Journal of Contemporary Applied Mathematics
V. 7, No 2, 2017, December
ISSN 2222-5498

Some symmetric Laguerre-Hahn linear functionals of class six at most

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Abstract. We show that if v is a symmetric regular Laguerre-Hahn linear form (functional), then the linear functional u defined by $u = -\lambda x^{-4}v + \frac{1}{2}\delta_0' + \delta_0$ is also regular and symmetric Laguerre-Hahn linear functional for every complex λ except for a discrete set of numbers depending on v. We explicitly give the coefficients of the second-order recurrence relation of the orthogonal sequence associated with u and the class of the linear functional u knowing that of v. Finally, we apply the above results to some examples.

Key Words and Phrases: Orthogonal polynomials, Laguerre-Hahn linear functionals.

2010 Mathematics Subject Classifications: 42C05; 33C45

1. Introduction

The Product of linear functional by a polynomials is one of the construction processes of linear functionals. Christoffel has proved that the product of a positive definite linear functional by a positive polynomial gives a positive definite linear functional [7]. This result has been generalized in [8]. It was proved that, on certain regularity conditions, the product of a regular linear functional u by a polynomial R gives a regular linear functional. In particular, if u is a semiclassical [15] (resp., Laguerre-Hahn [2, 9]), then the linear functional Ru, if it is regular is also semiclassical (resp., Laguerre-Hahn). Then it is interesting to consider the inverse problem, which consists in determining all regular linear functionals u, satisfying $Ru = -\lambda v$, where v is a given regular linear functional and $\lambda \neq 0$. When $R(x) = x - c$, x^2 (resp., $R(x) = x^3$, x^4), Maroni [14, 17] (resp., Maroni and Nicolau [12, 13]) found necessary and sufficient conditions for u to be regular. Also, an explicit expression for the orthogonal polynomials (OP) with respect to u is proved. Finally, it was proved that, if v is semiclassical linear functional (see [1, 3, 13, 17]), then u is a semiclassical linear functional. See also [11]. In particular, in this paper, Marcellán and Prianes proved that if v is Laguerre-Hahn linear functional, then u is also a Laguerre-Hahn linear functional. When $R(x)$ is of degree two, Branquinho and Marcellán [4] found necessary and sufficient conditions for u to be regular. More generally, when $R(x)$ is any nonzero polynomial, Lee and Kwon [10] found a necessary and sufficient condition for u

to be regular and gave its corresponding OP in terms of the OP relative to v.

In this paper, we consider the same problem as in [12] in the symmetric Laguerre-Hahn case: given a symmetric Laguerre-Hahn linear functional v, find the linear functional u defined by

$$u = -\lambda x^{-4}v + \frac{1}{2}\delta_0'' + \delta_0 \ \Leftrightarrow\ x^4u = -\lambda v\ ,\ (u)_1 = (u)_3 = 0\ ,\ (u)_2 = 1.$$

Section 2 is devoted to the preliminary results and notations used in the sequel. In section 3, an explicit necessary and sufficient condition for the regularity of the new linear functional different from in [12] is given. We compute the exact class of the Laguerre-Hahn linear functional obtained by the above modification. Finally, we apply our results to some examples.

2. Notations and preliminary results

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle v, f\rangle$ the action of $v \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(v)_n := \langle v, x^n\rangle, n \geq 0$, the moments of v. For any linear functional v and any polynomial h let $Dv = v'$, hv, δ_0, and $x^{-1}v$ be the linear functionals defined by: $\langle v', f\rangle := -\langle v, f'\rangle$, $\langle hv, f\rangle := \langle v, hf\rangle$, $\langle \delta_0, f\rangle := f(0)$ and $\langle x^{-1}v, f\rangle := \langle v, \theta_0 f\rangle$ where $\big(\theta_0 f\big)(x) = \dfrac{f(x)-f(0)}{x}$, $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $f \in \mathcal{P}$ and $v \in \mathcal{P}'$, we have

$$x(x^{-1}v) = v\ , \tag{1}$$

$$\begin{aligned} x^{-1}(xv) &= v - (v)_0\delta_0\ , \\ x^{-1}\delta_0 &= -\delta_0'\ . \end{aligned} \tag{2}$$

We also defined the right multiplication of a linear functional by a polynomial with

$$(vh)(x) := \left\langle v, \frac{xh(x)-\xi h(\xi)}{x-\xi}\right\rangle = \sum_{k=0}^{n}\left(\sum_{j=k}^{n} a_j (v)_{j-k}\right)x^k\ ,\ h(x) = \sum_{j=0}^{n} a_j x^j\ .$$

Next, it is possible to define the product of two linear functionals through

$$\langle uv, f\rangle := \langle u, vf\rangle\ ,\ f \in \mathcal{P}\ .$$

Let us define the operator $\sigma : \mathcal{P} \to \mathcal{P}$ by $(\sigma f)(x) = f(x^2)$. Then, we define the even part σv of v by $\langle \sigma v, f\rangle := \langle v, \sigma f\rangle$.
Therefore, we have [16]

$$f(x)(\sigma v) = \sigma(f(x^2)v)\ . \tag{3}$$

A linear functional v is called regular if there exists a sequence of polynomials $\{S_n\}_{n\geq 0}$ ($\deg S_n \leq n$) such that

$$\langle v, S_n S_m \rangle = r_n \delta_{n,m} \quad , \quad r_n \neq 0, \quad n \geq 0 \, .$$

Then $\deg S_n = n, n \geq 0$ and we can always suppose each S_n is monic. In such a case, the sequence $\{S_n\}_{n\geq 0}$ is unique. It is said to be the sequence of monic orthogonal polynomials with respect to v.

