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The regularized trace formula of the spectrum of a Sturm-Liouville problem with turning point

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Abstract. In this work, we establish the summation formula of the infinite sequence of the eigenvalues generated by a Sturm-Liouville problem with turning points in a finite interval $[0, \pi]$.

Key Words and Phrases: Singular Sturm-liouville Problem, turning points, Discrete spectrum, Asymptotic solutions, Regularized trace formula

1. Introduction

The regularized trace formula occupied a great place in the study of ordinary differential equation. It was first obtained by Gelfand and Levitan (see [1]-[2])as they computed the trace of the following boundary value problem

$$-y'' + q(x)\, y = \lambda\, y, \tag{1.1}$$

$$y'(0) = 0, \qquad y'(\pi) = 0, \tag{1.2}$$

where $q(x) \in C_1\ [0, \pi]$. They obtained the following formula:

$$\sum_{n=0}^{\infty} (\mu_n - \lambda_n) = \frac{1}{4}\,(q(0) + q(\pi))$$

,

where μ_n are eigenvalues of the operator of the equation (1.1) and $\lambda_n = n^2$ are eigenvalues of the same operator with $q(x) = 0$. following them, Gelfand performed technical method using the trace of a resolvent to obtain traces (see [3]) and this work was continued by many authors (see [4]-[6]). In [7] Dikii gave a proof of the Gelfand-Levitan formula in [3] and after short time he estimated the trace of all higher orders for the Sturm-Liouville operator by constructing the fractional powers of the operator in closed form and calculated its analytic extension for its zeta function see [8].After some years later, Levitan suggested one more method for computing the traces of the Sturm-Liouville operator [9].The first paper discussed and computed a singular differential operator with discrete spectrum was done by Gasymov [10]. The work was continued by different method like Dirac operators, differential operators with abstract operator-valued coefficients, and the case of matrix-valued Sturm-Liouville operators (see [11]-[15]). The author in [16] discussed the following boundary value problem

$$y'' + (\lambda + q(x))y = 0 \qquad 0 \leq x \leq \pi, \tag{1.3}$$

$$y'(0) - h\, y(0) = 0, \qquad y'(\pi) + H\, y(\pi) = 0, \tag{1.4}$$

where $q(x) \in C_2(0,\pi)$, h, and H are finite numbers.
Suppose that $\{\lambda_n\}_0^\infty$ is a sequence of eigenvalues of Sturm-Liouville (1.3)-(1.4) have the series $S_\lambda = \sum_{n=0}^{\infty}(\lambda_n - n^2 - C)$, $c = \frac{2}{\pi}(h + H + \frac{1}{2}\int_0^\pi q(x)\ dx)$ is convergent and $S_\lambda = \frac{1}{4}[q(0) + q(\pi)] - \frac{H}{\pi} - \frac{H^2}{2}$.
Many author completed this work under different conditions such as estimating the trace of Sturm -Liouville with spectral parameter see [17]-[19].
Consider the following Sturm-Liouville problem

$$-y'' + q(x)\ y = \lambda\ \rho(x)y \quad 0 \leq x \leq \pi \tag{1.5}$$

$$y(0) = 0 \quad , \quad y'(\pi) + H\ y(\pi) = 0, \tag{1.6}$$

where the non-negative real function $q(x)$ has a second piecewise integrable derivatives on $(0,\pi)$, H is positive number, λ is a spectral parameter and weighted function or the explosive factor $\rho(x)$ is of the form

$$\rho(x) = \begin{cases} 1 \ ; & 0 \leq x \leq a < \pi, \\ -1 \ ; & a < x \leq \pi. \end{cases} \tag{1.7}$$

Our aim in this paper is to evaluate the summation of discrete eigenvalues in finite interval of problem (1.5)-(1.6) using contour integration method as in [20], and [21] noticing that the right hand side of our problem aries more anallytical difficulty because it contains $\rho(x) = \pm 1$. Moreover according to the asymptotic formula integrations which is given in [22] we forced to calculate more integrations to obtain the trace formula.
Following [22], we state the basic results that are needed in the following calculation. Recall $W(\lambda) = \Psi(s)$, where $s = \sqrt{\lambda}$, which has the following asymptotic formula

$$W(\lambda) = L_0(s) + \frac{L_1(s)}{s} + \frac{L_2(s)}{s^2} + \frac{L_3(s)}{s^3} + O\left(\frac{e^{|Ims|a+|Res|(\pi-a)}}{|s|^4}\right), \tag{1.8}$$

where

$$L_0(s) = -\sin sa \sinh s(\pi - a) - \cos sa \cosh s(\pi - a),$$

$$\begin{aligned} L_1(s) &= -p_1 \sin sa \cosh s(\pi - a) - p_2 \cos sa \sinh s(\pi - a), \\ L_2(s) &= -Q_1 \sin sa \sinh s(\pi - a) - Q_2 \cos sa \cosh s(\pi - a), \\ L_3(s) &= -R_1 \sin sa \cosh s(\pi - a) - R_2 \cos sa \sinh s(\pi - a), \end{aligned} \tag{1.9}$$

and

$$\begin{aligned} p_1 &= \alpha_1(a) + \gamma_2(a), \\ p_2 &= \gamma_2(a) - \alpha_1(a), \\ Q_1 &= \alpha_2(a) + \delta_(a) + \alpha_1(a)\gamma_2(a), \\ Q_2 &= \beta_1(a) + \gamma_3(a) - \alpha_1(a)\gamma_2(a), \\ R_1 &= \beta_2(a) + \delta_2(a) + \alpha_1(a)\gamma_2(a) + \alpha_1(a)\gamma_3(a), \\ R_2 &= -\alpha_1(a)\delta_1(a) + \beta_1(a)\gamma_2(a). \end{aligned} \tag{1.10}$$

We prove that the roots of $\Psi(s) = 0$ are simple and coincide with the eigenvalues of problem (1.5)-(1.6). The asymptotic formula for the non negative and negative eigenvalues $\lambda_n^{\pm}$, where $n = 0, 1, 2, 3, ...$ have the following asymptotic formulas:

$$\begin{aligned} \lambda_n^+ &= \frac{\pi^2}{a^2}\left(n - \frac{1}{4}\right)^2 + \frac{2k_o\pi}{a} + \left(\frac{2k_1\pi}{a}\right)\frac{1}{n} + \frac{2\pi}{a}\left(\frac{a}{2\pi}k_o^2 + \frac{1}{4}k_1 + k_2\right)\frac{1}{n^2} + O\left(\frac{1}{n^3}\right), \\ \lambda_n^- &= -\frac{\pi^2}{(\pi - a)^2}\left(n + \frac{1}{4}\right)^2 - \frac{2h_o\pi}{\pi - a} - \left(\frac{(2h_1 + \frac{1}{2}h_o)\pi}{\pi - a}\right)\frac{1}{n} \\ &- \frac{2\pi}{\pi - a}\left(\frac{\pi - a}{2\pi}h_o^2 - \frac{1}{4}h_1 + h_2\right)\frac{1}{n^2} + O\left(\frac{1}{n^3}\right), \end{aligned} \tag{1.11}$$

