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Necessary and sufficient conditions for the boundedness of comutators of B-Riesz potentials in Lebegues spaces

Gulnara A. Abasova, Lale R. Aliyeva, Javanshir J. Hasanov, Esmira S. Shirinova

Abstract. In this paper the authors obtain necessary and sufficient conditions for the boundedness of the commutator $[b, I_\gamma^\alpha]$ defined by

$$[b, I_\gamma^\alpha]f(x) = \int_{\mathbb{R}^n_{k,+}} [b(x) - b(y)]f(y)T^y|x|^{\alpha-Q}\,(y')^\gamma dy$$

in the $L_{p,\gamma}$ spaces, where $0 < \alpha < n+|\gamma|$ and b is a locally integrable function on $\mathbb{R}^n_{k,+}$. It is proved that the commutator $[b, I_\gamma^\alpha]$ is bounded from the spaces $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$, $1 < p < \frac{n+|\gamma|}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$, and from the spaces $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$,$p = 1$, $1 - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$ if and only if $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$. Furthermore, it is shown that If $1 < p = \frac{Q}{\alpha}$, then the commutator $[b, \widetilde{I}_\gamma^\alpha]$ formed by the modified $B-$Riesz potential

$$\widetilde{I}_\gamma^\alpha f(x) = \int_{\mathbb{R}^n_{k,+}} \left(T^y|x|^{\alpha-Q} - |y|^{\alpha-Q}\chi_{E^*(0,1)}(y)\right) f(y)(y')^\gamma dy,$$

is bounded from $L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ to $BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$.

Key Words and Phrases: Commutator, Generalized shift operator, B-Riesz potential, B-maximal function, BMO_γ space.

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1. Introduction

Let b be a locally integrable function on $\mathbb{R}^n$ and T be a Calderon-Zygmund operator. The commutator is defined for smooth functions f by $[b, T]f = bT(f) - T(bf)$. Coifman, Rochberg and Weiss [2] stated that $[b, T]$ is a bounded operator on $L_p(\mathbb{R}^n), 1 < p < \infty$, if and only if $b \in BMO(\mathbb{R}^n)$. Chanillo [1] proved that the commutator with Reisz potentials $[b, I^\alpha]$ is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$, $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, if and only if

$b \in BMO(\mathbb{R}^n)$. Ding [3], Paluszynski [19], Perez [20], and Lu, Wu and Yang [16] have obtained various results on the boundedness of commutators. In this paper, we consider the boundedness of commutators defined by the B-Riesz potential

$$I_\gamma^\alpha f(x) = \int_{\mathbb{R}_{k,+}^n} f(y) T^y |x|^{\alpha-n-|\gamma|} (y')^\gamma dy$$

and modified B-Riesz potential

$$\widetilde{I}_\gamma^\alpha f(x) = \int_{\mathbb{R}_{k,+}^n} \left(T^y|x|^{\alpha-Q} - |y|^{\alpha-Q}\chi_{E^*(0,1)}(y)\right) f(y)(y')^\gamma dy,$$

and a locally integrable function $b \in \mathbb{R}_{k,+}^n$. We give a characterization of $BMO_\gamma(\mathbb{R}_{k,+}^n)$ via the boundedness of the commutators $[b, I_\gamma^\alpha]$ and $[b, \widetilde{I}_\gamma^\alpha]$ on $L_{p,\gamma}$ spaces. It is proved that the commutator $[b, I_\gamma^\alpha]$ is bounded from the spaces $L_{p,\gamma}(\mathbb{R}_{k,+}^n)$ to $L_{q,\gamma}(\mathbb{R}_{k,+}^n)$, $1 < p < \frac{n+|\gamma|}{\alpha}$, $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$, and from the spaces $L_{1,\gamma}(\mathbb{R}_{k,+}^n)$ to $WL_{q,\gamma}(\mathbb{R}_{k,+}^n)$,$p = 1$, $1 - \frac{1}{q} = \frac{\alpha}{n+|\gamma|}$ if and only if $b \in BMO_\gamma(\mathbb{R}_{k,+}^n)$. Furthermore, for the case $p = \frac{n+|\gamma|}{\alpha}$ we show that the commutator $[b, \widetilde{I}_\gamma^\alpha]$, formed by the modified $B-$ Riesz potential

