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Some Properties and Homomorphisms of Pseudo-Q-Algebras

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Abstract. In this paper, we study pseudo-Q-algebras and investigate some of their properties. We also consider ideals, minimal elements and center of pseudo-Q-algebras. Additionally, we define homomorphism of these algebras.

Key Words and Phrases: Q-algebras, subalgebras, ideals, pseudo-Q-algebras, minimal element, centre, homomorphism.

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1. Introduction

BCK-algebras and BCI-algebras were introduced by Imai. and Iseki as two classes of abstract algebras in 1966 [**7**], [**8**]. It is known that the class of BCK-algebras is a proper subclass of BCI-algebras. In 1983, BCH-algebras as a wide class of abstract algebras were introduced by Hu and Li [**12**], [**18**]. In their study, it is given that the class of BCI-algebras are proper subclasses of BCH-algebras. In 1999, the notion of d-algebras that is another useful generalization of BCK-algebras was introduced by Neggers and Kim [**19**]. In 2001, a new notion called a Q-algebras was introduced by J. Neggers, S. S. Ahn and H. S. Kim [**11**]. At the same time pseudo-BCK-algebras as an extension of BCK-algebras was introduced by G. Geordscu, and A. Iorgulescu [**3**]. In 2008, pseudo-BCK-algebras as a natural generalization of BCI-algebras and pseudo-BCK-algebras were introduced by W. A. Dudek and Y. B. Jun [**14**]. These algebras have also connections with other algebras of logics such as pseudo-MV-algebras and pseudo-BL-algebras defined by G. Georgesuc and A. Iorgulescu [**5**] and [**4**], respectively. As a generalization of many algebras, these pseudo algebras has been studied by many researchers [**1**], [**2**], [**16**] and [**9**], [**15**]. In this paper, we consider pseudo-Q-algebras. We state some basic properties of pseudo-Q-algebras and provide some characterization of these algebras. Additionally, we consider the ideals and homomorphisms of pseudo-Q-algebras.

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2. Preliminaries

Definition 2.1. [11] A Q-algebra $(X; *, 0)$ is a nonempty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

(I) $x * x = 0$;

(II) $x * 0 = x$;

(III) $(x * y) * z = (x * z) * y$ for all $x, y, z \in X$.

For brevity, we also call X a Q-algebra. On X we can define a binary relation $\leq$ by $x \leq y$ if and only if $x * y = 0$ for all $x, y \in X$. Recently, Ahn and Kim [13] introduced the notion of QS-algebras. A Q-algebra X is said to be a QS-algebra if it satisfies the additional relation:

(IV) $(x * y) * (x * z) = z * y$, for any $x, y, z \in X$.

Definition 2.2. [11] Let $(X; *, 0)$ be a Q-algebra and $I(\neq \emptyset) \subseteq X$. The set I is called an ideal of X if for any $x, y, z \in X$ the following hold:

(1) $0 \in I$;

(2) $x * y \in I$ and $y \in I$ imply $x \in I$.

Obviously, $\{0\}$ and X are ideals of X. We call $\{0\}$ and X the zero ideal and the trivial ideal of X, respectively. An ideal I is said to be proper if $I \neq X$.

Definition 2.3. [11] An ideal I of a Q-algebra $(X; *, 0)$ is said to be implicative if $(x * y) * z \in I$ and $y * z \in I$, then $x * z \in I$, for any $x, y, z, \in X$.
+99+9++9+99+96+

Example 2.4. *[11] Let* $X = \{0, 1, 2, 3\}$ *be a set with the following table:*

*	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X; *, 0)$ *is a Q-algebra, which is not a BCH/BCI /BCK-algebra. Neggers and Kim [6] introduced the related notion of B-algebra, that is, algebras* $(X; *, 0)$ *which satisfy (I)* $x * x = 0$*; (II)* $x * 0 = x$*; (V)* $(x * y) * z = x * (z * (0 * y))$*, for any* $x, y, z \in X$*. It is easy to see that B-algebras and Q-algebras are different notions. For instance, this example illustrates a Q-algebra, but not a B-algebra, since* $(3 * 2) * 1 = 0 \neq 3 = 3 * (1 * (0 * 2))$*.*