where

$$\begin{aligned} k_o &= \frac{1}{2\pi}\int_0^a q(t)dt \\ k_1 &= \frac{a}{2\pi^2}\left[\alpha_2(a) - \delta_1(a) - \beta_1(a) - \gamma_3(a)\right] \\ k_2 &= \frac{a^2}{2\pi^3}[\alpha_1(a)\left((\frac{\alpha_1^2(a)}{3} - \frac{2\alpha_1(a)}{a}) + \delta_1(a) + \gamma_2(a) + \gamma_3(a) - 2\alpha_2(a)\right) \\ &+ \beta_2(a) - \delta_2(a) - \beta_1(a)\gamma_2(a)] \\ h_o &= \frac{1}{2\pi}\left[H + \int_a^\pi q(t)dt\right] \\ h_1 &= \frac{\pi - a}{a}k_1 \\ h_2 &= \frac{(\pi - a)^2}{2\pi^3}\left[\gamma_2(a^2)(\frac{1}{3}\gamma_2(a) - 2\alpha_1(a)(\pi - a) + 2\alpha_1(a)) - \alpha_1(a)(\gamma_2(a) + \gamma_3(a) - \delta_1(a))\right] \end{aligned} \tag{1.12}$$

$$- \quad \frac{(\pi-a)^2}{2\pi^3}\left[\beta_2(a)-\beta_1(a)\gamma_2(a)-\delta_2(a)-\frac{2\gamma_2(a)}{(\pi-a)}(\gamma_2(a)-\delta_1(a)-\alpha_2(a))\right],$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \gamma_3, \delta_1$ and δ_2 are given by

$$\begin{aligned}
\alpha_1(x) &= \frac{1}{2}\int_0^x q(t)\ dt,\\
\alpha_2(x) &= \frac{1}{4}(\int_0^x q(t)\ dt)^2 \ + \ \frac{1}{4}[q(x) \ + \ q(0)],\\
\beta_1(x) &= \frac{-1}{4}(\int_0^x q(t)\ dt)^2 \ + \ \frac{1}{4}[q(x) \ - \ q(0)],\\
\beta_2(x) &= \frac{1}{8}(\int_0^x q(t)\ dt)^3 \ + \ \frac{1}{8}[q(0)\int_0^x q(t)\ dt \ + \ q'(0) \ + \ q'(x)],\\
\gamma_1(x) &= \frac{1}{2}\int_x^\pi q(t)\ dt,\\
\gamma_2(x) &= H \ + \ \frac{1}{2}\int_x^\pi q(t)\ dt,\\
\gamma_3(x) &= \frac{1}{4}(\int_x^\pi q(t)\ dt)^2 \ + \ \frac{1}{4}[2H \int_x^\pi q(t)\ dt \ + \ q(\pi) \ - \ q(x)],\\
\delta_1(x) &= \frac{-1}{4}(\int_x^\pi q(t)\ dt)^2 \ - \ \frac{1}{4}[2H \int_x^\pi q(t)\ dt \ + \ q(\pi) \ + \ q(x)],\\
\delta_2(x) &= \frac{1}{8}(\int_x^\pi q(t)\ dt)^3 \ + \ \frac{H}{4}[(\int_x^\pi q(t)\ dt)^2 \ + \ q(x) \ - \ q(\pi)]\\
&\quad + \ \frac{1}{8}[\int_x^\pi q(t)\ dt(q(x))] \ + \ \frac{1}{8}(q'(x) \ - \ q'(\pi)).
\end{aligned} \tag{1.13}$$

2. The calculations of the trace formula

Now we prove the essential theorem of the present work which concerns calculating regularized trace formula for Sturm-Liouville problem (1.5)-(1.6).

Theorem 1 Suppose that $q(x)$ has a second order piecewise integrable derivatives on $[0, \pi]$ then, in view of the introduced notations, (1.8) and (1.11), the following regularized trace formula takes place

$$\sum_{k=0}^{\infty}(\lambda_k^+ - \lambda_k^{o+} - d^+) + \sum_{k=0}^{\infty}(\lambda_k^- - \lambda_k^{o-} - d^-)$$

$$= D_1 \ + \ D_2\ ln\left[2+\frac{a^4+(\pi-a)^4}{a^2(\pi-a)^2}\right] + D_3\ tan^{-1}\left[\frac{\pi-a}{\pi}\right] + \ D_4\ ln\left[\frac{a}{\pi-a}\right], \tag{2.1}$$

where the constants d^+ , d^-, D_1 , D_2 , D_3 and D_4 are given by

$$d^+ = \frac{2}{a}\int_0^a q(t)dt, \qquad d^- = \frac{-4}{(\pi - a)}\left[H + \frac{1}{2}\int_a^\pi q(t)dt\right],$$

$$\begin{aligned}
D_1 &= -\frac{1}{4}\left(\int_a^\pi q(t)dt\right)^2 - \frac{1}{4}\left(\int_0^a q(t)dt\right)^2 + \frac{1}{2a}\left(\int_0^a q(t)dt\right) \\
&- \left(\int_a^\pi q(t)dt\right)\left[\frac{1}{2(\pi - a)}\right] + \left[H^2 - \frac{H}{(\pi - a)} - \frac{1}{2}q(a) - \frac{1}{4}q(\pi)\right], \\
D_2 &= \frac{2H}{\pi}\left(\int_0^a q(t)dt\right) + \frac{1}{\pi}\left(\int_0^a q(t)dt\right)\left(\int_a^\pi q(t)dt\right), \\
D_3 &= \frac{1}{\pi}\left(\int_0^a q(t)dt\right)^2, \\
D_4 &= -\frac{1}{\pi}\left(\int_0^a q(t)dt\right)^2 - \frac{2H}{\pi}\left(\int_0^a q(t)dt\right) - \frac{1}{\pi}\left(\int_0^a q(t)dt\right)\left(\int_a^\pi q(t)dt\right) \\
&- \frac{1}{\pi}(q(a) + q(0)).
\end{aligned} \tag{2.2}$$

Proof

To calculate the trace formula we use the familiar relation of summation of eigenvalues from the theory of functions of a complex a variable. According to formula of $\Psi(s)$ in (1.8), we have

$$2\sum_{k=0}^{n}\left[\lambda_k^+ + \lambda_k^-\right] = \frac{1}{2\pi i}\oint_{\Gamma_n} s^2 \, \frac{\Psi'(s)}{\Psi(s)}\, ds = \frac{1}{2\pi i}\oint_{\Gamma_n} s^2 \; d\; ln\Psi(s), \tag{2.3}$$

where the contour Γ_n is the quadratic contour on the s-domain as defined in [21]

$$\Gamma_n = \left\{|Re\ s| < \frac{\pi}{a}(n - \frac{1}{4}) + \frac{\pi}{2a}\ ,\ |Im\ s| \le \frac{\pi}{\pi - a}(n - \frac{1}{4}) + \frac{\pi}{2(\pi - a)}\right\}. \tag{2.4}$$