$$\widetilde{I}_\gamma^\alpha f(x) = \int_{\mathbb{R}_{k,+}^n} \left(T^y|x|^{\alpha-Q} - |y|^{\alpha-Q}\chi_{E^*(0,1)}(y)\right) f(y)(y')^\gamma dy,$$

is bounded from $L_{p,\gamma}\left(\mathbb{R}_{k,+}^n\right)$ to $BMO_\gamma\left(\mathbb{R}_{k,+}^n\right)$.

2. Preliminaries

Let $\mathbb{R}_{k,+}^n = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : \ x_1 > 0, \dots, x_k > 0, \ 1 \le k \le n\}$ and let $E(x,r) = \{y \in \mathbb{R}_{k,+}^n \ ; \ |x-y| < r\}$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k}$. For a measurable set $E \subset \mathbb{R}_{k,+}^n$ suppose $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E(0,r)|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int_{E(0,1)} (x')^\gamma dx = \frac{\pi^{\frac{n-k}{2}}}{2^k} \, \Gamma^{-1}\left(\frac{Q+2}{2}\right) \prod_{i=1}^k \Gamma\left(\frac{\gamma_i+1}{2}\right).$$

Denote by $L_{p,\gamma} \equiv L_{p,\gamma}(\mathbb{R}_{k,+}^n)$ the set of all classes of measurable functions f with finite norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}_{k,+}^n} |f(x)|^p (x')^\gamma dx\right)^{\frac{1}{p}}, \quad 1 \le p < \infty.$$

We denote by $WL_{p,\gamma}(\mathbb{R}_{k,+}^n)$, the weak $L_{p,\gamma}$ space defined as the set of all classes of locally integrable functions f with finite norms

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left|\left\{x \in \mathbb{R}_{k,+}^n \ : \ |f(x)| > r\right\}\right|_\gamma^{1/p}, \quad 1 \le p < \infty.$$

The generalized shift operator T^y is defined by (see, for example [14, 15])

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \ldots \int_0^\pi f\left((x',y')_\beta, x''-y''\right)\, d\nu(\beta),$$

where $d\nu(\beta) = \prod_{i=1}^{k} \sin^{\gamma_i-1}\beta_i \quad d\beta_1\ldots d\beta_k$, $x' = (x_1,\ldots,x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1},\ldots,x_n) \in \mathbb{R}^{n-k}$, $(x_i,y_i)_{\beta_i} = (x_i^2 - 2x_iy_i\cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x',y')_\beta = ((x_1,y_1)_{\beta_1},\ldots,(x_k,y_k)_{\beta_k})$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}}\,\Gamma^{-1}\left(\frac{|\gamma|}{2}\right)\prod_{i=1}^{k}\Gamma\left(\frac{\gamma_i+1}{2}\right) = \frac{2^{k-1}|\gamma|}{\pi}\left(\frac{|\gamma|}{2}+1\right)\omega(2,k,\gamma).$$

It is well known that T^y is closely related to the Laplace-Bessel differential operator $\Delta_B = \sum\limits_{i=1}^{k} B_i + \sum\limits_{i=k+1}^{n}\frac{\partial^2}{\partial x_i^2}$, where $B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}$, $i=1,\ldots,k$. Furthermore, T^y generates the corresponding B-convolution

$$(f\otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)\ T^y g(x)(y')^\gamma dy$$

for which the Young inequality

$$\|f\otimes g\|_{L_{r,\gamma}} \le \|f\|_{L_{p,\gamma}}\,\|g\|_{L_{q,\gamma}}, \quad 1\le p,q,r\le\infty, \quad \frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$$

holds.