Consider the following example. Let $X = \{0, 1, 2, 3, 4, 5\}$ be a set with the following table [11]:

*	0	1	2	3	4	5
0	0	2	1	3	4	5
1	1	0	2	4	5	3
2	2	1	0	5	3	4
3	3	4	5	0	2	1
4	4	5	3	1	0	2
5	5	3	4	2	1	0

Then $(X; *, 0)$ is a B-algebra but not Q-algebra since $(5 * 3) * 4 = 3 \neq 4 = (5 * 4) * 3$.

The following example shows that a Q-algebra may not satisfy the associative law.

Example 2.5. *[11] (a) Let $X = \{0, 1, 2\}$ with the table as follows:*

*	0	1	2
0	0	2	1
1	1	0	2
2	2	1	0

Then X is a Q-algebra, but associativity does not hold, since $(0 * 1) * 2 = 0 \neq 1 = 0 * (1 * 2)$.

(b) Let Z and R be the set of all integers and real numbers, respectively. Then $(Z; -, 0)$ and $(R; \div, 1)$ are nonassociative Q-algebras where " $-$ " is the usual subtraction and $\div$ is the usual division.

Theorem 2.6. *[11] Every Q-algebra $(X; *, 0)$ satisfying the associative law is a group under the operation $*$.*

Proof. Putting $x = y = z$ in the associative law $(x * y) * z = x * (y * z)$ and using (I) and (II) of Definition 2.1, we obtain $0 * x = x * 0 = x$. This means that 0 is the zero element of X. By (I), every element x of X has as its inverse the element x itself. Therefore $(X; *)$ is a group. □

3. Pseudo-Q-algebras

Definition 3.1. ([10]) A pseudo-BCH-algebra is an algebra $(X; *, \diamond, 0)$ of type $(2; 2; 0)$ satisfying for all $x, y, z \in X$ the axioms:

(pBCH-1) $x * x = x \diamond x = 0$;

(pBCH-2) $(x * y) \diamond z = (x \diamond z) * y$;

(pBCH-3) $x * y = y \diamond x = 0 \Rightarrow x = y$;

(pBCH-4) $x * y = 0 \Longleftrightarrow x \diamond y = 0$.

Definition 3.2. ([14]) A pseudo-BCI-algebra is a structure $X = (X; \leqslant; *, \diamond, 0)$, where $\leqslant$ is a binary relation on the set X, $*$ and $\diamond$ are binary operations on X and 0 is an element of X, satisfying the axioms for all $x, y, z \in X$:

(pBCI-1) $(x * y) \diamond (x * z) \leqslant z * y$, $(x \diamond y) * (x \diamond z) \leqslant z \diamond y$;

(pBCI-2) $x * (x \diamond y) \leqslant y$, $x \diamond (x * y) \leqslant y$;

(pBCI-3) $x \leqslant x$;

(pBCI-4) $x \leqslant y$, $y \leqslant x \Rightarrow x = y$;

(pBCI-5) $x \leqslant y \Rightarrow x * y = 0 \Rightarrow x \diamond y = 0$.
A pseudo-BCI-algebra X is called a pseudo-BCK-algebra if it satisfies the identity
(pBCK) $0 * x = 0 \diamond x = 0$.

† 1. ([12]) Every pseudo-BCH-algebra satisfies (pBCI-2) - (pBCH-5)

Definition 3.3. [17] A pseudo-Q-algebra is a non-empty set X with a constant 0 and two binary operations $*$ and $\diamond$ satisfying the following axioms:

(PQ1) $x * x = x \diamond x = 0$;

(PQ2) $x * 0 = x \diamond 0 = x$;

(PQ3) $(x * y) \diamond z = (x \diamond z) * y$, for all $x, y, z \in X$.

