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Solvability of the initial-boundary value problem for a second order parabolic operator-differential equation with multiple characteristics

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Abstract. In this paper, a class of parabolic operator-differential equations of the second order with multiple characteristics is considered, for which the initial boundary value problem on the semiaxis is well-posed and uniquely solvable in the Sobolev type-space.

Key Words and Phrases: operator-differential equation, multiple characteristic, initial-boundary value problem, Hilbert space, regular solvability.

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Suppose that H is a separable Hilbert space with a scalar product (x, y), $x, y \in H$, A is a self-adjoint positive definite operator on H ($A = A^* \geq cE$, $c > 0$, E is the identity operator). By H_γ ($\gamma \geq 0$) we denote the scale of Hilbert spaces generated by the operator A, i.e. $H_\gamma = Dom(A^\gamma)$, $(x, y)_\gamma = (A^\gamma x, A^\gamma y)$, x, $y \in Dom\,(A^\gamma)$. For $\gamma = 0$ we consider that $H_0 = H$, $(x, y)_0 = (x, y)$, $x, y \in H$.

By $L_2\,([a, b]; H)$ $(-\infty \leq a < b \leq +\infty)$ we denote the space of measurable functions (see [1]) with values in H and the norm

$$\|f\|_{L_2([a,b];H)} = \left(\int_a^b \|f(t)\|^2\, dt\right)^{1/2},$$

$W_2^2\,([a, b]; H)$ is the space of all functions with values in H such that $\frac{d^2u(t)}{dt^2}$, $A^2u(t) \in L_2\,([a, b]; H)$ with the norm

$$\|u\|_{W_2^2([a,b];H)} = \left(\left\|\frac{d^2u}{dt^2}\right\|^2_{L_2([a,b];H)} + \left\|A^2u\right\|^2_{L_2([a,b];H)}\right)^{1/2}.$$

For more details about the space $W_2^2\,([a, b]; H)$, see [2, Ch.1]. We assume that

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for $a=-\infty$, $b=+\infty$

$$L_2\left((-\infty,+\infty);H\right)\equiv L_2\left(R;H\right), W_2^2\left((-\infty,+\infty);H\right)\equiv W_2^2\left(R;H\right), R=(-\infty,+\infty),$$

for $a=0$, $b=+\infty$

$$L_2\left([0,+\infty);H\right)\equiv L_2\left(R_+;H\right), W_2^2\left([0,+\infty);H\right)\equiv W_2^2\left(R_+;H\right), R_+=[0,+\infty).$$

For the whole article, all the derivatives are understood in the sense of distributions.

Consider the following problem on the semiaxis R_+:

$$\frac{d^2u(t)}{dt^2}+2A\frac{du(t)}{dt}+A^2u(t)=f\left(t\right), t\in R_+, \tag{1}$$

$$u(0)=\frac{du(0)}{dt}=0, \tag{2}$$

where $A=A^*\geq cE,\quad c>0$, $f\left(t\right)\in L_2\left(R_+;H\right)$, $u\left(t\right)\in W_2^2\left(R_+;H\right)$.

Definition 1. If the vector function $u\left(t\right)\in W_2^2\left(R_+;H\right)$ satisfies equation (1) almost everywhere in R_+, then it is called a regular solution of equation (1).

Definition 2. If for any $f\left(t\right)\in L_2\left(R_+;H\right)$ there exists a regular solution $u\left(t\right)$ of equation (1) satisfying the boundary conditions (2) in the sense of relations

$$\lim_{t\to 0}\left\|A^{3/2}u\left(t\right)\right\|=0, \lim_{t\to 0}\left\|A^{1/2}\frac{du\left(t\right)}{dt}\right\|=0,$$

and the following inequality holds:

$$\|u\|_{W_2^2(R_+;H)}\leq const\,\|f\|_{L_2(R_+;H)},$$

then we say that say that problem (1), (2) is *regularly solvable.*

In this paper, the regular solvability of the initial boundary value problem (1), (2) is established.

In spite of the fact that, since 50th years of the last century, there are sufficiently many papers and books dedicated to the solvability of parabolic operator-differential equations in Banach spaces , in particular, Hilbert spaces, interest in such equations does not abate until recently (see, for example, [3-5]). However, in the mathematical literature studies on the parabolic operator-differential equations with multiple characteristics are almost unaffected, although they have a wide application in some problems in mechanics and mathematical physics. Note that equation (1) is an abstract parabolic equation in Hilbert space that has multiple characteristics.