Lemma 1. *For all $x\in\mathbb{R}^n_{k,+}$ the following equality is valid*

$$\int_{E(x,t)} g(y)(y')^\gamma dy = C_{\gamma,k}^{-1}\int_{B((x,0),t)} g\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2}, z''\right) d\mu(z,\overline{z'}),$$

where $B((x,0),t) = \{(z,\overline{z'})\in\mathbb{R}^n\times(0,\infty)^k \ : \ |(x_1-\sqrt{z_1^2+\overline{z}_1^2},\ldots,x_k-\sqrt{z_k^2+\overline{z}_k^2}, x''-z'')| < t\}$, $d\mu(z,\overline{z'}) = (\overline{z'})^{\gamma-1}dz d\overline{z'}$, $d\overline{z'} = d\overline{z}_1\cdots d\overline{z}_k$, $(\overline{z'})^{\gamma-1} = (\overline{z}_1)^{\gamma_1-1}\cdots(\overline{z}_k)^{\gamma_k-1}$.

Lemma 2. *For all $x\in\mathbb{R}^n_{k,+}$ the following equality is valid*

$$\int_{E_t} T^y g(x)(y')^\gamma dy = \int_{E((x,0),t)} g\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2}, z''\right) d\mu(z,\overline{z'}),$$

where $E((x,0),t) = \{(z,\overline{z'})\in\mathbb{R}^n\times(0,\infty)^k \ : \ \left|(x-z,\overline{z'})\right| < t\}$.

The proof of Lemmas 1, 2 is straightforward via the following substitutions

$$z''=y'', z_i = y_i\cos\alpha_i, \quad \overline{z_i} = y_i\sin\alpha_i, \quad 0\le\alpha_i<\pi,\ i=1,\ldots,k,$$
$$y\in\mathbb{R}^n_{k,+},\ \overline{z'} = (\overline{z}_1,\ldots,\overline{z}_k),\ (z,\overline{z'})\in\mathbb{R}^n\times(0,\infty)^k,\ 1\le k\le n.$$

Lemma 3. *[10] Let $1 \le p \le \infty$. Then for all $y \in \mathbb{R}^n_{k,+}$*

$$\|T^y f(\cdot)\|_{L_{p,\gamma}} \le \|f\|_{L_{p,\gamma}}.$$

Lemma 4. *[8] Let $0 < \alpha < Q$. Then for $2|x| \le |y|$ the following inequality is valid*

$$\left|T^y |x|^{\alpha-Q} - |y|^{\alpha-Q}\right| \le 2^{Q-\alpha+1} |y|^{\alpha-Q-1} |x|.$$

We define $B-BMO$ space, $BMO_\gamma(\mathbb{R}^n_{k,+})$, as the set of locally integrable functions f with finite norm

$$\|f\|_{BMO_\gamma} = \sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E(0,t)|_\gamma^{-1} \int_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|(y')^\gamma dy < \infty,$$

or

$$\|f\|_{BMO_\gamma} = \inf_C \sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E(0,t)|_\gamma^{-1} \int_{E(0,t)} |T^y f(x) - C|(y')^\gamma dy < \infty,$$

where $f_{E(0,t)}(x) = |E(0,t)|_\gamma^{-1} \int_{E(0,t)} T^y f(x)(y')^\gamma dy$.

We define $BMO\,(X,\mu)$ space, as the set of locally integrable functions f with finite norm

$$\|f\|_* = \sup_{t>0,\ x\in X} \mu(B(x,t))^{-1} \int_{B(x,t)} |f(y) - f_{B(x,t)}| d\mu(y) < \infty,$$

where $f_{B(x,t)} = \mu(B(x,t))^{-1} \int_{B(x,t)} f(y) d\mu(y)$.

Theorem 1. *[17] Let $f \in BMO\,(X,\mu)$ and μ doubling measure. Then for any $r > 0$,*

$$\mu\left|\left\{y \in B(x,t) \ :\ |f(x) - f_{B(x,t)}| > r\right\}\right| \le C\mu(B(x,t)) e^{\frac{-cr}{\|f\|_*}},$$

where the constants C and c do not depend on f or r.