Definition 3.4. Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebras and let $\emptyset \neq I \subset X$. I is called a pseudo subalgebra of X if $x * y, x \diamond y \in I$ whenever $x, y \in I$. I is called ideal of X if it satisfies:

(I1) $0 \in I$;

(I2) $x * y$ or $x \diamond y \in I$ and $y \in I$ imply $x \in I$ for all $x, y \in X$.

We will denote by $\mathrm{Id}(X)$ the set of all ideals of X. Obviously, $\{0\}, X \in \mathrm{Id}(X)$.

Example 3.5. *Let $X = \{0, 1, 2, 3\}$. Define the binary operations " $*$ " and " $\diamond$ " on X by the following tables:*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	0	0	0
3	3	2	3	0

$\diamond$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	2	0	0
3	3	3	1	0

Then it is easy to show that $(X; *, 0)$ and $(X; \diamond, 0)$ are not Q-algebras,but $(X; *, \diamond, 0)$ is a pseudo-Q-algebras. Let $I = \{0, 1\}$. Then I is a pseudo subalgebra of X ,but not an ideal of X, since $2 * 1 = 0$, and $0, 1 \in I$ but $2 \notin I$.

Proposition 3.6. *If $(X; *, \diamond, 0)$ is a pseudo-Q-algebra, then (PQ4) $(x * (x \diamond y)) \diamond y = (x \diamond (x * y)) * y = 0$, for any $x, y \in X$.*

Proof. Let x, y be in X. Then by (PQ1) and (PQ3), we have $(x*(x\diamond y))\diamond y = (x\diamond y)*(x\diamond y) = 0$ and $(x \diamond (x * y)) * y = (x * y) \diamond (x * y) = 0$.

□

We now investigate some relations between pseudo-Q-algebras and pseudo-BCH-algebras. The following theorems are easily proven, and we omit their proofs.

Theorem 3.7. *Every pseudo-BCH-algebra X is a pseudo-Q-algebra. Every pseudo-Q-algebra X satisfying condition*

$$(pBCH)(x * y) = (y \diamond x) = 0 \quad \textit{implies} \quad x = y$$

is a pseudo-BCH-algebra.

Theorem 3.8. *[17] Every pseudo-Q-algebra X satisfying*

$$x * (x \diamond y) = x * y$$

or

$$x \diamond (x * y) = x \diamond y$$

for all $x, y \in X$ is a trivial algebra.

Proof. Putting $x = y$ in the equation $x * (x \diamond y) = x * y$ or $x \diamond (x * y) = x \diamond y$, we obtain $x * 0 = 0$ or $x \diamond 0 = 0$. By (PQ2), we have $x = 0$. Hence X is a trivial algebra. □

Definition 3.9. Let $(X;*,\diamond,0)$ be a pseudo-Q-algebras. Define the relation " $\leq$ " on X by

$x \leq y$ if and only if $x * y = 0$ (or equivalently, $x \diamond y = 0$)

for all $x, y \in X$.

Proposition 3.10. *In a pseudo-Q-algebra $(X;*,\diamond,0)$ the following properties hold for all $x, y \in X$:*

(1) $x \leq 0 \Rightarrow x = 0$;

(2) $x * (x \diamond y) \leq y, x \diamond (x * y) \leq y$; *[17]*

(3) $0 * x = 0 \diamond x$; *[17]*

(4) $x \leq y \Rightarrow 0 * x = 0 \diamond y$;

(5) $0 \diamond (0 * (0 \diamond x)) = 0 \diamond x$, $0 * (0 \diamond (0 * x)) = 0 * x$;

(6) $0 * (x * y) = (0 \diamond x) \diamond (0 * y)$; *[17]*

(7) $0 \diamond (x \diamond y) = (0 * x) * (0 \diamond y)$ *[17].*

Proof. (1) Let $x \leq 0$. Then we get $x * 0 = 0$. By (PQ2) we have $x * 0 = x \diamond 0 = x$. So $x * 0 = 0 = x \diamond 0 = x$. Hence we get $x = 0$.