It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (see, for instance, the monograph by Chihara [5])

$$\begin{aligned} &S_{n+2}(x) = (x - \xi_{n+1}) S_{n+1}(x) - \rho_{n+1} S_n(x) \, , \quad n \geq 0 \, , \\ &S_1(x) = x - \xi_0 \, , \qquad S_0(x) = 1 \, , \end{aligned} \tag{4}$$

with $\big(\xi_n, \rho_{n+1}\big) \in \mathbb{C} \times \mathbb{C} - \{0\} \, , \quad n \geq 0$. By convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the associated sequence of first order for the sequence $\{S_n\}_{n\geq 0}$ satisfying the recurrence relation

$$\begin{aligned} &S^{(1)}_{n+2}(x) = (x - \xi_{n+2}) S^{(1)}_{n+1}(x) - \rho_{n+2} S^{(1)}_n(x), \quad n \geq 0, \\ &S^{(1)}_1(x) = x - \xi_1, \qquad S^{(1)}_0(x) = 1, \quad (S^{(1)}_{-1}(x) = 0). \end{aligned} \tag{5}$$

Another important representation of $S_n^{(1)}(x)$ is, (see [6]),

$$S_n^{(1)}(x) := \left\langle v, \frac{S_{n+1}(x) - S_{n+1}(\zeta)}{x - \zeta} \right\rangle \, .$$

Also, let $\{S_n(., \mu)\}_{n\geq 0}$ be the co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [6]

$$S_n(x, \mu) = S_n(x) - \mu S^{(1)}_{n-1}(x), \quad n \geq 0 \, . \tag{6}$$

A linear functional v is called symmetric if $(v)_{2n+1} = 0, \; n \geq 0$. In 4, we have $\xi_n = 0, \; n \geq 0$ [5].

Throughout this paper, unless otherwise mentioned, the linear functionals v will be supposed normalized, (i.e., $(v)_0 = 1$), symmetric and regular.

Let us consider the decomposition of $\{S_n\}_{n\geq 0}$ [5, 16]:

$$S_{2n}(x) = \tilde{P}_n(x^2) \, , \quad S_{2n+1}(x) = x \tilde{R}_n(x^2) \, , \tag{7}$$

The sequences $\{\tilde{P}_n\}_{n\geq 0}$ and $\{\tilde{R}_n\}_{n\geq 0}$ are respectively orthogonal with to σv and $x\sigma v$. We also have

$$\begin{aligned} &\tilde{P}_{n+2}(x) = (x - \xi^{\tilde{P}}_{n+1}) \tilde{P}_{n+1}(x) - \rho^{\tilde{P}}_{n+1} \tilde{P}_n(x) \, , \quad n \geq 0 \, , \\ &\tilde{P}_1(x) = x - \xi^{\tilde{P}}_0 \, , \qquad \tilde{P}_0(x) = 1 \, , \end{aligned} \tag{8}$$

with

$$\xi^{\tilde{P}}_0 = \rho_1 \, , \; \xi^{\tilde{P}}_{n+1} = \rho_{2n+2} + \rho_{2n+3} \, , \; \rho^{\tilde{P}}_{n+1} = \rho_{2n+1} \rho_{2n+2} \, , \; n \geq 0 \, . \tag{9}$$

By virtue of 4, with $\xi_n = 0$, we get $S_{n+2}(0) = -\rho_{n+1}S(0)$. Consequently,

$$S_0(0) = \tilde{P}_0(0) = 1\ ,\ S_{2n+2}(0) = \tilde{P}_{n+1}(0) = (-1)^{n+1}\prod_{\nu=0}^{n}\rho_{2\nu+1}\ ,\ n \geq 0\ . \tag{10}$$

Proposition 2.1. *[15] v is regular if and only if σv and $x\sigma v$ are regular.*

3. The Laguerre-Hahn case

For a $\lambda \in \mathbb{C} - \{0\}$, we can define a new linear functional in $\mathcal{P}'$,

$$u = -\lambda x^{-4}v + +\frac{1}{2}\delta_0' + \delta_0\ . \tag{11}$$

From 1, 2 and 11, we have

$$x^4 u = -\lambda v\ ,\ (u)_1 = (u)_3 = 0\ ,\ (u)_2 = 1\ . \tag{12}$$

Remark 3.1. *The above problem was partially treated by P. Maroni and I. Nicolau [12] and we are going to handle it differently.*

Proposition 3.2. *u is regular if and only if*

$$\tilde{P}_n(0,\lambda)\Delta_n \neq 0\ ,\ n \geq 0\ .$$

where $\tilde{P}_n$ is defined by 7 and

$$\Delta_n = \tau_n\{\lambda + \sum_{\nu=0}^{n}\frac{\tilde{P}_\nu^2(0,\lambda)}{\tau_\nu}\}\ ,\ n \geq 0\ ,$$

$$\tau_n := \left\langle \sigma v, \tilde{P}_n^2 \right\rangle = \prod_{\nu=0}^{n}\rho_\nu^{\tilde{P}} = \prod_{\nu=0}^{2n}\rho_\nu\ ,\ n \geq 0\ .$$

Proof. Applying the operator σ for 12 and using 3, we obtain

$$x\sigma u = -\lambda x^{-1}\sigma v + \delta_0\ . \tag{13}$$

From 2 and 13, we get

$$\sigma u = -\lambda x^{-2}\sigma v - \delta_0' + \delta_0\ . \tag{14}$$

From 11, it is plain that u is symmetric linear functional. Then, according to , u is regular if and only if $x\sigma u$ and σu are regular. But $x\sigma u = -\lambda x^{-1}\sigma v + \delta_0$ is regular if and only if $\lambda \neq 0$ and $\tilde{P}_n(0,\lambda) \neq 0\ ,\ n \geq 0$ (see [17]). So u is regular if and only if $\tilde{P}_n(0,\lambda) \neq 0$ and $\sigma u = -\lambda x^{-2}\sigma v - \delta_0' + \delta_0$ is regular. Or, it was shown in [3] that the form $-\lambda x^{-2}\sigma v - \delta_0' + \delta_0$ is regular if and only if $\Delta_n \neq 0\ ,\ n \geq 0$. Then, we deduce the desired result. □

Remark 3.3. *(i) In fact, we have the well-known identity (see [5])*

$$\tilde{P}_{n+1}(0)\tilde{P}^{(1)}_{n+1}(0) - \tilde{P}_{n+2}(0)\tilde{P}^{(1)}_{n}(0) = \prod_{\nu=0}^{n} \rho^{\tilde{P}}_{\nu+1} \ , \ n \geq 0 \ .$$