The equation (1.8) can be written in the form

$$\Psi(s) = L_o(s)\left[1 + r(s)\right], \tag{2.5}$$

$$r(s) = \frac{L_1(s)}{s\ L_o(s)} + \frac{L_2(s)}{s^2\ L_o(s)} + \frac{L_3(s)}{s^3\ L_o(s)} + O\left(\frac{e^{|Ims|a + |Res|(\pi - a)}}{s^4\ L_o(s)}\right), \tag{2.6}$$

with $L_o(s), L_1(s), L_2(s)$ and $L_3(s)$ are given by (1.9). The term $L_o(s)$ holds the following inequality from below on the contour Γ_n

$$|L_o(s)| \ge C\ e^{|Ims|a + |Res|(\pi - a)}, \tag{2.7}$$

C is constant.Substituting from (2.7) into (2.6), we obtain

$$r(s) \;=\; \frac{L_1(s)}{s\;L_o(s)} + \frac{L_2(s)}{s^2\;L_o(s)} + \;\frac{L_3(s)}{s^3\;L_o(s)} + O\left(\frac{1}{s^4}\right). \tag{2.8}$$

From (2.5) into the equation (2.3), we have

$$2\sum_{k=0}^{n}\left[\lambda_k^+ + \lambda_k^-\right] \;= \frac{1}{2\pi i}\oint_{\Gamma_n} s^2\; d\; ln\; L_o(s) \;+\; \frac{1}{2\pi i}\oint_{\Gamma_n} s^2\; d\; ln[1 + r(s)], \tag{2.9}$$

moreover, the first integral given by the next formula

$$\frac{1}{2\pi i}\oint_{\Gamma_n} s^2\; d\; ln\; L_o\;(s) \;=\; 2\sum_{k=0}^{n}\left[\lambda_k^{o+} + \lambda_k^{o-}\right]. \tag{2.10}$$

Since $\lambda_k^{o\pm}$ are the eigenvalues of (1.5)-(1.6) for $q(x) = 0$. Integrating by part the second integral of (2.9) we have

$$\frac{1}{2\pi i}\oint_{\Gamma_n} s^2\; d\; ln[1 + r(s)] = \frac{-1}{2\pi i}\oint_{\Gamma_n} 2s\;\left[r(s) - \frac{1}{2}r^2(s) + \frac{1}{3}r^3(s) + O(\frac{1}{s^4})\right]\; ds. \tag{2.11}$$

Substituting from (2.8) into (2.11) after some simplifications we obtain,

$$\frac{1}{2\pi i}\oint_{\Gamma_n} s^2\; d\; ln[\;1 \;+\; r(s)\;] =$$

$$\begin{aligned} &\frac{-1}{2\pi i}\oint_{\Gamma_n} 2s\left[\frac{L_1(s)}{sL_o(s)} + \frac{L_2(s)}{s^2L_o(s)} - \frac{L_1^2(s)}{2s^2L_o^2(s)} + \frac{L_3(s)}{s^3L_0(s)} - \frac{L_1(s)L_2(s)}{s^3L_0^2(s)}\right.\\ &\left. + \tfrac{1}{3}\tfrac{L_1^3(s)}{s^3L_0^3(s)} + O\left(\frac{1}{|s|^4}\right)\right] ds. \end{aligned} \tag{2.12}$$

From (2.10) and (2.12), into the equation (2.9) we obtain

$$\begin{aligned} &2\textstyle\sum_{k=0}^{n}\left[(\lambda_k^+ - \lambda_k^{o+})\right] \;-\; 2\textstyle\sum_{k=0}^{n}\left[\lambda_k^- - \lambda_k^{o-}\right] =\\ &\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{L_1(s)}{L_o(s)}ds - \frac{1}{\pi i}\oint_{\Gamma_n}\frac{L_2(s)}{sL_o(s)}ds + \frac{1}{2\pi i}\oint_{\Gamma_n}\frac{L_1^2(s)}{sL_o^2(s)}ds - \frac{1}{\pi i}\oint_{\Gamma_n}\frac{L_3(s)}{s^2L_o(s)}ds\\ &\quad + \frac{1}{\pi i}\oint_{\Gamma_n}\frac{L_1(s)L_2(s)}{s^2L_o^2(s)}ds - \frac{1}{3\pi i}\oint_{\Gamma_n}\frac{L_1^3(s)}{s^2L_o^3(s)}ds + \oint_{\Gamma_n} O\left(\frac{1}{|s|^3}\right) ds. \end{aligned} \tag{2.13}$$

We want to calculate each integral term of (2.13), its easy to shown that

$$\oint_{\Gamma_n} O\left(\frac{1}{|s|^4}\right) ds \to 0 \quad as\ n \to \infty \tag{2.14}$$

Now our purpose to compute six integrals of equation (2.13), starting with calculation of the first integration $\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{L_1(s)}{L_o(s)}ds$, since $\frac{L_1(s)}{L_o(s)}$ is odd, therefore from (1.9) we obtain

$$\begin{aligned}\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{L_1(s)}{L_o(s)}ds &= \frac{-1}{\pi i}\oint_{\Gamma_n}\frac{P_1 \sin\ sa\ \cosh\ s(\pi-a)+P_2\cos\ sa\ \sinh\ s(\pi-a)}{\sin\ sa\ \sinh\ s(\pi-a)+\cos\ sa\ \cosh\ s(\pi-a)}ds\\ &= \frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{P_1\sin\,(\sigma_n+i\xi)a\ \cosh\,(\sigma_n+i\xi)(\pi-a)+P_2\cos\,(\sigma_n+i\xi)a\ \sinh\,(\sigma_n+i\xi)(\pi-a)}{\sin\,(\sigma_n+i\xi)a\ \sinh\,(\sigma_n+i\xi)(\pi-a)+\cos\,(\sigma_n+i\xi)a\ \cosh\,(\sigma_n+i\xi)(\pi-a)}\,d\xi\\ &+\frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\frac{P_1\sin\,(\sigma+i\xi_n)a\ \cosh\,(\sigma+i\xi_n)(\pi-a)+P_2\cos\,(\sigma+i\xi_n)a\ \sinh\,(\sigma+i\xi_n)(\pi-a)}{\sin\,(\sigma+i\xi_n)a\ \sinh\,(\sigma+i\xi_n)(\pi-a)+\cos\,(\sigma+i\xi_n)a\ \cosh\,(\sigma+i\xi_n)(\pi-a)}\,d\sigma.\end{aligned} \tag{2.15}$$

After simplifying equation (2.15) becomes in the form

$$\begin{aligned}\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{L_1(s)}{L_o(s)}ds &= I_1^* + I_1^{**},\\ where\quad I_1^* &= \frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{P_1\tan\,(\sigma_n+i\xi)a+P_2\tanh\,(\sigma_n+i\xi)(\pi-a)}{\tan\,(\sigma_n+i\xi)a\ \tanh\,(\sigma_n+i\xi)(\pi-a)+1}\,d\xi\\ I_1^{**} &= \frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\frac{P_1\tan\,(\sigma+i\xi_n)a+P_2\tanh\,(\sigma+i\xi_n)(\pi-a)}{\tan\,(\sigma+i\xi_n)a\ \tanh\,(\sigma+i\xi_n)(\pi-a)+1}\,d\sigma.\end{aligned} \tag{2.16}$$

Noticing that I_1^* the function $tan(\sigma_n+i\xi)$ is bounded while

$$\tanh(\sigma_n+i\xi)(\pi-a) = 1 + O\left(e^{-2\sigma_n(\pi-a)}\right), \tag{2.17}$$

therefore I_1^* turned to the next form

$$I_1^* = \frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{P_2 + P_1\tan\,(\sigma_n+i\xi)a}{\tan\,(\sigma_n+i\xi)a + 1}\,d\xi + o(1). \tag{2.18}$$