(2) By using (PQ3) and (PQ1), we have

$$(x * (x \diamond y)) \diamond y = (x \diamond y) * (x \diamond y) = 0$$

. Hence $x * (x \diamond y) \leq y$. Similarly, $x \diamond (x * y) \leq y$.

(3) By using (PQ1) and (PQ3), we get

$$0 * x = (x \diamond x) * x = (x * x) \diamond x = 0 \diamond x$$

.

(4) Let $x \leq y$. Then $x \diamond y = 0$ and therefore,

$$0 * x = (x \diamond y) * x = (x * x) \diamond y = 0 \diamond y$$

.

(5) From (2) it follows that $0 * (0 \diamond x) \leq x$ and $0 \diamond (0 * x) \leq x$. Hence, using (3) and (4) we obtain

$$0 \diamond (0 * (0 \diamond x)) = 0 \diamond x,$$

$$0 * (0 \diamond (0 * x)) = 0 * x$$

.

(6) By using (PQ1) and (PQ3), we have

$$\begin{aligned}
(0 \diamond x) \diamond (0 * y) &= (((x * y) * (x * y)) \diamond x) \diamond (0 * y) \\
&= (((x \diamond x) * y) * (x * y)) \diamond (0 * y) \\
&= ((0 * y) * (x * y)) \diamond (0 * y) \\
&= ((0 * y) \diamond (0 * y)) * (x * y) \\
&= 0 * (x * y)
\end{aligned}$$

.

(7) The proof is similar to the proof of (6). □

† **2.** Every pseudo-Q-algebra satisfies (pBCI-2) and (pBCI-3).

Theorem 3.11. *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebra. The following statements are equivalent:*

(i) $x * (y * z) = (x * y) * z$, for all $x, y, z \in X$;

(ii) $0 * x = x = 0 \diamond x$, for every $x \in X$;

(iii) $x * y = x \diamond y = y * x$, for all $x, y \in X$;

(iv) $x \diamond (y \diamond z) = (x \diamond y) \diamond z$, for all $x, y, z \in X$.

Proof. (i)⇒ (ii) Let $x \in X$. We have $x = x * 0 = x * (x * x) = (x * x) * x = 0 * x$. By (3), we have $0 \diamond x = x$.

(iv) ⇒ (ii) The proof is similar to the above proof.

(ii) ⇒ (iii) Let (ii) hold and $x, y \in X$. By using proposition 3.10(6) and (PQ3) we obtain

$$\begin{aligned}
x * y &= 0 * (x * y) = (0 \diamond x) \diamond (0 * y) \\
&= x \diamond y. \\
&= (0 * x) \diamond y = (0 \diamond y) * x = y * x
\end{aligned}$$

.

(iii) ⇒ (i) Let $x, y, z \in X$. Using (iii) and (PQ3) we get

$x * (y * z) = (y \diamond z) * x = (y * x) \diamond z = (x * y) * z.$

(iii) $\Rightarrow$ (iv) has a proof similar to the proof of (iii) $\Rightarrow$ (i).

Hence all the conditions are equivalent.

□

Theorem 3.12. *Every pseudo-Q-algebra $(X; *, \diamond, 0)$ satisfying the associative law is a group under each operation " $*$ " and " $\diamond$ ".*

Proof. Putting $x = y = z$ in the associative law $(x * y) * z = x * (y * z)$ and using (I) and (II), we obtain $0 * x = x * 0 = x$. This means that 0 is the zero element of X. By (I), every element x of X has as its inverse the element x itself. Therefore $(X; *)$ and $(X; \diamond)$ are a group.

□

Definition 3.13. An element a of a pseudo-Q-algebra X is said to be *minimal* if for every $x \in X$ the following implication

$$x \leq a \Rightarrow x = a$$

holds.