It is clear that $BMO(X,\mu) = BMO_p(X,\mu)$ if the John-Nirenberg inequality holds.

Theorem 2. *Let $f \in L^{loc}_{1,\gamma}(\mathbb{R}^n_{k,+})$. If*

$$\sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} \left(|E(0,t)|_\gamma^{-1} \int_{E(0,t)} |T^y f(x) - f_{E(0,t)}(x)|^p (y')^\gamma dy \right)^{1/p} = \|f\|_{BMO_{p,\gamma}} < \infty,$$

then for any $1 < p < \infty$,

$$\|f\|_{BMO_\gamma} \le \|f\|_{BMO_{p,\gamma}} \le A_p \|f\|_{BMO_\gamma},$$

where the constant A_p depends only on p.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y, d, ν). By this we mean a topological space $Y = \mathbb{R}^n \times (0,\infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x,\overline{x'}),2r)) \le C_1 \nu(E((x,\overline{x'}),r))$$

with a constant C_1 independent of $(x,\overline{x'})$ and $r > 0$. Here $E((x,\overline{x'}),r) = \{(y,\overline{y'}) \in Y : d(((x,\overline{x'}),(y,\overline{y'})) < r\}$, $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1} dy\, d\overline{y'}$, $(\overline{y'})^{\gamma-1} = (\overline{y_1})^{\gamma_1-1}\cdots(\overline{y_k})^{\gamma_k-1}$, $d((x,\overline{x'}),(y,\overline{y'})) = |(x,\overline{x'})-(y,\overline{y'})| \equiv (|x-y|^2 + (\overline{x'}-\overline{y'})^2)^{\frac{1}{2}}$.

Indeed, Lemma 2

$$\int_{E_r} T^y \left| f\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2}, z''\right)\right| (y')^\gamma\, dy$$
$$= \int\limits_{E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),r\right)} \left|\overline{f}(y,\overline{y'})\right|\, d\nu(y,\overline{y'})$$

where $\overline{f}(x,\overline{x'}) = f\left(\sqrt{x_1^2+\overline{x}_1^2},\dots,\sqrt{x_k^2+\overline{x}_k^2}, x''\right)$ and

$$|E_r|_\gamma = \nu E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2}, z'', 0\right), r\right).$$

Using Lemma 2 we have

$$\sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E(0,t)|_\gamma^{-1} \int_{E(0,t)} |T^y f(x) - C|(y')^\gamma dy$$
$$= \sup_{t>0,\ x\in Y} (\nu E((x,0),t))^{-1} \int\limits_{E((x,0),t)} \left|\overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2},\dots,\sqrt{z_k^2+\overline{z}_k^2}, z'', 0\right) - C\right|\, d\nu(y,\overline{y'}).$$

The Theorem 2 is proved. □

We define the B–Riesz potential $I_\gamma^\alpha f$ (see [5, 6, 21]) as

$$I_\gamma^\alpha f(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y |x|^{\alpha-Q}\, (y')^\gamma dy,$$

where $0 < \alpha < Q$, and the modified B-Riesz potential $\widetilde{I}_\gamma^\alpha f$ as

$$\widetilde{I}_\gamma^\alpha f(x) = \int_{\mathbb{R}^n_{k,+}} \left(T^y|x|^{\alpha-Q} - |y|^{\alpha-Q}\chi_{E^*(0,1)}(y)\right) f(y)(y')^\gamma dy,$$

where $E^*(0,1) = \mathbb{R}^n_{k,+} \backslash E(0,1)$, and B-maximal operator $M_\gamma f$ as

$$M_\gamma f(x) = \sup_{r>0} |E(0,r)|_\gamma^{-1} \int\limits_{E(0,r)} T^y |f(x)| (y')^\gamma dy.$$

In the following two theorems the boundedness of the B-Riesz potential $I_\gamma^\alpha f$, B-maximal operator $M_\gamma f$ and modified B-Riesz potential $\widetilde{I}_\gamma^\alpha f$ are given.