By virture of the relations

$$\begin{aligned}\cos\,(\sigma_n+i\xi)a &= \frac{(-1)^n}{\sqrt{2}}\,[\cosh\xi\ a - i\ \sinh\ \xi a],\\ \sin\,(\sigma_n+i\xi)a &= \frac{(-1)^n}{\sqrt{2}}\,[\cosh\xi a + i\ \sinh\ \xi a],\end{aligned} \tag{2.19}$$

Then the equation (2.18) becomes

$$I_1^* = \frac{-2}{\pi-a}(P_1+P_2)(n+\frac{1}{4}) + o(1). \tag{2.20}$$

On the other hand, to estimate I_1^{**} we obtain $\tanh(\sigma+i\xi_n)(\pi-a)$ is bounded and

$$tan(\sigma+i\xi_n)a = i + O(e^{-2\xi_n a}), \tag{2.21}$$

then, I_1^{**} becomes of the form

$$I_1^{**} = \frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n} \frac{P_2 \sinh{(\sigma+i\xi_n)(\pi-a)} + i\, P_1 \cosh{(\sigma+i\xi_n)(\pi-a)}}{\cosh{(\sigma+i\xi_n)(\pi-a)} + i \sinh{(\sigma+i\xi_n)(\pi-a)}}\, d\sigma \tag{2.22}$$

$+ \ o(1)$.

Urging as before, applying the following relations

$$\begin{aligned} \cosh{(\sigma+i\xi_n)(\pi-a)} &= \tfrac{(-1)^n}{\sqrt{2}}\,[\cosh\sigma\,(\pi-a) + i\, \sinh\, \sigma(\pi-a)] \\ \sinh{(\sigma+i\xi_n)(\pi-a)} &= \tfrac{(-1)^n}{\sqrt{2}}\,[\sinh\sigma(\pi-a) + i\, \cosh\, \sigma(\pi-a)] \end{aligned} \tag{2.23}$$

to (2.22), we obtain

$$I_1^{**} = \frac{2}{a}(P_1-P_2)(n+\frac{1}{4}) + o(1). \tag{2.24}$$

Substitutig from (2.20) and (2.24) into (2.16), we get

$$\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{Z_1(s)}{Z_o(s)}ds = \left[\frac{2(P_1-P_2)}{a} - \frac{2(P_1+P_2)}{\pi-a}\right](n+\frac{1}{4}) + o(1). \tag{2.25}$$

In similar way, to estimate the second integral of the equation (2.13) $\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{L_2(s)}{sL_o(s)}ds$ where $\frac{L_2(s)}{sL_o(s)}$ is odd function, then

$$\begin{aligned} &\tfrac{-1}{\pi i}\oint_{\Gamma_n}\tfrac{L_2(s)}{sL_o(s)}ds = \tfrac{-1}{\pi i}\oint_{\Gamma_n}\tfrac{Q_1\ \sin\ sa\ \sinh\ s(\pi-a)+Q_2\ \cos\ sa\ \cosh\ s(\pi-a)}{s\,[\sin\ sa\ \sinh\ s(\pi-a)+\cos\ sa\ \cosh\ s(\pi-a)]}ds \\ &= \tfrac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\tfrac{Q_1\ \sin\,(\sigma_n+i\xi)a\ \sinh\,(\sigma_n+i\xi)(\pi-a)+Q_2\ \cos\,(\sigma_n+i\xi)a\ \cosh\,(\sigma_n+i\xi)(\pi-a)}{(\sigma_n+i\xi)[\sin\,(\sigma_n+i\xi)a\ \sinh\,(\sigma_n+i\xi)(\pi-a)+\cos\,(\sigma_n+i\xi)a\ \cosh\,(\sigma_n+i\xi)(\pi-a)]}\,d\xi \\ &+\tfrac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\tfrac{Q_1\ \sin\,(\sigma+i\xi_n)a\ \sinh\,(\sigma+i\xi_n)(\pi-a)+Q_2\ \cos\,(\sigma+i\xi_n)a\ \cosh\,(\sigma+i\xi_n)(\pi-a)}{(\sigma+i\xi_n)[\sin\,(\sigma+i\xi_n)a\ \sinh\,(\sigma+i\xi_n)(\pi-a)+\cos\,(\sigma+i\xi_n)a\ \cosh\,(\sigma+i\xi_n)(\pi-a)]}\,d\sigma. \end{aligned} \tag{2.26}$$

From (2.26) we get

$$\begin{aligned} &\tfrac{-1}{\pi i}\oint_{\Gamma_n}\tfrac{L_2(s)}{sL_o(s)}ds = I_2^{*} + I_2^{**}, \\ &where\quad I_2^{*} = \tfrac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\tfrac{Q_1\ \tan\,(\sigma_n+i\xi)a\ \ \tanh\,(\sigma_n+i\xi)(\pi-a)+Q_2}{(\sigma_n+i\xi)[\tan\,(\sigma_n+i\xi)a\ \ \tanh\,(\sigma_n+i\xi)(\pi-a)+1]}\,d\xi, \\ &\qquad\quad I_2^{**} = \tfrac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\tfrac{Q_1\ \tan\,(\sigma+i\xi_n)a\ \ \tanh\,(\sigma+i\xi_n)(\pi-a)+Q_2}{(\sigma+i\xi_n)[\tan\,(\sigma+i\xi_n)a\ \ \tanh\,(\sigma+i\xi_n)(\pi-a)+1]}\,d\sigma. \end{aligned} \tag{2.27}$$

remember that (2.17) $\tan{(\sigma+i\xi_n)a}$ is bounded and using (2.17, then we obtain

$$I_2^{*} = \frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{Q_1\ \sin{(\sigma_n+i\xi)a} + Q_2\ \cos{(\sigma_n+i\xi)a}}{(\sigma_n+i\xi)\ [\ \sin{(\sigma_n+i\xi)a} + \cos{(\sigma_n+i\xi)a}\ ]}\,d\xi + o(1). \tag{2.28}$$

By the aid of (2.19), after some simplifications , I_2^{*} becomes

$$I_2^{*} = -\frac{2(Q_1+Q_2)}{\pi}\ \tan^{-1}\left(\frac{a}{\pi-a}\right) + \frac{(Q_2-Q_1)}{\pi}\ ln\left[1 + \frac{a^2}{(\pi-a)^2}\right] \tag{2.29}$$

+ o(1).

Now we calculate I_2^{**}, since $tanh(\sigma + i\xi_n)(\pi - a)$ is bounded, from (2.27) we have

$$I_2^{**} = \frac{-2}{\pi i}\int_{-\sigma_n}^{\sigma_n} \frac{Q_2\ \cosh\,(\sigma + i\xi_n)(\pi - a)\ + iQ_1\ \sinh\,(\sigma + i\xi_n)(\pi - a)}{(\sigma + i\xi_n)\ [\ \cosh\,(\sigma + i\xi_n)(\pi - a)\ +\ i\ \ \sinh\,(\sigma + i\xi_n)(\pi - a)\ ]}\, d\sigma \tag{2.30}$$

+ o(1).