Proposition 3.14. *Let X be a pseudo-Q-algebra and let $a \in X$. Then the following conditions are equivalent for every $x \in X$:*

(i) a is minimal;

(ii) $x \diamond (x * a) = a$;

(iii) $0 \diamond (0 * a) = a$;

(iv) $a * x = (0 * x) \diamond (0 * a)$;

(v) $a * x = 0 \diamond (x * a)$.

Proof. (i)$\Rightarrow$ (ii) By proposition 3.10(1), $x \diamond (x * a) \leq a$ for all $x \in X$. Since a is minimal, we get (ii).

(ii) $\Rightarrow$ (iii) Obvious.

(iii)$\Rightarrow$ (iv) We have $a * x = (0 \diamond (0 \diamond a)) * x = (0 * x) \diamond (0 * a)$.

(iv)⇒ (v) Using proposition 3.10(3) and 3.10(6) we see that $0 \diamond (x * a) = 0 * (x * a) = (0 * x) * (0 * a) = (0 * x) \diamond (0 * a) = a * x$.

(v) ⇒ (i) Let $x \leq a$. Then $x * a = 0$ and hence $a * x = 0 \diamond (x * a) = 0$. Thus $a \leq x$. Consequently, $x = a$.

□

Replacing " $*$ " by " $\diamond$ " and " $\diamond$ " by $*$ in Proposition 3.14 we obtain

Proposition 3.15. *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebra and let $a \in X$. Then for every $x \in X$ the following conditions are equivalent:*

(i) a is minimal;

(ii) $x * (x \diamond a) = a$;

(iii) $0 * (0 \diamond a) = a$;

(iv) $a \diamond x = (0 \diamond x) * (0 \diamond a)$;

(v) $a \diamond x = 0 * (x \diamond a)$.

Proposition 3.16. *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebras and let $a \in X$. Then a is minimal if and only if there is an element $x \in X$ such that $a = 0 * x$.*

Proof. Let a be a minimal element of $(X; *, \diamond, 0)$. By Proposition 3.14, we have $a = 0 * (0 \diamond a)$. If we set $x = 0 \diamond a$, then $a = 0 * x$.
Conversely, suppose that $a = 0 * x$ for some $x \in X$. Using Proposition 3.10 (5) we get $0 * (0 \diamond a) = 0 * (0 \diamond (0 * x)) = 0 * x = a$. From Proposition 3.15 it follows that a is minimal. □

Definition 3.17. For $x \in X$, set

$$\overline{x} = 0 * (0 \diamond x).$$

By Proposition 3.10 (3), $\overline{x} = 0 * (0 * x) = 0 \diamond (0 \diamond x) = 0 * (0 \diamond x)$.

Proposition 3.18. *Let X be a pseudo-Q-algebra. For any $x, y \in X$ we have:*

(a) $\overline{x * y} = \overline{x} * \overline{y}$;

(b) $\overline{x \diamond y} = \overline{x} \diamond \overline{y}$;

(c) $\overline{\overline{x}} = \overline{x}$.

Proof. (a) Using proposition 3.10(6) and 3.10(7) we get

$$\begin{aligned}\overline{x*y} &= 0\diamond(0*(x*y)) = 0\diamond[(0\diamond x)\diamond(0*y)]\\ &= [0*(0\diamond x)]*[0\diamond(0*y)] = \overline{x}*\overline{y}.\end{aligned}$$

(b) Has a proof similar to (a).

(c) By Proposition 3.10(5), $0*(0\diamond(0*x)) = 0*x$, that is, $0*\overline{x} = 0*x$. Hence $\overline{\overline{x}} = 0\diamond(0*\overline{x}) = 0\diamond(0*x) = \overline{x}$. □

Definition 3.19. Following the terminology from BCH-algebras (see [10], Definition 5) the set $\{x\in X : x = \overline{x}\}$ will be called *the centre of* $(X;*,\diamond,0)$. We shall denote it by $CenX$. By Proposition 3.14, $CenX$ is the set of all minimal elements of X. We have

$$CenX = \{\overline{x} : x\in X\}.$$

Proposition 3.20. *Let $(X;*,\diamond,0)$ be a pseudo-Q-algebra. Then CenX is a subalgebra of $(X;*,\diamond,0)$.*