Dividing the above equation by $\tilde{P}_{n+2}(0)\tilde{P}_{n+1}(0)$ and using 7, 9 and 10, we obtain

$$\frac{\tilde{P}^{(1)}_{n+1}(0)}{\tilde{P}_{n+2}(0)} - \frac{\tilde{P}^{(1)}_{n}(0)}{\tilde{P}_{n+1}(0)} = -\prod_{\nu=0}^{n+1} \frac{\rho_{2\nu}}{\rho_{2\nu+1}} \ , \ n \geq 0 \ .$$

This leads to

$$\tilde{P}^{(1)}_{n}(0) = -\tilde{P}_{n+1}(0)\Omega_n \ , \ n \geq 0 \ , \tag{15}$$

with

$$\Omega_n = \sum_{\nu=0}^{n} \frac{\tau^e_\nu}{\tau^o_\nu} \ , \ n \geq 0 \ ,$$

and

$$\tau^e_n = \prod_{\nu=0}^{n} \rho_{2\nu} \ , \quad \tau^o_n = \prod_{\nu=0}^{n} \rho_{2\nu+1} \ , \ n \geq 0 \ .$$

Using 6 and 15, we can easily find that u is regular if and only if

$$(1+\lambda\Omega_n)\Big(\lambda^2\Theta^{(2)}_n + \lambda(1+2\Theta^{(1)}_n) + 1 + \Theta^{(0)}_n\Big) \neq 0 \ , \ n \geq 0 \ , \tag{16}$$

where for $i \in \{0,1,2\}$

$$\Theta^{(i)}_n = \sum_{\nu=0}^{n-1} \frac{\tau^o_\nu}{\tau^e_{\nu+1}}\Omega^i_\nu \ , \ n \geq 0 \ , \ \left(\sum_{\nu=0}^{-1}\right) = 0 \ .$$

(ii) If v is a symmetric positive definite functional and $\lambda > 0$, then from 16, the linear functional u is regular.

When u is regular, let $\{Z_n\}_{n\geq 0}$ be the corresponding sequence satisfying the recurrence relation

$$\begin{aligned} &Z_{n+2}(x) = xZ_{n+1}(x) - \gamma_{n+1}Z_n(x) \ , \quad n \geq 0 \ , \\ &Z_1(x) = x \ , \qquad Z_0(x) = 1 \ . \end{aligned} \tag{17}$$

Let us now consider the quadratic decomposition of the sequence $\{Z_n\}_{n\geq 0}$

$$Z_{2n}(x) = P_n(x^2) \ , \quad Z_{2n+1}(x) = xR_n(x^2) \ , \quad n \geq 0 \ . \tag{18}$$

From 13 and 14, we can deduce the following results.

Proposition 3.4. [17] *The polynomials of the sequence* $\{R_n\}_{n\geq 0}$ *satisfy the relation*

$$R_{n+1}(x) = \tilde{P}_{n+1}(x) + \tilde{a}_n \tilde{P}_n(x) \; , \quad n \geq 0 \; , \tag{19}$$

where

$$\tilde{a}_n = -\frac{\tilde{P}_{n+1}(0,\lambda)}{\tilde{P}_n(0,\lambda)} \; , \quad n \geq 0 \; . \tag{20}$$

Proposition 3.5. *[3] The polynomials of the sequence* $\{P_n\}_{n\geq 0}$ *satisfy the relation*

$$\begin{aligned} &P_{n+2}(x) = \tilde{P}_{n+2}(x) + \tilde{c}_{n+1}\tilde{P}_{n+1}(x) + \tilde{b}_n \tilde{P}_n(x) \; , \quad n \geq 0 \; , \\ &P_1(x) = \tilde{P}_1(x) + \tilde{c}_0 \; , \end{aligned} \tag{21}$$

where

$$\tilde{b}_n = \frac{\Delta_{n+1}}{\Delta_n} \; , \quad n \geq 0 \; ,$$

and

$$\tilde{c}_0 = \rho_1 - 1 \; , \quad \tilde{c}_{n+1} = \rho_{2n+2} + \rho_{2n+3} - \frac{\tilde{P}_{n+1}(0,\lambda)\tilde{P}_n(0,\lambda)}{\Delta_n} \; , \quad n \geq 0 \; .$$

Lemma 3.6.

$$\begin{aligned} &Z_{n+4}(x) = S_{n+4}(x) + b_{n+2}S_{n+2}(x) + a_n S_n(x) \; , \quad n \geq 0 \; , \\ &Z_3(x) = S_3(x) + b_1 S_1(x) \; , \\ &Z_2(x) = S_2(x) + b_0 \; . \end{aligned} \tag{22}$$

with

$$\begin{aligned} &a_{2n} = \tilde{b}_n \; , \quad a_{2n+1} = \rho_{2n+2}\tilde{a}_{n+1} \; , \quad n \geq 0 \; , \\ &b_{2n+2} = \tilde{c}_{n+1} \; , \quad b_{2n+3} = \rho_{2n+4} + \tilde{a}_{n+1} \; , \quad n \geq 0 \; , \\ &b_1 = \rho_2 + \tilde{a}_0 \; , \quad b_0 = \tilde{c}_0 \; . \end{aligned} \tag{23}$$

Proof. From 18, 19 and 21, we have for $n \geq 0$

$$\begin{aligned} &Z_{2n+5}(x) = x\tilde{P}_{n+2}(x^2) + x\tilde{a}_{n+1}\tilde{P}_{n+1}(x^2) \; , \\ &Z_{2n+4}(x) = \tilde{P}_{n+2}(x^2) + \tilde{c}_{n+1}\tilde{P}_{n+1}(x^2) + \tilde{b}_n\tilde{P}_n(x^2) \; , \\ &Z_3(x) = x\tilde{P}_1(x^2) + \tilde{a}_0 \; , \\ &Z_2(x) = \tilde{P}_1(x^2) + \tilde{c}_0 \; . \end{aligned}$$