Theorem 3. *[5],[7] Let* $0 < \alpha < Q$ *and* $1 \le p < \frac{Q}{\alpha}$. *Then*

1) If $1 < p < \frac{Q}{\alpha}$, *then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ *is necessary and sufficient for the boundedness* I_γ^α *from* $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ *to* $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p = 1$, *then condition* $1 - \frac{1}{q} = \frac{\alpha}{Q}$ *is necessary and sufficient for the boundedness* I_γ^α *from* $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ *to* $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

3) If $1 < p = \frac{Q}{\alpha}$, *then the operator* $\widetilde{I}_\gamma^\alpha$ *is bounded from* $L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *to* $BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$.

Moreover, for $f \in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *the integral* $I_\gamma^\alpha f$ *exists almost everywhere, then* $I_\gamma^\alpha \in BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$ *and the following inequality is valid*

$$\|I_\gamma^\alpha f\|_{BMO_\gamma} \le C\|f\|_{L_{p,\gamma}},$$

where $C > 0$ *does not depend on* f.

Theorem 4. *[5],[7]*

1. If $f \in L_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, *then* $M_\gamma f \in WL_{1,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *and*

$$\|M_\gamma f\|_{WL_{1,\gamma}} \le C_{1,\gamma}\|f\|_{L_{1,\gamma}},$$

where $C_{1,\gamma}$ *depends only on* γ,k *and* n.

2. If $f \in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $1 < p < \infty$, *then* $M_\gamma f \in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ *and*

$$\|M_\gamma f\|_{L_{p,\gamma}} \le C_{p,\gamma}\|f\|_{L_{p,\gamma}},$$

where $C_{p,\gamma}$ *depends only on* p,γ,k *and* n.

3. Main Result

Let $f \in L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and b is a locally integrable function on $\mathbb{R}^n_{k,+}$. Consider the commutators

$$[b, I_{\alpha,\gamma}]f(x) = \int_{\mathbb{R}^n_{k,+}} [b(x) - b(y)]f(y)T^y|x|^{\alpha-Q}\,(y')^\gamma dy, \quad 0 < \alpha < Q$$

and

$$[b, \widetilde{I}_{\alpha,\gamma}]f(x) = \int_{\mathbb{R}^n_{k,+}} [b(x) - b(y)]\left(T^y|x|^{\alpha-Q} - |y|^{\alpha-Q}\chi_{E^*(0,1)}(y)\right) f(y)(y')^\gamma dy$$

formed by the $B-$Riesz potential $I_{\alpha,\gamma}$ and modified $B-$Riesz potential $\widetilde{I}_{\alpha,\gamma}$ respectively, where $E^*(0,1) = \mathbb{R}^n_{k,+}\backslash E(0,1)$.

In this section we show that the condition $b \in BMO_\gamma$ is the necessary and sufficient for the boundedness of $[b, I_{\alpha,\gamma}]$ from the spaces $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p < \frac{Q}{\alpha}$, and from the spaces $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ for $p = 1$. We prove that the commutator

$$[a, \widetilde{I}_{\alpha,\gamma}]f(x) = \int_{\mathbb{R}^n_{k,+}} [b(x) - b(y)]\left(T^y|x|^{\alpha-Q} - |y|^{\alpha-Q}\chi_{E^*(0,1)}(y)\right) f(y)(y')^\gamma dy$$

is bounded from $L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ to $BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$.

Theorem 5. *Let $0<\alpha<Q$ and $1\le p<\frac{Q}{\alpha}$. Then*

1) If $1<p\le\frac{Q}{\alpha}$ and $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{Q}$, then the commutator $[b,I_{\alpha,\gamma}]$ is bounded from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$ if and only if $b\in BMO_\gamma(\mathbb{R}^n_{k,+})$.

2) If $p=1$ and $1-\frac{1}{q}=\frac{\alpha}{Q}$, then the commutator $[b,I_{\alpha,\gamma}]$ is bounded from $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$ if and only if $b\in BMO_\gamma(\mathbb{R}^n_{k,+})$.