Substituting from (2.23) into (2.30) , we have

$$I_2^{**} \ = \ -\frac{2(Q_1 + Q_2)}{\pi}\ \tan^{-1}\left(\frac{\pi - a}{a}\right) \ - \ \frac{(Q_2 - Q_1)}{\pi}\ ln\left[1 \ + \ \frac{(\pi - a)^2}{a^2}\right] \ + \ o(1). \tag{2.31}$$

By substitution from (2.31) and (2.29) into (2.27), we get

$$\frac{-1}{\pi i}\oint_{\Gamma_n} \frac{L_2(s)}{sL_o(s)}ds \ = \ -Q_1 - Q_2 \ + \ \frac{Q_2 - Q_1}{\pi}\ ln\,\frac{a^2}{(\pi - a)^2} \ + \ o(1). \tag{2.32}$$

In this moment we estimate the third integral of (2.13), we have

$$\begin{aligned}
&\tfrac{1}{2\pi i}\oint_{\Gamma_n} \tfrac{L_1^2(s)}{s\,L_o^2(s)}ds = \tfrac{1}{2\pi i}\oint_{\Gamma_n} \tfrac{[\ P_1\ \sin\ sa\ \cosh\ s(\pi-a)+P_2\ \cos\ sa\ \sinh\ s(\pi-a)]^2}{s\ [\ \sin\ sa\ \cosh\ s(\pi-a)+\cos\ sa\ \sinh\ s(\pi-a)\ ]^2}\, ds \\
&= \tfrac{1}{\pi}\int_{-\xi_n}^{\xi_n} \tfrac{[P_1\ \sin\,(\sigma_n+i\xi)a\ \cosh\,(\sigma_n+i\xi)(\pi-a)+P_2\ \cos\,(\sigma_n+i\xi)a\ \sinh\,(\sigma_n+i\xi)(\pi-a)]^2}{(\sigma_n+i\xi)[\sin\,(\sigma_n+i\xi)a\ \sinh\,(\sigma_n+i\xi)(\pi-a)+\cos\,(\sigma_n+i\xi)a\ \cosh\,(\sigma_n+i\xi)(\pi-a)]^2}\, d\xi \\
&+\tfrac{-1}{\pi i}\int_{-\sigma_n}^{\sigma_n} \tfrac{[P_1\ \sin\,(\sigma+i\xi_n)a\ \cosh\,(\sigma+i\xi_n)(\pi-a)+Q_2\ \cos\,(\sigma+i\xi_n)a\ \sinh\,(\sigma+i\xi_n)(\pi-a)]^2}{(\sigma+i\xi_n)[\sin\,(\sigma+i\xi_n)a\ \sinh\,(\sigma+i\xi_n)(\pi-a)+\cos\,(\sigma+i\xi_n)a\ \cosh\,(\sigma+i\xi_n)(\pi-a)]^2}\, d\sigma.
\end{aligned} \tag{2.33}$$

After some calculations (2.33) becomes in the following form

$$\begin{aligned}
&\tfrac{1}{2\pi i}\oint_{\Gamma_n} \tfrac{L_1^2(s)}{sL_o^2(s)}ds \ = \ I_3^* \ + \ I_3^{**}, \\
&since \quad I_3^* = \ \tfrac{1}{\pi}\int_{-\xi_n}^{\xi_n}\left[\tfrac{P_1+P_2}{2} \ + \ i\tfrac{P_1-P_2}{2i}\tfrac{\sinh\ \xi a}{\cosh\ \xi a}\right]^2 \tfrac{1}{\sigma_n+i\xi}\, d\xi, \\
&\qquad I_3^{**} = \ \tfrac{-1}{\pi i}\int_{-\sigma_n}^{\sigma_n}\left[\tfrac{i(P_2-P_1)}{2i} \ + \ \tfrac{P_1+P_2}{2}\tfrac{\cosh\ \sigma(\pi-a)}{\sinh\ \sigma(\pi-a)}\right]^2 \tfrac{1}{\sigma+i\xi_n}\, d\sigma.
\end{aligned} \tag{2.34}$$

Simplify I_3^* we obtain

$$\begin{aligned}
I_3^* = \ &\tfrac{1}{4\pi}\int_{-\xi_n}^{\xi_n} \tfrac{(P_1+P_2)^2}{\sigma_n+i\xi}\, d\xi \ - \ \tfrac{i}{2\pi}\int_{-\xi_n}^{\xi_n} \tfrac{(P_1^2-P_2^2)}{\sigma_n+i\xi}\ \tfrac{\sinh\ \xi a}{\cosh\ \xi a}d\xi\ . \\
&- \ \tfrac{1}{4\pi}\int_{-\xi_n}^{\xi_n} \tfrac{(P_1-P_2)^2}{\sigma_n+i\xi}\ \tfrac{\sinh^2\ \xi a}{\cosh^2\ \xi a}\, d\xi
\end{aligned} \tag{2.35}$$

Compute each term in (2.35), we obtain

$$I_3^* \ = \ \frac{2P_1P_2}{\pi}\tan^{-1}\frac{a}{\pi - a} + \frac{(P_1^2 - P_2^2)}{2\pi}ln\,\left(1 + \frac{a^2}{(\pi - a)^2}\right) + o(1). \tag{2.36}$$

In simillar way, we have

$$I_3^{**} = \frac{1}{4\pi i}\int_{-\sigma_n}^{\sigma_n} \frac{(P_2-P_1)^2}{\sigma+i\xi_n}\,d\sigma + \frac{1}{2\pi i}\int_{-\sigma_n}^{\sigma_n} \frac{(P_2^2-P_1^2)}{\sigma+i\xi_n}\,\frac{\cosh\,\sigma(\pi-a)}{\sinh\,\sigma(\pi-a)}\,d\sigma - \frac{1}{4\pi i}\int_{-\sigma_n}^{\sigma_n} \frac{(P_1+P_2)^2}{\sigma+i\xi_n}\,\frac{\cosh^2\,\sigma(\pi-a)}{\sinh^2\,\sigma(\pi-a)}\,d\sigma\ . \tag{2.37}$$

After some simplifications, we have

$$I_3^{**} = \frac{P_1^2+P_2^2}{2\pi}\,\tan^{-1}\frac{\pi-a}{a} + \frac{(P_1^2-P_2^2}{2\pi}\,ln\left(1+\frac{(\pi-a)^2}{a^2}\right) + o(1). \tag{2.38}$$

Substituting from (2.36) and (2.38) into (2.34), we obtain

$$\frac{1}{2\pi i}\oint_{\Gamma_n}\frac{Z_1^2(s)}{sZ_o^2(s)}ds = \frac{(P_1-P_2)^2}{\pi}tan^{-1}\frac{\pi-a}{a} + P_1P_2 + \frac{P_1^2-P_2^2}{2\pi}\,ln\left(2+\frac{a^4+(\pi-a)^4)}{a^2(\pi-a)^2}\right) + o(1). \tag{2.39}$$