Then, from the above equations, 4 and 7, we get 22 and 23. ☐

Proposition 3.7. *[17]*

$$\begin{aligned} &\gamma_1 = 1 \; , \; \gamma_3 = \lambda\tfrac{\rho_1+\lambda}{\lambda+1} \; , \; \gamma_{2n+5} = \rho_{n+1}\tfrac{\tilde{a}_{n+1}\Delta_n}{\Delta_{n+1}} \; , \\ &\gamma_2 = -\lambda - 1 \; , \; \gamma_{2n+4} = \tfrac{\Delta_{n+1}}{\Delta_n} \; , ; n \geq 0 \; . \end{aligned} \tag{24}$$

Remark 3.8. *From 9, 17 and 24, the sequence $\{P_n\}_{n\geq 0}$ satisfies the recurrence relation 8 with*

$$\begin{aligned} &\beta_0^P = 1\ ,\ \beta_1^P = -\lambda - 1 + \lambda\tfrac{\rho+\lambda}{\lambda+1}\ ,\ \beta_{n+2}^P = \tfrac{\Delta_{n+1}}{\Delta_n} + \rho_{n+1}\tfrac{\tilde{a}_{n+1}\Delta_n}{\Delta_{n+1}}\ ,\ n \geq 0\ ,\\ &\gamma_1^P = -1 - \lambda\ ,\ \gamma_2^P = \lambda\tfrac{\rho+\lambda}{(\lambda+1)^2}\Delta_1\ ,\ \gamma_{n+3}^P = \rho_{n+1}\tfrac{\tilde{a}_{n+1}\Delta_n}{\Delta_{n+1}}\tfrac{\Delta_{n+2}}{\Delta_{n+1}}\ ,;n \geq 0\ . \end{aligned} \tag{25}$$

Definition 3.9. *[2] The regular linear functional v is called Laguerre-Hahn if its formal Stieltjes function $S(v)(z)$ satisfies the Riccati equation*

$$\tilde{\Phi}(z)S^{'}(v)(z) = \tilde{B}(z)S^2(v)(z) + \tilde{C}(z)S(v)(z) + \tilde{D}(z)\ , \tag{26}$$

where $\tilde{\Phi}$ is monic, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are polynomials and

$$S(v)(z) = -\sum_{n\geq 0}\frac{(v)_n}{z^{n+1}}\ .$$

It was shown in [9] that 26 is equivalent to

$$(\tilde{\Phi}(x)v)^{'} + \tilde{\Psi}v + \tilde{B}(x^{-1}v^2) = 0\ , \tag{27}$$

with

$$\tilde{\Psi}(x) = -\tilde{\Phi}^{'}(x) - \tilde{C}(x)\ .$$

We also have the following relation:

$$\tilde{D}(x) = -(v\theta_0\tilde{\Phi})^{'}(x) - (v\theta_0\tilde{\Psi})(x) - (v^2\theta_0^2\tilde{B})(x)\ .$$

Remark 3.10. *[15] When $\tilde{B} = 0$ in 26 or 27, the linear form v is semiclassical.*

Proposition 3.11. [1] *We define $\tilde{d} = \max(\deg(\tilde{\Phi}), \deg(\tilde{B}))$ and $\tilde{p} = \deg(\tilde{\Psi})$.*
The Laguerre-Hahn linear functional v satisfying 27 is of class $\tilde{s} = \max(\tilde{d}-2, \tilde{p}-1)$ if and only if

$$\prod_{c\in\mathcal{Z}}\{|\tilde{\Phi}^{'}(c) + \tilde{\Psi}(c)| + |\tilde{B}(c)| + |\left\langle v, \theta_c^2\tilde{\Phi} + \theta_c\tilde{\Psi} + v\theta_0\theta_c\tilde{B}\right\rangle|\} \neq 0\ ,$$

where $\mathcal{Z}$ denotes the set of zeros of $\tilde{\Phi}$.

Corollary 3.12. *The linear functional v satisfying 26 is of class $\tilde{s}$ if and only if*

$$\prod_{c\in\mathcal{Z}}\{|\tilde{C}(c)| + |\tilde{B}(c)| + |\tilde{D}(c)|\} \neq 0\ . \tag{28}$$

Remark 3.13. *(i) Equation 28 is equivalent to the fact the polynomial coefficients in 26 are coprime.*
(ii) Since v is symmetric linear functional, then if $\tilde{p}$ is even, $\deg(\tilde{\Phi})$ and $\deg(\tilde{B})$ are odd ; if $\tilde{p}$ is odd, $\deg(\tilde{\Phi})$ and $\deg(\tilde{B})$ are even.

Proposition 3.14. *If v is a Laguerre-Hahn linear functional satisfying 26, then for every $\lambda \in \mathbb{C} - \{0\}$ such that $\tilde{P}_n(0,\lambda)\Delta_n \neq 0,\ n \geq 0$, the linear functional u defined by 11 is regular and Laguerre-Hahn. It satisfies*

$$\Phi(z)S^{'}(u)(z) = B(z)S^2(u)(z) + C(z)S(u)(z) + D(z)\ , \tag{29}$$

where

$$\begin{aligned} &\Phi(z) = z^4\tilde{\Phi}(z)\ , \\ &B(z) = -\lambda^{-1}z^8\tilde{B}(z)\ , \\ &C(z) = z^4\tilde{C}(z) - 2\lambda^{-1}z^5(z^2+1)\tilde{B}(z) - 4z^3\tilde{\Phi}(z)\ , \\ &D(z) = z(z^2+1)\tilde{C}(z) - (3z^2+1)\tilde{\Phi}(z) - \lambda^{-1}z^2(z^2+1)^2\tilde{B}(z) - \lambda\tilde{D}(z)\ , \end{aligned} \tag{30}$$

and u is of class s such that $s =\leq \tilde{s}+8$.

Proof. We have [14]

$$\begin{aligned} S(v)(z) &= S(-\lambda^{-1}\xi^4 u)(z) \\ &= -\lambda^{-1}\{z^4S(u)(z) + (u\theta_0(\xi^4))(z)\} \\ &= -\lambda^{-1}\{z^4S(u)(z) + z^3 + z\}\ . \end{aligned} \tag{31}$$

By substitution 31 in 26, we easily find 29 and 30.