Proposition 3.21. *Let X be a pseudo-Q-algebra and let $x,y\in CenX$. Then for every $z\in X$ we have*

$$x\diamond(z*y) = y*(z\diamond x).$$

Proof. Let $z\in X$. Using Propositions 3.15 and 3.14, we obtain
$x\diamond(z*y) = [z*(z\diamond x)]\diamond(z*y) = [z\diamond(z*y)]*(z\diamond x) = y*(z\diamond x)$.

□

Proposition 3.22. *Let $(X;*,\diamond,0)$ be a pseudo-Q-algebra and let $I\in Id(X)$. For any $x,y\in X$, if $y\in I$ and $x\le y$, then $x\in I$.*

Proof. Straightforward. □

Proposition 3.23. *Let $(X;*,\diamond,,0)$ be a pseudo-Q-algebra and I be a subset of X satisfying (I1). Then I is an ideal of $(X;*,\diamond,0)$ if and only if for all $x,y\in X$,*
(I2”) if $x\diamond y\in I$ and $y\in I$, then $x\in I$.

Proof. Let I be an ideal of $(X;*,\diamond,0)$. Suppose that $x\diamond y\in I$ and $y\in I$. By proposition 3.10(2), $x*(x\diamond y)\le y$ and from Proposition 3.22 it follows that $x*(x\diamond y)\in I$. Therefore, since $x\diamond y\in I$ and I satisfies (I2), we obtain $x\in I$, that is, (I2”) holds. The proof of the implication (I2”) $\Rightarrow$ (I2) is analogous. □

Definition 3.24. An ideal I of a pseudo-Q-algebra $(X;*,\diamond,0)$ is said to be closed if $0*x\in I$ for every $x\in I$.

Theorem 3.25. *An ideal I of a pseudo-Q-algebra $(X; *, \diamond, 0)$ is closed if and only if I is a subalgebra of $(X; *, \diamond, 0)$.*

Proof. Suppose that I is a closed ideal of $(X; *, \diamond, 0)$ and let $x, y \in I$. By (PQ3) and (PQ1), we have

$$\begin{aligned} ((x*y)*(0*y))\diamond x &= [(x*y)\diamond x]*(0*y) \\ &= [(x\diamond x)*y]*(0*y) \\ &= (0*y)*(0*y) = 0 \end{aligned}$$

.

Hence $[(x*y)*(0*y)]\diamond x \in I$. Since $x, 0*y \in I$, we have $x*y \in I$. Similarly, $x\diamond y \in I$. Conversely, if I is a subalgebra of $(X; *, \diamond, 0)$, then $x \in I$ and $0 \in I$ imply $0*x \in I$. □

Definition 3.26. [17] Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebra. For any nonempty subset S of X, we define

$$G(S) = \{x \in S | 0*x = x = 0\diamond x\},$$

if $S = X$ then $G(x)$ is called the G-part of X.

Definition 3.27. For any pseudo-Q-algebra $(X; *, \diamond, 0)$, the set

$$B(X) = \{x \in X | 0*x = 0 = 0\diamond x\}$$

is called the $p-radical$ of X.

If $B(X) = \{0\}$, then we say that X is a $p-semisimple$ pseudo-Q-algebra. The following property is obvious:

$$G(X) \cap B(X) = \{0\}.$$

Proposition 3.28. *If $(X; *, \diamond, 0)$ is a pseudo-Q-algebra and $x, y \in X$, then $y \in B(X)$ if and only if $(x*y)\diamond x = 0 = (x\diamond y)*x$.*

Proof. By (PQ3) and (PQ1) we have $(x*y)\diamond x = (x\diamond x)*y = 0*y = 0$ and $(x\diamond y)*x = (x*x)\diamond y = 0\diamond y = 0$ if and only if $y \in B(X)$ □