Urging as before the fourth integral of (2.13), we have

$$\begin{aligned} &\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{L_3(s)}{s^2\,L_o(s)}ds = -\frac{1}{\pi i}\oint_{\Gamma_n}\frac{[\,R_1\,\sin\,sa\,\cosh\,s(\pi-a)+R_2\,\cos\,sa\,\sinh\,s(\pi-a)]}{s^2\,[\,\sin\,sa\,\sinh\,s(\pi-a)+\cos\,sa\,\cosh\,s(\pi-a)\,]}\,ds \\ &= \frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\frac{[R_1\,\sin\,(\sigma_n+i\xi)a\,\cosh\,(\sigma_n+i\xi)(\pi-a)+R_2\,\cos\,(\sigma_n+i\xi)a\,\sinh\,(\sigma_n+i\xi)(\pi-a)]}{(\sigma_n+i\xi)^2[\sin\,(\sigma_n+i\xi)A\,\sinh\,(\sigma_n+i\xi)(\pi-a)+\cos\,(\sigma_n+i\xi)a\,\cosh\,(\sigma_n+i\xi)(\pi-a)]}\,d\xi \\ &+\frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\frac{[R_1\,\sin\,(\sigma+i\xi_n)a\,\cosh\,(\sigma+i\xi_n)(\pi-a)+R_2\,\cos\,(\sigma+i\xi_n)a\,\sinh\,(\sigma+i\xi_n)(\pi-a)]}{(\sigma+i\xi_n)^2[\sin\,(\sigma+i\xi_n)a\,\sinh\,(\sigma+i\xi_n)(\pi-a)+\cos\,(\sigma+i\xi_n)a\,\cosh\,(\sigma+i\xi_n)(\pi-a)]}\,d\sigma \end{aligned} \tag{2.40}$$

After some calculations equation(2.40) can be written as

$$\begin{aligned} &\frac{-1}{\pi i}\oint_{\Gamma_n}\frac{L_3(s)}{s^2L_o(s)}ds = I_4^* + I_4^{**} \\ &where\quad I_4^* = \frac{-1}{\pi}\int_{-\xi_n}^{\xi_n}\left[(R_1+R_2) + i(R_1-R_2)\frac{\sinh\,\xi a}{\cosh\,\xi a}\right]\frac{1}{(\sigma_n+i\xi)^2}\,d\xi, \\ &\qquad I_4^{**} = \frac{-1}{\pi}\int_{-\sigma_n}^{\sigma_n}\left[(R_2-R_1) + i\,(R_1+R_2)\frac{\cosh\,\sigma(\pi-a)}{\sinh\,\sigma(\pi-a)}\right]\frac{1}{(\sigma+i\xi_n)^2}\,d\sigma. \end{aligned} \tag{2.41}$$

For I_4^* we have

$$I_4^* = \frac{-1}{\pi}\int_{-\xi_n}^{\xi_n}\frac{(R_1+R_2)}{(\sigma_n+i\xi)^2}\,d\xi - \frac{i}{\pi}\int_{-\xi_n}^{\xi_n}\frac{(R_1-R_2)}{(\sigma_n+i\xi)^2}\,\frac{\sinh\,\xi a}{\cosh\,\xi a}d\xi + o(1). \tag{2.42}$$

by estimating the integrations in (2.42), we get

$$I_4^* = \frac{2(R_2-R_1)\,a^3}{\pi^2(n+\frac{1}{4})\,[a^2+(\pi-a)^2]} - \frac{2(R_1+R_2)\,a^2(\pi-a)}{\pi^2(n+\frac{1}{4})\,[a^2+(\pi-a)^2]} + o(1). \tag{2.43}$$

On the Other hand, for I_4^{**} we get

$$I_4^{**} = -\frac{1}{\pi}\int_{-\xi_n}^{\xi_n} \frac{(R_2-R_1)}{(\sigma_n+i\xi)^2}\, d\sigma \;-\; \frac{i}{\pi}\int_{-\xi_n}^{\xi_n} \frac{(R_1+R_2)}{(\sigma_n+i\xi)^2}\,\frac{\cosh\ \xi a}{\sinh\ \xi a} d\sigma \;+\; o(1). \tag{2.44}$$

Then calculate the integrals we obtain in (2.44), we obtain

$$I_4^{**} \;=\; \frac{2(R_2-R_1)\ a\ (\pi-a)^2}{\pi^2(n+\frac{1}{4})\,[a^2+(\pi-a)^2]} \;-\; \frac{2(R_1+R_2)\ (\pi-a)^3}{\pi^2(n+\frac{1}{4})\,[a^2+(\pi-a)^2]} \;+o(1). \tag{2.45}$$

Substituting from (2.43) and (2.45) into (2.40), we obtain

$$I_4 \;=\; \frac{2a(R_2-R_1)-2(\pi-a)(R_1+R_2)}{\pi^2\ (n+\frac{1}{4})} \;+o(1). \tag{2.46}$$

Regard as we did before we calculate the fifth integral of (2.13)

$$\begin{aligned}
&\frac{1}{\pi i}\oint_{\Gamma_n} \frac{L_1(s)\ L_2(s)}{s^2\ L_o^2(s)} ds = \\
&\frac{1}{\pi i}\oint_{\Gamma_n} \frac{[\ P_1\ \sin\ sa\ \cosh\ s(\pi-a)+P_2\ \cos\ sa\ \sinh\ s(\pi-a)][\ Q_1\ \sin\ sa\ \sinh\ s(\pi-a)+Q_2\ \cos\ sa\ \cosh\ s(\pi-a)]}{s^2\ [\ \sin\ sa\ \sinh\ s(\pi-a)+\cos\ sa\ \cosh\ s(\pi-a)\ ]^2}\ ds \\
&= \frac{-2}{\pi}\int_{-\xi_n}^{\xi_n}\left[\frac{[P_1\ \sin\ (\sigma_n+i\xi)a\ \cosh\ (\sigma_n+i\xi)(\pi-a)+P_2\ \cos\ (\sigma_n+i\xi)a\ \sinh\ (\sigma_n+i\xi)(\pi-a)]}{(\sigma_n+i\xi)[\sin\ (\sigma_n+i\xi)a\ \sinh\ (\sigma_n+i\xi)(\pi-a)+\cos\ (\sigma_n+i\xi)a\ \cosh\ (\sigma_n+i\xi)(\pi-a)]}\right] \\
&\times\left[\frac{[Q_1\ \sin\ (\sigma_n+i\xi)a\ \sinh\ (\sigma_n+i\xi)(\pi-a)+Q_2\ \cos\ (\sigma_n+i\xi)a\ \cosh\ (\sigma_n+i\xi)(\pi-a)]}{(\sigma_n+i\xi)[\sin\ (\sigma_n+i\xi)a\ \sinh\ (\sigma_n+i\xi)(\pi-a)+\cos\ (\sigma_n+i\xi)a\ \cosh\ (\sigma_n+i\xi)(\pi-a)]}\right]\ d\xi \\
&+\frac{2}{\pi i}\int_{-\sigma_n}^{\sigma_n}\left[\frac{[P_1\ \sin\ (\sigma+i\xi_n)a\ \cosh\ (\sigma+i\xi_n)(\pi-a)+P_2\ \cos\ (\sigma+i\xi_n)a\ \sinh\ (\sigma+i\xi_n)(\pi-a)]}{(\sigma+i\xi_n)[\sin\ (\sigma+i\xi_n)a\ \sinh\ (\sigma+i\xi_n)(\pi-a)+\cos\ (\sigma+i\xi_n)a\ \cosh\ (\sigma+i\xi_n)(\pi-a)]}\right] \\
&\times\left[\frac{Q_1\ \sin\ (\sigma+i\xi_n)a\ \sinh\ (\sigma+i\xi_n)(\pi-a)+Q_2\ \cos\ (\sigma+i\xi_n)a\ \cosh\ (\sigma+i\xi_n)(\pi-a)]}{(\sigma+i\xi_n)[\sin\ (\sigma+i\xi_n)a\ \sinh\ (\sigma+i\xi_n)(\pi-a)+\cos\ (\sigma+i\xi_n)a\ \cosh\ (\sigma+i\xi_n)(\pi-a)]}\right]\ d\sigma.
\end{aligned} \tag{2.47}$$