The linear functional u satisfies the distribution equation

$$(\Phi(x)u)^{'} + \Psi u + B(x^{-1}u^2) = 0\ , \tag{32}$$

where Φ and B are the polynomials defined by 30 and

$$\Psi(x) = -\Phi^{'}(x) - C(x) = x^4(\tilde{\Psi}(x) + 2\lambda^{-1}x(x^2+1)\tilde{B}(x))\ . \tag{33}$$

Then $\deg(\Phi) \leq \tilde{s}+6$, $\deg(B) \leq \tilde{s}+10$ and $\deg(\Psi) := p \leq \tilde{s}+9$.

Thus $d = \max(\deg(\Phi), \deg(B)) \leq \tilde{s}+10$ and $s = \max(d-2, p-1) \leq \tilde{s}+8$. □

Proposition 3.15. *The class of u depends only on the zero $x = 0$ of Φ.*

Proof. Since v is a Laguerre-Hahn linear functional of class $\tilde{s}$, $S(v)(z)$ satisfies 26, where the polynomials $\tilde{\Phi}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are coprime. Let Φ , B , C and D be as in . Let c be a zero of Φ different from 0, this implies that $\tilde{\Phi}(c) = 0$. We know that $|\tilde{C}(c)| + |\tilde{B}(c)| + |\tilde{D}(c)| \neq 0$,

(i) if $\tilde{B}(c) \neq 0$, then $B(c) \neq 0$,

(ii) if $\tilde{B}(c) = 0$ and $\tilde{C}(c) \neq 0$, then $C(c) \neq 0$,

(iii) if $\tilde{B}(c) = \tilde{C}(c) = 0$, then $\tilde{D}(c) \neq 0$, whence $|C(c)| + |B(c)| + |D(c)| \neq 0$. □

Concerning the class of u, we have the following result.

Proposition 3.16. *Let $t = \deg(\Phi)$, $r = \deg(B)$, $p = \deg(\Psi)$, $X(z) = \tilde{\Phi}(z) + \lambda \tilde{D}(z)$ and $Y(z) = 3\tilde{\Phi}(z) + \lambda^{-1}\tilde{B}(z)$. Under the conditions of , for the class of u, the following four different cases hold:*
(1) If $X(0) \neq 0$, then

$$s = \begin{cases} \tilde{s}+8 & if\, \tilde{p} < \tilde{r}+1,\ \tilde{t} \leq \tilde{r}\,, \\ \tilde{s}+6 & if\, \tilde{p} = \tilde{r}+1,\ \tilde{t} \leq \tilde{r}+2 \text{ or } \tilde{p} < \tilde{r}+1\,,\, \tilde{t} = \tilde{r}+2\,, \\ \tilde{s}+4 & if \tilde{p} \leq \tilde{r}+1\,,\, \tilde{t} \geq \tilde{r}+4 \text{ or } \tilde{p} \geq \tilde{r}+3\,. \end{cases} \tag{34}$$

(2) If $X(0) = 0$ and $\tilde{C}(0) - X'(0) \neq 0$, then

$$s = \begin{cases} \tilde{s}+7 & if\, \tilde{p} < \tilde{r}+1,\ \tilde{t} \leq \tilde{r}\,, \\ \tilde{s}+5 & if\, \tilde{p} = \tilde{r}+1,\ \tilde{t} \leq \tilde{r}+2 \text{ or } \tilde{p} < \tilde{r}+1\,,\, \tilde{t} = \tilde{r}+2\,, \\ \tilde{s}+3 & if \tilde{p} \leq \tilde{r}+1\,,\, \tilde{t} \geq \tilde{r}+4 \text{ or } \tilde{p} \geq \tilde{r}+3\,. \end{cases} \tag{35}$$

(3) If $X(0) = \tilde{C}(0) - X'(0) = 0$ and $\tilde{C}'(0) - Y(0) - \frac{1}{2}X''(0) \neq 0$, then

$$s = \begin{cases} \tilde{s}+6 & if\, \tilde{p} < \tilde{r}+1,\ \tilde{t} \leq \tilde{r}\,, \\ \tilde{s}+4 & if\, \tilde{p} = \tilde{r}+1,\ \tilde{t} \leq \tilde{r}+2 \text{ or } \tilde{p} < \tilde{r}+1\,,\, \tilde{t} = \tilde{r}+2\,, \\ \tilde{s}+2 & if \tilde{p} \leq \tilde{r}+1\,,\, \tilde{t} \geq \tilde{r}+4 \text{ or } \tilde{p} \geq \tilde{r}+3\,, \end{cases} \tag{36}$$

(4) If $X(0) = \tilde{C}(0) - X'(0) = \tilde{C}'(0) - Y(0) - \frac{1}{2}X''(0) = 0$ and $\tilde{\Phi}(0) \neq 0$, then

$$s = \begin{cases} \tilde{s}+5 & if\, \tilde{p} < \tilde{r}+1,\ \tilde{t} \leq \tilde{r}\,, \\ \tilde{s}+3 & if\, \tilde{p} = \tilde{r}+1,\ \tilde{t} \leq \tilde{r}+2 \text{ or } \tilde{p} < \tilde{r}+1\,,\, \tilde{t} = \tilde{r}+2\,, \\ \tilde{s}+1 & if \tilde{p} \leq \tilde{r}+1\,,\, \tilde{t} \geq \tilde{r}+4 \text{ or } \tilde{p} \geq \tilde{r}+3\,, \end{cases} \tag{37}$$

where the polynomials $\tilde{\Phi}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ are defined in (26).

Proof. (1) If $X(0) \neq 0$, then from 30 and 33 we have

$$t = \tilde{t} + 4\,, \quad r = \tilde{r} + 8\,, \quad p \leq \max(\tilde{p}+4, \tilde{r}+7)\,.$$

We will distinguish three cases:

(a) $\tilde{p} < \tilde{r}+1$, then $p = \tilde{r}+7$ and $s = \max(\tilde{t}+2, \tilde{r}+7)$. If $\tilde{t} \leq \tilde{r}$, then $s = \tilde{r}+6 = \tilde{s}+8$. If $\tilde{t} = \tilde{r}+2$, then $s = \tilde{r}+6 = \tilde{s}+6$, since $\tilde{s} = \tilde{r}$ in this case. If $\tilde{t} \geq \tilde{r}+4$, then $s = \tilde{t}+2 = \tilde{s}+4$.