Proposition 3.29. *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebra. Then $B(X)$ is an ideal of X.*

Proof. Since $(0*0)*0 = 0$, by Proposition 3.28, we get $0 \in B(X)$. Let $x*y \in B(X)$ and $y \in B(X)$. Then by Proposition 3.28, we have $((x*y)*x)\diamond(x*y) = 0$. By (PQ3), $((x*y)\diamond(x*y))*x = 0*x = 0$. Hence $x \in B(X)$. Therefore $B(X)$ is an ideal of X. □

Proposition 3.30. *If S is a subalgebra of a pseudo-Q-algebra $(X; *, \diamond, 0)$, then $G(X)\cap S = G(S)$.*

Proof. It is obvious that $G(X) \cap S \subseteq G(S)$. If $x \in G(S)$, then $0 * x = x$ and $x \in S \subseteq X$. Then $x \in G(X)$ and so $x \in G(X) \cap S$, which proves the proposition. □

Theorem 3.31. *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebra. If $G(X) = X$,then X is p-semisimple.*

Proof. Assume that $G(X) = X$.
By $G(X) \cap B(X) = \{0\}$, we have $\{0\} = G(X) \cap B(X) = X \cap B(X) = B(X)$. Hence X is p-semisimple. □

4. Homomorphism

Definition 4.1. Let X and Y be a pseudo-Q-algebras. A mapping $f : X \to Y$ is called a *homomorphism* of pseudo-Q-algebras if

$$f(x * y) = f(x) * f(y) \text{ and } f(x \diamond y) = f(x) \diamond f(y) \text{ for all } x, y \in X.$$

Note that if $f : X \to Y$ is a homomorphism of pseudo-Q-algebras, then $f(0_X) = 0_Y$ where 0_X and 0_Y are zero elements of X and Y, respectively.

Example 4.2. *Let $(X; *, \diamond, 0)$ be a pseudo-Q-algebras, then the function $f : X \to X$ such that $f(x) = 0 \diamond x$ for any $x \in X$ is a homomorphism of pseudo-Q-algebras.*

$$\begin{aligned} f(x) * f(y) &= (0 \diamond x) * (0 \diamond y) \\ &= (0 \diamond x) * (0 * y) \\ &= (0 * (0 * y)) \diamond x \\ &= (0 \diamond (0 * y)) \diamond x \\ &= (0 \diamond x) \diamond (0 * y) \\ &= 0 * (x * y) \\ &= 0 \diamond (x * y) = f(x * y) \end{aligned}$$

$$\begin{aligned} f(x) \diamond f(y) &= (0 \diamond x) \diamond (0 \diamond y) \\ &= (0 * x) \diamond (0 \diamond y) \\ &= (0 \diamond (0 \diamond y)) * x \\ &= (0 * (0 \diamond y)) * x \\ &= (0 * x) * (0 \diamond y) \\ &= 0 \diamond (x \diamond y) = f(x \diamond y) \end{aligned}$$

Example 4.3. *Define $\Phi : X \to CenX$ by $\Phi(x) = \overline{x}$ for all $x \in X$. By Proposition 3.18, Φ is a homomorphism from X onto $CenX$.*

Theorem 4.4. *Let $f : X \to Y$ be a homomorphism of pseudo-Q-algebras. If B is a pseudo strong ideal of Y , then $f^{-1}(B)$ is a pseudo strong ideal of X.*

Proof. Assume that B is a pseudo strong ideal of Y . Obviously, $0_x \in f^{-1}(B)$. Let $x, y, z \in X$ be such that $(x * y) \diamond z, (x \diamond y) * z, y \in f^{-1}(B)$. Then $(f(x) * f(y)) \diamond f(z) = f((x * y) \diamond z), f(y) \in B$. Sine B is a pseudo strong ideal of Y , it follows from (PI3) and (PI3') that $f(x * z) = f(x) * f(z), f(x \diamond z) = f(x) \diamond f(z) \in B$ so that $x * z, x \diamond z \in f^{-1}(B)$. Hence $f^{-1}(B)$ is a pseudo strong ideal of X. □

Theorem 4.5. *Let $f : X \to Y$ be a homomorphism of pseudo-Q-algebras.*
(i) If B is a pseudo ideal of Y, then $f^{-1}(B)$ is a pseudo ideal of X.
(ii) If f is surjective and I is a pseudo ideal of X, then $f(I)$ is a pseudo ideal of Y.