After some calculations equation (2.47) becomes

$$\frac{1}{\pi i}\oint_{\Gamma_n}\frac{L_1(s)L_2(s)}{s^2L_o^2(s)}ds \;=\; I_5^* \;+\; I_5^{**},$$
where

$$\begin{aligned}
I_5^* &= \frac{-1}{2\pi}\int_{-\xi_n}^{\xi_n}\Bigg[(P_1+P_2)(Q_1+Q_2)-2i(P_1Q_1-P_2Q_2)\frac{\sinh\xi a}{\cosh\xi a} \\
&\qquad -(P_1-P_2)(Q_1-Q_2)\frac{\sinh^2\xi a}{\cosh^2\xi a}\Bigg]\ \frac{1}{(\sigma_n+i\xi)^2}\ d\xi \\
I_5^{**} &= \frac{-1}{2\pi i}\int_{-\sigma_n}^{\sigma_n}\Bigg[(P_2-P_1)(Q_1+Q_2)-2(P_1Q_2+P_2Q_1)\frac{\cosh\sigma(\pi-a)}{\sinh\sigma(\pi-a)} \\
&\qquad + i(P_1+P_2)(Q_2-Q_1)\frac{\cosh^2\sigma(\pi-a)}{\sinh^2\sigma(\pi-a)}\Bigg]\ \frac{1}{(\sigma+i\xi_n)^2}\ d\sigma.
\end{aligned} \tag{2.48}$$

For I_5^* we have

$$\begin{aligned} I_5^* \;=\; & \tfrac{-1}{2\pi}\int_{-\xi_n}^{\xi_n} \tfrac{(P_1+P_2)(Q_1+Q_2)}{(\sigma_n+i\xi)^2}\, d\xi \;-\; \tfrac{i}{\pi}\int_{-\xi_n}^{\xi_n} \tfrac{(P_1Q_1-P_2Q_2)}{(\sigma_n+i\xi)^2}\, \tfrac{\sinh\ \xi a}{\cosh\ \xi a} \\ & +\tfrac{1}{2\pi}\int_{-\xi_n}^{\xi_n} \tfrac{(P_1-P_2)(Q_1-Q_2)}{(\sigma_n+i\xi)^2}\, \tfrac{\sinh^2\ \xi a}{\cosh^2\ \xi a} d\xi \;+\; o(1). \end{aligned} \tag{2.49}$$

by calculating the integrations in (2.49), we obtain

$$I_5^* \;=\; \frac{2(P_2Q_2 - P_1Q_1)\ a^3}{\pi^2(n+\frac{1}{4})\,[a^2+(\pi-a)^2]} \;-\; \frac{2(P_1Q_2+P_2Q_1)\ a^2(\pi-a)}{\pi^2(n+\frac{1}{4})\,[a^2+(\pi-a)^2]} \;+ o(1). \tag{2.50}$$

On the Other hand, for I_5^{**} we get

$$\begin{aligned} I_5^{**} \;=\; & \tfrac{-1}{2\pi}\int_{-\sigma_n}^{\sigma_n} \tfrac{(P_2-P_1)(Q_1+Q_2)}{(\sigma+i\xi_n)^2}\, d\sigma \;+\; \tfrac{1}{\pi i}\int_{-\sigma_n}^{\sigma_n} \tfrac{(P_1Q_2+P_2Q_1)}{(\sigma+i\xi_n)^2}\, \tfrac{\cosh\ \sigma(\pi-a)}{\sinh\ \sigma(\pi-a)} \\ & -\tfrac{1}{2\pi}\int_{-\sigma_n}^{\sigma_n} \tfrac{(P_1+P_2)(Q_2-Q_2)}{(\sigma+i\xi_n)^2}\, \tfrac{\cosh^2\ \sigma(\pi-a)}{\sinh^2\ \sigma(\pi-a)} d\sigma \;+\; o(1). \end{aligned} \tag{2.51}$$

from which after simplifications we obtain

$$I_5^{**} \;=\; \frac{2(P_2Q_2 - P_1Q_1)\ a\ (\pi-a)^2}{\pi^2(n+\frac{1}{4})\,[a^2+(\pi-a)^2]} \;-\; \frac{2(P_1Q_2+P_2Q_1)\ (\pi-a)^3}{\pi^2(n+\frac{1}{4})\,[a^2+(\pi-a)^2]} \;+ o(1). \tag{2.52}$$

Substituting from (2.50) and (2.52) into (2.48), we obtain

$$I_5 \;=\; \frac{2a(P_2Q_2 - P_1Q_1) - 2(\pi-a)(P_1Q_2+P_2Q_1)}{\pi^2\ (n+\frac{1}{4})} \;+ o(1). \tag{2.53}$$

Finally we evaluate the sixth integral of (2.13)

$$\begin{aligned} & \tfrac{-1}{3\pi i}\oint_{\Gamma_n} \tfrac{L_1^3(s)}{s^2\ L_o^3(s)} ds = -\tfrac{1}{3\pi i}\oint_{\Gamma_n} \tfrac{[\ P_1\ \sin\ sa\ \cosh\ s(\pi-a)+P_2\ \cos\ sa\ \sinh\ s(\pi-a)]^3}{s^2\ [\ \sin\ sa\ \sinh\ s(\pi-a)+\cos\ sa\ \cosh\ s(\pi-a)\ ]3}\ ds \\ & = \tfrac{-2}{3\pi}\int_{-\xi_n}^{\xi_n} \tfrac{[P_1\ \sin\ (\sigma_n+i\xi)a\ \cosh\ (\sigma_n+i\xi)(\pi-a)+P_2\ \cos\ (\sigma_n+i\xi)a\ \sinh\ (\sigma_n+i\xi)(\pi-a)]^3}{(\sigma_n+i\xi)^2[\sin\ (\sigma_n+i\xi)A\ \sinh\ (\sigma_n+i\xi)(\pi-a)+\cos\ (\sigma_n+i\xi)a\ \cosh\ (\sigma_n+i\xi)(\pi-a)]^3}\ d\xi \\ & +\tfrac{2}{3\pi i}\int_{-\sigma_n}^{\sigma_n} \tfrac{[P_1\ \sin\ (\sigma+i\xi_n)a\ \cosh\ (\sigma+i\xi_n)(\pi-a)+P_2\ \cos\ (\sigma+i\xi_n)a\ \sinh\ (\sigma+i\xi_n)(\pi-a)]^3}{(\sigma+i\xi_n)^2[\sin\ (\sigma+i\xi_n)a\ \sinh\ (\sigma+i\xi_n)(\pi-a)+\cos\ (\sigma+i\xi_n)a\ \cosh\ (\sigma+i\xi_n)(\pi-a)]^3}\ d\sigma \end{aligned} \tag{2.54}$$