(b) $\tilde{p} = \tilde{r}+1$, then $p = \tilde{r}+7 = \tilde{p}+6$ and $s = \max(\tilde{t}+2, \tilde{p}+5)$. If $\tilde{t} \leq \tilde{p}+1$, then $s = \tilde{p}+5 = \tilde{s}+6$. If $\tilde{t} \geq \tilde{p}+3$, then $s = \tilde{t}+2 = \tilde{s}+4$.

(c) $\tilde{p} \geq \tilde{r}+3$, then $p \leq \tilde{p}+4$ and $s = \max(\tilde{t}+2, \tilde{p}+3)$.
Since $\tilde{t}+2 = \tilde{s}+4$ or $\tilde{p}+3 = \tilde{s}+4$, then $s = \tilde{s}+4$.

Therefore, from the above situations, we deduce 34.

(2) If $X(0)=0$, then from 30 we have $B(0)=C(0)=D(0)=0$, therefore 29-30 is divisible by z. Thus, u fulfils 29 with

$$\begin{cases} \Phi(z)=z^3\tilde{\Phi}(z)\ , \\ B(z)=-\lambda^{-1}z^7\tilde{B}(z)\ , \\ C(z)=z^3\tilde{C}(z)-2\lambda^{-1}z^4(z^2+1)\tilde{B}(z)-4z^2\tilde{\Phi}(z)\ , \\ D(z)=(z^2+1)\tilde{C}(z)-3z\tilde{\Phi}(z)-\lambda^{-1}z(z^2+1)^2\tilde{B}(z)-(\theta_0(\tilde{\Phi}+\lambda\tilde{D}))(z)\ . \end{cases} \tag{38}$$

Whether $\tilde{C}(0)-X'(0)\neq 0$, it is not possible to simplify, which means that the class of u verifies **??**.

(3) If $X(0)=\tilde{C}(0)-X'(0)=0$, then it is possible to simplify 29-38 by z. Thus, u fulfils 29 with

$$\begin{cases} \Phi(z)=z^2\tilde{\Phi}(z)\ , \\ B(z)=-\lambda^{-1}z^6\tilde{B}(z)\ , \\ C(z)=z^2\tilde{C}(z)-2\lambda^{-1}z^3(z^2+1)\tilde{B}(z)-4z\tilde{\Phi}(z)\ , \\ D(z)=z\tilde{C}(z)+(\theta_0\tilde{C})(z)-3\tilde{\Phi}(z)-\lambda^{-1}(z^2+1)^2\tilde{B}(z)-(\theta_0^2(\tilde{\Phi}+\lambda\tilde{D}))(z)\ . \end{cases} \tag{39}$$

Thus, if $\tilde{C}'(0)-Y(0)-\frac{1}{2}X''(0)\neq 0$, it is not possible to simplify, which means that the class of u verifies 37.

(4) If $X(0)=\tilde{C}(0)-X'(0)=\tilde{C}'(0)-Y(0)-\frac{1}{2}X''(0)=0$, then it is possible to simplify 29-39 by z. Thus, u fulfils 29 with

$$\begin{cases} \Phi(z)=z\tilde{\Phi}(z)\ , \\ B(z)=-\lambda^{-1}z^5\tilde{B}(z)\ , \\ C(z)=z\tilde{C}(z)-2\lambda^{-1}z^2(z^2+1)\tilde{B}(z)-4\tilde{\Phi}(z)\ , \\ D(z)=\tilde{C}(z)+(\theta_0^2\tilde{C})(z)-(\theta_0(3\tilde{\Phi}+\lambda^{-1}\tilde{B}))(z)-\lambda^{-1}z\tilde{B}(z)-(\theta_0^3(\tilde{\Phi}+\lambda\tilde{D}))(z)\ . \end{cases}$$

Therefore, if $\tilde{\Phi}(0)\neq 0$, it is not possible to simplify, which means that the class of u verifies 37. □

4. Examples

Example 4.1. *Let v be the associated linear functional of the first order of Hermite. Here [9, 11]*

$$\rho_{2n+1}=n+1\ ,\quad \rho_{2n+2}=\frac{2n+3}{2}\ ,\quad n\geq 0\ , \tag{40}$$

$$\begin{gathered} \tilde{\Phi}(x)=1\ ,\quad \tilde{\Psi}(x)=2x\ ,\quad \tilde{B}(x)=-1\ , \\ \tilde{C}(x)=-2x\ ,\quad \tilde{D}(x)=-2\ . \end{gathered} \tag{41}$$

In this case, the linear functional v is a Laguerre-Hahn linear functional of class $\tilde{s}=0$.

From 10 and 40, we have

$$\tilde{P}_n(0) = (-1)^n \Gamma(n+1)\ , \quad n \geq 0\ . \tag{42}$$

From 5, 8, 9 and 40, we get the second -order recurrence relation satisfied by $\{\tilde{P}_n\}_{n\geq 0}$. *Using this relation, we deduce by induction*

$$\tilde{P}_n^{(1)}(0) = 2(-1)^{n+1}\left(\Gamma(n+2) - \frac{2\Gamma(n+\frac{5}{2})}{\sqrt{\pi}}\right)\ , \quad n \geq 0\ . \tag{43}$$