We put

$$\overset{o}{W}{}_2^2\left(R_+;H\right)=\left\{u(t):\ u(t)\ \in W_2^2\left(R_+;H\right)\ ,\ u(0)=\frac{du(0)}{dt}=0\right\}$$

and denote by Q_0 the operator acting from the space $\overset{o}{W}{}_2^2\left(R_+;H\right)$ to the space $L_2\left(R_+;H\right)$ as follows:

$$Q_0 u\left(t\right) \equiv \frac{d^2u(t)}{dt^2} + 2A\frac{du(t)}{dt} + A^2u(t), u(t) \in \overset{o}{W}{}_2^2\left(R_+;H\right).$$

Then the following theorem holds

Theorem. *The operator Q_0 is an isomorphism between the spaces $\overset{o}{W}{}_2^2(R_+;H)$ and $L_2\left(R_+;H\right)$.*

Proof. Obviously, the equation $Q_0 u\left(t\right) = 0$, has only the trivial solution $u(t) \in \overset{o}{W}{}_2^2\left(R_+;H\right)$. Indeed, the equation $Q_0 u\left(t\right) = 0$ has a solution in the space $W_2^2\left(R_+;H\right)$ of the form

$$u_0(t) = e^{-tA}\varphi_0 + tAe^{-tA}\varphi_1,$$

where the vectors φ_0, $\varphi_1 \in H_{3/2}$. Taking into account conditions (2), we have the following system:

$$\begin{cases} \varphi_0 = 0, \\ -A\varphi_0 + A\varphi_1 = 0. \end{cases}$$

From this system we obtain that $\varphi_0 = \varphi_1 = 0$ and $u_0(t) = 0$. Therefore, the equation $Q_0 u\left(t\right) = 0$ has only the trivial solution in the space $\overset{o}{W}{}_2^2\left(R_+;H\right)$.

Now we shall show that the equation $Q_0 u\left(t\right) = f\left(t\right)$ for any $f\left(t\right) \in L_2\left(R_+;H\right)$ has a solution in the space $\overset{o}{W}{}_2^2\left(R_+;H\right)$. We extend the function $f\left(t\right)$ to zero for $t < 0$, then, applying the direct and inverse Fourier transform, it becomes clear that

$$u_1\left(t\right) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(i\xi E + A\right)^{-2}\left(\int_0^{+\infty} f\left(s\right)e^{-i\xi s}ds\right)e^{i\xi t}d\xi, t \in R,$$

satisfies the equation $Q_0 u\left(t\right) = f\left(t\right)$ almost everywhere in R. We show that $u_1\left(t\right) \in W_2^2\left(R;H\right)$. From Parseval's equality we obtain:

$$\left\|u_1\right\|_{W_2^2(R;H)}^2 = \left\|\frac{d^2u_1}{dt^2}\right\|_{L_2(R;H)}^2 + \left\|A^2u_1\right\|_{L_2(R;H)}^2 =$$

$$= \left\|-\xi^2\tilde{u}_1\left(\xi\right)\right\|_{L_2(R;H)}^2 + \left\|A^2\tilde{u}_1\left(\xi\right)\right\|_{L_2(R;H)}^2 =$$

$$= \left\|-\xi^2\left(i\xi E + A\right)^{-2}\tilde{f}\left(\xi\right)\right\|_{L_2(R;H)}^2 + \left\|A^2\left(i\xi E + A\right)^{-2}\tilde{f}\left(\xi\right)\right\|_{L_2(R;H)}^2 \leq$$

$$\leq \sup_{\xi\in R}\left\|-\xi^2\left(i\xi E + A\right)^{-2}\right\|_{H\to H}^2\left\|\tilde{f}\left(\xi\right)\right\|_{L_2(R;H)}^2 +$$

$$+\sup_{\xi\in R}\left\|A^2\left(i\xi E+A\right)^{-2}\right\|^2_{H\to H}\left\|\tilde{f}\left(\xi\right)\right\|^2_{L_2(R;H)}, \tag{3}$$