After simplification (2.54) becomes

$$\begin{aligned} & \tfrac{-1}{3\pi i}\oint_{\Gamma_n} \tfrac{L_1^3(s)}{s^2 L_o^3(s)} ds \;=\; I_6^* \;+\; I_6^{**}, \\ & where \\ & I_6^* \;=\; \tfrac{-2}{3\pi}\int_{-\xi_n}^{\xi_n} \left[\ \tfrac{(\ P_1+\ P_2)}{2} + \tfrac{i(P_1-P_2)}{2}\tfrac{\sinh\ \xi a}{\cosh\ \xi a}\right]^3 \tfrac{1}{(\sigma_n+i\xi)^2}\ d\xi, \\ & I_6^{**} \;=\; \tfrac{2}{3\pi i}\int_{-\sigma_n}^{\sigma_n} \left[\tfrac{(P_2-P_1)}{2i} + \tfrac{(P_1+P_2)}{2}\tfrac{\cosh\ \sigma(\pi-a)}{\sinh\ \sigma(\pi-a)}\right]^3 \tfrac{1}{(\sigma+i\xi_n)^2}\ d\sigma. \end{aligned} \tag{2.55}$$

For I_6^* we obtain

$$\begin{aligned} I_6^* &= -\tfrac{1}{12\pi}\int_{-\xi_n}^{\xi_n}\tfrac{(P_1+P_2)^3}{(\sigma_n+i\xi)^2}\,d\xi \;-\; \tfrac{i}{4\pi}\int_{-\xi_n}^{\xi_n}\tfrac{(P_1-P_2)(P_1+P_2)^2}{(\sigma_n+i\xi)^2}\,\tfrac{\sinh\ \xi a}{\cosh\ \xi a}\,d\xi \\ &+\tfrac{1}{4\pi}\int_{-\xi_n}^{\xi_n}\tfrac{(P_1+P_2)(P_1-P_2)^2}{(\sigma_n+i\xi)^2}\,\tfrac{\sinh^2\ \xi a}{\cosh^2\ \xi a}\,d\xi \;+\; \tfrac{i}{12\pi}\int_{-\xi_n}^{\xi_n}\tfrac{(P_1-P_2)^3}{(\sigma_n+i\xi)^2}\,\tfrac{\sinh^3\ \xi a}{\cosh^3\ \xi a}\,d\xi + o(1). \end{aligned} \tag{2.56}$$

by calculating the integrations in (2.56) we obtain

$$\begin{aligned} I_6^* &= \frac{\left[-3(P_1-P_2)\ (P_1+P_2)^2+(P_1-P_2)^3\right]a^3}{6\pi^2(n+\frac{1}{4})\left[a^2+(\pi-a)^2\right]} \\ &+ \frac{\left[3(P_1+P_2)(P_1-P_2)^2-(P_1+P_2)^3\right]\ a^2(\pi-a)}{6\pi^2(n+\frac{1}{4})\left[a^2+(\pi-a)^2\right]} + o(1). \end{aligned} \tag{2.57}$$

Urging as before, for I_6^{**} becomes

$$\begin{aligned} I_6^{**} &= \tfrac{1}{12\pi}\int_{-\sigma_n}^{\sigma_n}\tfrac{(P_2-P_1)^3}{(\sigma+i\xi_n)^2}\,d\sigma \;+\; \tfrac{i}{4\pi i}\int_{-\sigma_n}^{\sigma_n}\tfrac{(P_1+P_2)(P_2-P_1)^2}{(\sigma+i\xi_n)^2}\,\tfrac{\cosh\ \sigma(\pi-a)}{\sinh\ \sigma(\pi-a)} \\ &-\tfrac{1}{4\pi}\int_{-\sigma_n}^{\sigma_n}\tfrac{(P_2-P_1)(P_1+P_2)^2}{(\sigma+i\xi_n)^2}\,\tfrac{\cosh^2\ \sigma(\pi-a)}{\sinh^2\ \sigma(\pi-a)}\,d\sigma \;-\; \tfrac{i}{12\pi}\int_{-\sigma_n}^{\sigma_n}\tfrac{(P_1+P_2)^3}{(\sigma+i\xi_n)^2}\,\tfrac{\cosh^3\ \sigma(\pi-a)}{\sinh^3\ \sigma(\pi-a)}\,d\sigma + o(1). \end{aligned} \tag{2.58}$$

from which after simplifications we get

$$\begin{aligned} I_6^{**} &= \frac{\left[(P_1-P_2)^3-3(P_1-P_2)(P_1+P_2)^2\right]a\ (\pi-a)^2}{6\pi^2(n+\frac{1}{4})\left[a^2+(\pi-a)^2\right]} \\ &- \frac{\left[(P_1+P_2)^3-3(P_1+P_2)(P_2-P_1)^2\right](\pi-a)^3}{6\pi^2(n+\frac{1}{4})\left[a^2+(\pi-a)^2\right]} + o(1). \end{aligned} \tag{2.59}$$

Substituting from (2.57) and (2.59) into (2.55), we get

$$I_6 = \frac{a\left[(P_1-P_2)^3-3(P_1-P_2)(P_1+P_2)^2\right]-(\pi-a)\left[(P_1+P_2)^3-3(P_1+P_2)(P_2-P_1)^2\right]}{6\pi^2\ (n+\frac{1}{4})} \tag{2.60}$$

+o(1).

Substituting from (2.14), (2.25), (2.32), (2.39),(2.46), (2.53), and (2.60) into (2.13) we obtain

$$2\sum_{k=0}^{n}\left[(\lambda_k^+-\lambda_k^{o+})\right] \;+\; 2\sum_{k=0}^{n}\left[\lambda_k^--\lambda_k^{o-}\right]$$

$$
\begin{aligned}
&= \frac{2(P_1 - P_2)}{(a)}(n+\frac{1}{4}) \;-\; \frac{2(P_1 + P_2)}{(\pi - a)}(n+\frac{1}{4}) - Q_1 \;- Q_2 \;+ P_1 P_2 \\
&+ \frac{P_1^2 - P_2^2}{\pi} \ln\left[2 + \frac{a^4 + (\pi - a)^4}{a^2(\pi - a)^2}\right] + \frac{(P_1 - P_2)^2}{\pi}\tan^{-1}\left[\frac{\pi - a}{\pi}\right] + \frac{2(Q_2 - Q_1)}{\pi}\, ln\frac{a}{(\pi - a)} \\
&+ \frac{a}{6\pi^2(n+\frac{1}{4})}\left[12(R_2 - R_1) + 12(P_2Q_2 - P_1Q_1) + (P_1 - P_2)^3 - 3(P_1 - P_2)(P_1 + P_2)^2\right] \\
&- \frac{(\pi - a)}{6\pi^2(n+\frac{1}{4})}\left[12(R_1 + R_2) + 12(P_1Q_2 + P_2Q_1) + (P_1 + P_2)^3 - 3(P_1 + P_2)(P_1 - P_2)^2\right] \\
&+ o(1).
\end{aligned}
\tag{2.61}
$$

Letting $n \to \infty$, and by the help of (1.10),(1.12)and (1.13)we get the required formula in (2.1).

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