Using 6, 42 and 43, we obtain

$$\tilde{P}_n(0,\lambda) = \frac{(-1)^n}{\sqrt{\pi}} d_n(\lambda)\ , \quad n \geq 0\ , \tag{44}$$

and

$$\Delta_n = \frac{\Gamma(2n+2)}{2^{2n}}\left(\lambda^2 \Theta_n^{(2)} + \lambda(1 + 2\Theta_n^{(1)}) + 1 + \Theta_n^{(0)}\right)\ , \quad n \geq 0\ ,$$

with

$$d_n(\lambda) = (1-2\lambda)\sqrt{\pi}\Gamma(n+1) + 4\lambda\Gamma(n+\frac{3}{2})\ , \quad n \geq 0\ ,$$

and

$$\begin{cases} \Theta_n^{(2)} & = 2\displaystyle\sum_{\nu=0}^{n-1}(\sqrt{\pi}\frac{\Gamma(\nu+2)}{\Gamma(\nu+\frac{5}{2})} + \frac{4}{\sqrt{\pi}}\frac{\Gamma(\nu+\frac{5}{2})}{\Gamma(\nu+2)} - 4)\ , \\ \Theta_n^{(1)} & = 2\displaystyle\sum_{\nu=0}^{n-1}(1 - \sqrt{\pi}\frac{\Gamma(\nu+2)}{\Gamma(\nu+\frac{5}{2})})\ , \\ \Theta_n^{(0)} & = \frac{\sqrt{\pi}}{2}\displaystyle\sum_{\nu=0}^{n-1}\frac{\Gamma(\nu+2)}{\Gamma(\nu+\frac{5}{2})}\ . \end{cases}$$

Or,

$$\frac{\Gamma(\nu+2)}{\Gamma(\nu+\frac{5}{2})} = \sigma_{\nu+1} - \sigma_\nu\ , \quad \frac{\Gamma(\nu+\frac{5}{2})}{\Gamma(\nu+2)} = \omega_{\nu+1} - \omega_\nu\ ,$$

with

$$\begin{cases} \sigma_\nu = 2\frac{\Gamma(\nu+2)}{\Gamma(\nu+\frac{3}{2})}\ , & \nu \geq 0\ , \\ \omega_\nu = \frac{2}{3}\frac{\Gamma(\nu+\frac{5}{2})}{\Gamma(\nu+1)}\ , & \nu \geq 0\ . \end{cases}$$

Therefore for $n \geq 0$, we have

$$\Delta_n = \frac{\Gamma(2n+2)}{2^{2n}} e_n(\lambda)\ , \tag{45}$$

with

$$\begin{aligned} e_0(\lambda) &= \lambda + 1\ , \\ e_n(\lambda) &= 4(\sqrt{\pi}\tfrac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} + \tfrac{4}{3}\tfrac{\Gamma(n+\frac{5}{2})}{\Gamma(n+1)} - 2n - 3)\lambda^2 + (18n + 17 - 4\sqrt{\pi}\tfrac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})})\lambda \\ &\quad + \sqrt{\pi}\tfrac{\Gamma(n+2)}{\Gamma(n+\frac{3}{2})} - 1\ ,\ n \geq 1\ . \end{aligned}$$

Then, u is regular for every $\lambda \neq 0$ such that

$$d_n(\lambda)e_n(\lambda) \neq 0\ , \quad n \geq 0\ . \tag{46}$$

From 20 and 44, we get

$$\tilde{a}_n = \frac{d_{n+1}(\lambda)}{d_n(\lambda)}\ , \quad n \geq 0\ . \tag{47}$$

Using 24, 45 and 47, we obtain

$$\begin{cases} \gamma_1 = 1\ ,\ \gamma_2 = -\lambda - 1\ ,\ \gamma_3 = \lambda\ , \\ \gamma_{2n+4} = (n+1)(2n+3)\frac{e_{n+1}(\lambda)}{2e_n(\lambda)}\ ,\ n \geq 0\ , \\ \gamma_{4n+5} = 2\frac{(n+1)d_{2n+2}(\lambda)e_{2n}(\lambda)}{(4n+3)(2n+1)d_{2n+1}(\lambda)e_{2n+1}(\lambda)}\ ,\ n \geq 0\ , \\ \gamma_{4n+7} = \frac{(2n+3)d_{2n+3}(\lambda)e_{2n+1}(\lambda)}{(4n+5)(2n+2)d_{2n+2}(\lambda)e_{2n+2}(\lambda)}\ ,\ n \geq 0\ . \end{cases}$$

Taking into account that the linear functional v is Laguerre-Hahn and by virtue of , the linear functional u is also Laguerre-Hahn. It satisfies 29 and 32 with

$$\begin{cases} \Phi(x) = x^4\ , \quad \Psi(x) = -2\lambda^{-1}x^5(x^2+1-\lambda)\ , \quad B(x) = \lambda^{-1}x^8\ , \\ C(x) = 2\lambda^{-1}x^5(x^2+1-\lambda) - 4x^3\ , \quad D(x) = (\lambda^{-1}x^2-2)(x^2+1)^2 - x^2 + 2\lambda + 1\ . \end{cases}$$

From 41, we have

$$X(0) = 1 - 2\lambda\ , \quad \tilde{C}(0) - X'(0) = 0\ , \quad \tilde{C}'(0) - Y(0) - \frac{1}{2}X''(0) = -5 - \lambda^{-1}\ .$$

Now it is enough to use to obtain the following.

(i) If λ satisfies 46 and $\lambda^{-1} \neq 2$, then the class of is $s = 6$.

(ii) If $\lambda^{-1} = 2$, then the class of is $s = 4$.

Example 4.2. *Let v be the associated linear functional of the first order of $\mathcal{J}(\alpha, -\alpha)$. Here [9, 11]*

$$\begin{aligned} \rho_{2n+1} &= \tfrac{(2n+\alpha+2)(2n-\alpha+2)}{(4n+3)(4n+5)}\ , \\ \rho_{2n+2} &= \tfrac{(2n+\alpha+3)(2n-\alpha+3)}{(4n+5)(4n+7)}\ , \quad n \geq 0\ , \end{aligned} \tag{48}$$

$$\begin{aligned} \tilde{\Phi}(x) = x^2 - 1\ , \quad \tilde{\Psi}(x) = -4x\ , \quad \tilde{B}(x) = \frac{1-\alpha^2}{3}\ , \\ \tilde{C}(x) = 2x\ , \quad \tilde{D}(x) = 3\ . \end{aligned} \tag{49}$$

We assume $(1-\alpha^2) \neq 0$, then v is a Laguerre-Hahn linear functional of class $\tilde{s} = 0$.