where $\tilde{u}_1\left(\xi\right)$ and $\tilde{f}\left(\xi\right)$ are the Fourier transforms of the functions $u_1\left(t\right)$ and$f\left(t\right)$, respectively. $\sigma\left(A\right)$ denotes the spectrum of the operator A. From the spectral decomposition of the operator A for $\xi\in R$, we have:

$$\left\|-\xi^2\left(i\xi E+A\right)^{-2}\right\|=\sup_{\mu\in\sigma(A)}\left|-\xi^2\left(i\xi+\mu\right)^{-2}\right|\le$$

$$\le\sup_{\mu\in\sigma(A)}\frac{\xi^2}{\xi^2+\mu^2}\le 1, \tag{4}$$

$$\left\|A^2\left(i\xi E+A\right)^{-2}\right\|=\sup_{\mu\in\sigma(A)}\left|\mu^2\left(i\xi+\mu\right)^{-2}\right|\le$$

$$\le\sup_{\mu\in\sigma(A)}\frac{\mu^2}{\xi^2+\mu^2}\le 1. \tag{5}$$

Taking into account (4) and (5) into (3) we obtain:

$$\|u_1\|^2_{W_2^2(R;H)}\le 2\left\|\tilde{f}\left(\xi\right)\right\|^2_{L_2(R;H)}=2\left\|f(t)\right\|^2_{L_2(R_+;H)}.$$

Therefore, $u_1\left(t\right)\in W_2^2\left(R;H\right)$. Now we denote the restriction of the function $u_1\left(t\right)$ to R_+ by $v\left(t\right)$. Obviously, $v\left(t\right)\in W_2^2\left(R_+;H\right)$. By the theorem n traces [2, Ch.1] $v\left(0\right)\in H_{3/2}$, $\frac{dv(0)}{dt}\in H_{1/2}$. Then we will find a solution to the equation $Q_0u\left(t\right)=f\left(t\right)$ of the form

$$u(t)=v(t)+e^{-tA}\psi_0+tAe^{-tA}\psi_1,$$

where the vectors ψ_0, $\psi_1\in H_{3/2}$. For a clear determination of ψ_0, ψ_1, from conditions (2) we obtain the following system:

$$\begin{cases} v(0)+\psi_0=0,\\ \frac{dv(0)}{dt}-A\psi_0+A\psi_1=0.\end{cases}$$

And from this system we have

$$\psi_0=-v(0)\in H_{3/2},$$

$$\psi_1=-v(0)-A^{-1}\frac{dv(0)}{dt}\in H_{3/2}.$$

Thus, $u(t)\in \overset{o}{W}{}_2^2\left(R_+;H\right)$.

It is clear that the operator Q_0 is bounded and acts from the space $\overset{o}{W}{}_2^2\left(R_+;H\right)$ to the space $L_2\left(R_+;H\right)$. Indeed, taking into account the theorem on intermediate derivatives [2, Ch.1], we have:

$$\|Q_0u\|_{L_2(R_+;H)} = \left\| \frac{d^2u}{dt^2} + 2A\frac{du}{dt} + A^2u \right\|_{L_2(R_+;H)} \leq$$
$$\leq \left\| \frac{d^2u}{dt^2} \right\|_{L_2(R_+;H)} + 2\left\| A\frac{du}{dt} \right\|_{L_2(R_+;H)} + \|A^2u\|_{L_2(R_+;H)} \leq$$
$$\leq const\, \|u\|_{W_2^2(R_+;H)} .$$

Thus, the operator $Q_0 : \overset{o}{W}{}_2^2(R_+; H) \to L_2(R_+; H)$ is bijective and bounded. Hence, using the Banach inverse operator theorem, the operator $Q_0^{-1} : L_2(R_+; H) \to \overset{o}{W}{}_2^2(R_+; H)$ is bounded. Therefore, the operator Q_0 is an isomorphism between the spaces $\overset{o}{W}{}_2^2(R_+; H)$ and $L_2(R_+; H)$.

The theorem is proved.

Corollary 1. It follows from Theorem 1 that initial-boundary value problem (1), (2) is regularly solvable.

Corollary 2. It follows from Theorem 1 that the norms $\|u\|_{W_2^2(R_+;H)}$ and $\|Q_0u\|_{L_2(R_+;H)}$ are equivalent on the space $\overset{o}{W}{}_2^2(R_+; H)$.

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