega_1)^{-1}\left(1-\left|\inf_{\|\xi\|=1}\frac{Q(\xi)}{(\omega_2-\omega_1)(1+|\omega_1\omega_2|\,\|C\xi\|^2}\right|^2\right)^{-1/2},$$

if $\inf\limits_{\|\xi\|=1}Q(\xi)<0$, where

$$Q(\xi)=4\,|\omega_1\omega_2|\,Re(C\xi,\xi)+(\omega_1+\omega_2)\left(1-|\omega_1\omega_2|\,\|C\xi\|^2\right),\ \ \|\xi\|=1.$$

In the present paper the theorem on the completeness of decreasing elementary solutions of the equation (2) with the boundary condition (3) is proved in the space of all regular solutions of the boundary value problem (2), (3) with $A^{-1} \in \sigma_p(H)$. We use the research methods of the paper [4] here.

We note that if $A^{-1} \in \sigma_p(H)$ and $\|B\| < N_T^{-1}$, then the operator pencil (1) has only discrete spectrum with unique limit point at infinity.

Let λ_n, $n = 1, 2, ...$, be the eigenvalues of the operator pencil (1) from the left semi-plane and $e_{0,n}, e_{1,n}, ..., e_{m,n}$ be the eigen and adjoint vectors, corresponding to the eigenvalues λ_n, i.e.

$$P(\lambda_n)e_{0,n} = 0 \ (e_{0,n} \neq 0),$$

$$P(\lambda_n)e_{1,n} + \frac{P'(\lambda_n)}{1!}e_{0,n} = 0,$$

$$P(\lambda_n)e_{k,n} + \frac{P'(\lambda_n)}{1!}e_{k-1,n} + \frac{P''(\lambda_n)}{2!}e_{k-2,n} = 0, \ k = 2, 3, ..., m.$$

Then obviously, the vector-functions

$$u_{h,n}(t) = e^{\lambda_n t}\left(\frac{t^h}{h!}e_{0,n} + \frac{t^{h-1}}{(h-1)!}e_{1,n} + ... + e_{h,n}\right), \ h = 0, 1, ..., m,$$

belong to the space $W_2^2(R_+; H)$ and satisfy the equation (2). They are called the elementary solutions of the equation (2). Obviously, the elementary solutions $u_{h,n}(t)$ satisfy the following boundary conditions:

$$u_{0,n}(0) - Tu'_{0,n}(0) = e_{0,n} - \lambda_n T e_{0,n} \equiv \psi_{0,n}, \tag{4}$$

$$u_{h,n}(0) - Tu'_{h,n}(0) = e_{h,n} - \lambda_n T e_{h-1,n} \equiv \psi_{h,n}, \ h = 1, 2, ..., m. \tag{5}$$

The system $\{\psi_{h,n}\}_{n=1}^{\infty}$ is called the derivative chain of the eigen and adjoint vectors of the operator pencil (1), corresponding to the boundary value problem (2), (3).

The following theorem takes place.

Theorem 2. *Let* $\frac{1}{\omega_1} \notin \sigma(C)$, $C = A^{3/2}TA^{-1/2}$, $\|B\| < N_T^{-1}$, *where the number* N_T *is defined in the theorem 1, and one of the following conditions:*

$$1) \ A^{-1} \in \sigma_p(H)(0 < p \leq 1);$$

$$2) \ A^{-1} \in \sigma_p(H)(0 < p < \infty), B \in \sigma_\infty(H)$$

is satisfied. Then the system of elementary solutions of the boundary value problem (2), (3) *is complete in the space of all its regular solutions.*

Proof. First we denote by $W_2(P)$ the set of all regular solutions of the boundary value problem (2), (3). From the theorems on intermediate derivatives and on traces [1, ch. 1] it follows that $W_2(P)$ is the closed subspace of the space $W_2^2(R_+; H)$.

Let $u(t) \in W_2(P)$. Obviously, $u(t) \in W_2^2(R_+; H)$ and $u(0) - Tu'(0) = \varphi \in H_{3/2}$. Thus, if the conditions of the theorem are satisfied, i.e. $\frac{1}{\omega_1} \notin \sigma(C)$, $C = A^{3/2}TA^{-1/2}$, $\|B\| < N_T^{-1}$, then according to the theorem 1 $u(t)$ will be the regular solution of the boundary value problem (2), (3).

From the theorem on traces [1, ch. 1] and from the uniqueness of the solution of the boundary value problem (2), (3) we obtain the following inequalities:

$$c_1 \|\varphi\|_{H_{3/2}} \leq \|u\|_{W_2^3(R_+;H)} \leq c_2 \|\varphi\|_{H_{3/2}}. \tag{6}$$

The completeness of the system $\{\psi_{h,n}\}_{n=1}^{\infty}$ in the space $H_{3/2}$ has been determined in the paper [3] for satisfying all conditions of the theorem. As the system is complete $\{\psi_{h,n}\}_{n=1}^{\infty}$ in $H_{3/2}$, then for given $\varepsilon > 0$ there exist a number N and the numbers $c_{h,n}^N$ such that

$$\left\| \varphi - \sum_{n=1}^{N} \sum_{h} c_{h,n}^N \psi_{h,n} \right\|_{H_{3/2}} < \varepsilon. \tag{7}$$

Taking into consideration the equalities (3), (4), (5) in (6), and from the inequality (7) we obtain

$$\left\| u(t) - \sum_{n=1}^{N} \sum_{h} c_{h,n}^N u_{h,n}(t) \right\|_{W_2^2(R_+;H)} < c_2 \varepsilon.$$

It means that the system of elementary solutions is complete in $W_2(P)$. Theorem is proved.

We note that the problem of the completeness of elementary solutions for the boundary value problem (2), (3) for $\omega_1 = -1$, $\omega_2 = 1$ has been investigated in the paper [4].

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