By applying the same process as we did to obtain 44, 45 and using the above results, we can get for $n \geq 0$

$$\tilde{P}_n(0, \lambda) = \frac{(-1)^n}{\Gamma(2n+\alpha+\frac{5}{2})}f_n(\lambda)\ , \tag{50}$$

and

$$\Delta_n = \frac{3\pi\Gamma(2n+\alpha+2)\Gamma(2n-\alpha+2)}{2^{4n+2}(4n+3)\Gamma(\alpha+2)\Gamma(2-\alpha)\Gamma^2(2n+\frac{3}{2})} g_n(\lambda)\ , \tag{51}$$

where

$$f_n(\lambda) = (1-3\lambda)\frac{\Gamma(n+1+\frac{\alpha}{2})\Gamma(n+1-\frac{\alpha}{2})}{\Gamma(1+\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2})} + 3\lambda\frac{\Gamma(n+\frac{3}{2}+\frac{\alpha}{2})\Gamma(n+\frac{3}{2}-\frac{\alpha}{2})}{\Gamma(\frac{3}{2}+\frac{\alpha}{2})\Gamma(\frac{3}{2}-\frac{\alpha}{2})}\ ,\ n \geq 0\ ,$$

and

$$\begin{aligned} g_0(\lambda) &= \lambda+1\ , \\ g_n(\lambda) &= \tfrac{2}{3}\pi\Gamma(2+\alpha)(2-\alpha)\Bigg\{ \tfrac{(1-3\lambda)^2}{\Gamma^2(1+\frac{\alpha}{2})\Gamma^2(1-\frac{\alpha}{2})}\left(\tfrac{\Gamma(n+2+\frac{\alpha}{2})\Gamma(n+2-\frac{\alpha}{2})}{\Gamma(n+\frac{3}{2}+\frac{\alpha}{2})\Gamma(n+\frac{3}{2}-\frac{\alpha}{2})} - \tfrac{\Gamma(2+\frac{\alpha}{2})\Gamma(2-\frac{\alpha}{2})}{\Gamma(\frac{3}{2}+\frac{\alpha}{2})\Gamma(\frac{3}{2}-\frac{\alpha}{2})}\right) \\ &\quad + \tfrac{3\lambda^2}{\Gamma^2(\frac{3}{2}+\frac{\alpha}{2})\Gamma^2(\frac{3}{2}-\frac{\alpha}{2})}\left(\tfrac{\Gamma(n+\frac{5}{2}+\frac{\alpha}{2})\Gamma(n+\frac{5}{2}-\frac{\alpha}{2})}{\Gamma(n+1+\frac{\alpha}{2})\Gamma(n+1-\frac{\alpha}{2})} - \tfrac{\Gamma(\frac{5}{2}+\frac{\alpha}{2})\Gamma(\frac{5}{2}-\frac{\alpha}{2})}{\Gamma(1+\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2})}\right) \\ &\quad + \tfrac{3\lambda(1-3\lambda)n(2n+5)}{2\Gamma(1+\frac{\alpha}{2})\Gamma(1-\frac{\alpha}{2})\Gamma(\frac{3}{2}+\frac{\alpha}{2})\Gamma(\frac{3}{2}-\frac{\alpha}{2})}\Bigg\}\ , \quad n \geq 1\ . \end{aligned}$$

Then, u is regular for every $\lambda \neq 0$ such that

$$f_n(\lambda)g_n(\lambda)\ ,\ n \geq 0\ . \tag{52}$$

From 20 and 50, we obtain

$$\tilde{a}_n = \frac{f_{n+1}(\lambda)}{(2n+\alpha+\frac{7}{2})(2n+\alpha+\frac{5}{2})f_n(\lambda)}\ ,\ n \geq 0\ .$$

Using 48, 50, 51 and 24, we get

$$\left\{ \begin{array}{l} \gamma_1 = 1\ ,\ \gamma_2 = -\lambda-1\ ,\ \gamma_3 = \frac{\lambda(\alpha^2+15\lambda-4)}{15(\lambda+1)}\ , \\ \gamma_{2n+4} = \frac{\Delta_{n+1}}{\Delta_n}\ ,\ n \geq 0\ , \\ \gamma_{4n+5} = \frac{(2n+\alpha+2)(2n-\alpha+2)\tilde{a}_{2n+1}\Delta_{2n}}{(4n+3)(4n+5)\Delta_{2n+1}}\ ,\ n \geq 0\ , \\ \gamma_{4n+7} = \frac{(2n+\alpha+3)(2n-\alpha+3)\tilde{a}_{2n+2}\Delta_{2n+1}}{(4n+5)(4n+7)\Delta_{2n+2}}\ ,\ n \geq 0\ . \end{array} \right.$$

According to , the linear functional u is also Laguerre-Hahn. It satisfies 29 and 32 with

$$\left\{ \begin{array}{l} \Phi(x) = x^4(x^2-1)\ , \quad \Psi(x) = \frac{2}{3}\lambda^{-1}(1-\alpha^2)x^5(x^2+1) - 4x^5\ , \quad B(x) = -\frac{\lambda^{-1}}{3}(1-\alpha^2)x^8\ , \\ C(x) = -\frac{2}{3}\lambda^{-1}(1-\alpha^2)x^5(x^2+1) - 2x^3(x^2-2)\ , \\ D(x) = -\frac{\lambda^{-1}(1-\alpha^2)}{3}x^2(x^2+1)^2 - (x^2-2)^2 + 5 - 3\lambda\ . \end{array} \right.$$

From 49, we have

$$X(0) = 3\lambda - 1\ , \quad \tilde{C}(0) - X'(0) = 0\ , \tilde{C}'(0) - Y(0) - \frac{1}{2}X''(0) = 4 - \frac{\lambda^{-1}}{3}(1-\alpha^2)\ , \quad \tilde{\Phi}(0) = -1\ .$$

Now it is enough to to obtain the following.

(i) If λ *satisfies* (52) *and* $\lambda^{-1} \neq 3$*, then the class of is* $s = 6$*.*

(ii) If $\lambda^{-1} = 3$ *and* $3 + \alpha^2 \neq 0$*, then the class of is* $s = 4$*.*

(iii) If $\lambda^{-1} = 3$ *and* $3 + \alpha^2 = 0$*, then the class of is* $s = 3$*.*

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Received 31 June 2017
Accepted 25 July 2017