Proof. (i) Straightforward.
(ii) Assume that f is surjective and let I be a pseudo ideal of X. Obviously, $0_Y \in f(I)$. For every $y \in f(I)$, let $a, b \in Y$ be such that $a * y \in f(I), b \diamond y \in f(I)$. Then there exist $x_*, x_\diamond \in I$ such that $f(x_*) = a * y$ and $f(x_\diamond) = b \diamond y$. Since $y \in f(I)$, there exists $x_y \in I$ such that $f(x_y) = y$. Also f is surjective, there exist $x_a, x_b \in X$ such that $f(x_a) = a$ and $f(x_b) = b$. Hence $f(x_a * x_y) = f(x_a) * f(x_y) = a * y \in f(I)$ and $f(x_b \diamond x_y) = f(x_b) \diamond f(x_y) = b \diamond y \in f(I)$, which imply that $x_a * x_y \in I$ and $x_b * x_y \in I$. Since I is a pseudo ideal of X, we get $x_a, x_b \in I$ and thus $a = f(x_a), b = f(x_b) \in f(I)$. Therefore $f(I)$ is a pseudo ideal of X □

Corollary 4.6. *Let $f : X \to Y$ be a homomorphism of pseudo-Q-algebras. Then Kerf $= \{x \in X | f(x) = 0\}$ is a pseudo strong ideal(ideal) of X.*

Proposition 4.7. *Let $f : (X; *_1, \diamond_1, 0) \to (Y; *_2, \diamond_2, 0)$ be a homomorphism of pseudo-Q-algebras. Then $x *_1 y, y \diamond_1 x \in$ Kerf if $f(x) = f(y), \forall x \in X$.*

Proof. Assume that $f(x) = f(y), \forall x \in X$. Then $f(x) *_2 f(y) = f(x *_1 y) = 0$ and $f(x) \diamond_2 f(y) = f(x \diamond_1 y) = 0$. Hence $x *_1 y, y \diamond_1 x \in \mathrm{Ker} f$. □

Proposition 4.8. *Let $f : (X; *_1, \diamond_1, 0) \to (Y; *_2, \diamond_2, 0)$ be a homomorphism of pseudo-Q-algebras. If $y \in$ Kerf, then $x *_1 (x *_1 y), (x *_1 y) *_1 x, x \diamond_1 (x *_1 y), (x *_1 y) \diamond_1 x, x *_1 (x \diamond_1 y), (x *_1 y) *_1 x, x \diamond_1 (x \diamond_1 y), (x \diamond_1 y) \diamond_1 x \in$ Kerf.*

Lemma 4.9. *Let $f : X \to Y$ be a homomorphism of pseudo-Q-algebras. Then f is a monomorphism if and only if Kerf $= \{0\}$.*

Theorem 4.10. *Let X, Y and Z be pseudo-Q-algebras, and $h : X \to Y$ be an onto homomorphism of pseudo-Q-algebras and $g : X \to Z$ be a homomorphism of pseudo-Q-algebras. If Kerh $\subset$ Kerg, then there exists a unique homomorphism of pseudo-Q-algebras $f : Y \to Z$ satisfying $f \circ h = g$.*

Theorem 4.11. *Let X, Y and Z be pseudo-Q-algebras, and $g : X \to Z$ be a homomorphism of pseudo-Q-algebras and $h : Y \to Z$ be an one-to-one homomorphism of pseudo-Q-algebras. If Img $\subset$ Imh, then there exists a unique homomorphism of pseudo-Q-algebras $f : X \to Y$ satisfying $h \circ f = g$.*

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