3) If $1<p=\frac{Q}{\alpha}$, then the commutator $[b,\widetilde{I}_{\alpha,\gamma}]$ is bounded from $L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ to $BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$.

Moreover, for $f\in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ if the integral $[b,I_{\alpha,\gamma}]f$ exists almost everywhere, then $[b,I_{\alpha,\gamma}]\in BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$, and the following inequality holds

$$\|[b,I_{\alpha,\gamma}]f\|_{BMO_\gamma}\le C\|b\|_{BMO_\gamma}\|f\|_{L_{p,\gamma}},$$

where $C>0$ does not depend on f.

Proof. 1) *Sufficiency* : Let $f\in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Then we write

$$[b,I_{\alpha,\gamma}]f(x)=\left(\int_{E(0,t)}+\int_{\mathbb{R}^n_{k,+}\setminus E(0,t)}\right)T^y([b-b(x)]f(x))|y|^{\alpha-Q}(y')^\gamma dy\equiv F_1(x,t)+F_2(x,t). \quad (1)$$

Firstly, we estimate $F_1(x,t)$. By the Hölder's inequality we have

$$|F_1(x,t)|\le\int_{E(0,t)}T^y|[b-b(x)]f(x)||y|^{\alpha-Q}(y')^\gamma dy$$

$$\le\sum_{j=-\infty}^{-1}\left(2^jt\right)^{\alpha-Q}\int_{E(0,2^{j+1}t)\setminus E(0,2^jt)}T^y|[b-b(x)]f(x)|(y')^\gamma dy$$

$$\le\sum_{j=-\infty}^{-1}\left(2^jt\right)^{\alpha-Q}\left(\int_{E(0,2^{j+1}t)\setminus E(0,2^jt)}|T^yb(x)-b|^{r'}(y')^\gamma dy\right)^{1/r'}$$

$$\times\left(\int_{E(0,2^{j+1}t)\setminus E(0,2^jt)}T^y\left|f(x)\right|^r(y')^\gamma dy\right)^{1/r}$$

$$\le\sum_{j=-\infty}^{-1}\left(2^jt\right)^{Q}\left(2^jt\right)^{\alpha-Q}\|b\|_{BMO_\gamma}\left(M_\gamma(|f|^r)(x)\right)^{1/r}.$$

Hence

$$|F_1(x,t)|\le C_1t^\alpha\|b\|_{BMO_\gamma}\left(M_\gamma(|f|^r)(x)\right)^{1/r}. \quad (2)$$

Secondly, we estimate $F_2(x,t)$. By the Hölder's inequality and Lemma 1 we have

$$|F_2(x,t)| \le \int_{\mathbb{R}^n_{k,+}\setminus E(0,t)} T^y(|b-b(x)||f(x)|)|y|^{\alpha-Q}(y')^\gamma dy$$

$$\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-Q} \int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y(|b-b(x)||f(x)|)(y')^\gamma dy$$

$$\le \sum_{j=0}^{\infty} \left(2^j t\right)^{\alpha-Q} \left(\int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} |T^y b(x)-b|^{p'}(y')^\gamma dy\right)^{1/p'}$$

$$\times \left(\int_{E(0,2^{j+1}t)\setminus E(0,2^j t)} T^y \left|f(x)\right|^p (y')^\gamma dy\right)^{1/p}$$

$$\le 2^{\frac{Q}{p'}} t^{\alpha-\frac{Q}{p}} \|b\|_{BMO_\gamma} \|f\|_{L_{p,\gamma}} \sum_{j=0}^{\infty} 2^{j\left(\alpha-\frac{Q}{p}\right)}.$$

Hence

$$|F_2(x,t)| \le C_2 t^{\alpha-\frac{Q}{p}} \|b\|_{BMO_\gamma} \|f\|_{L_{p,\gamma}}. \tag{3}$$

Thus, from (2) and (3) we have

$$|[b,I_{\alpha,\gamma}]f(x)| \le C_1 t^\alpha \|b\|_{BMO_\gamma} (M_\gamma(|f|^r)(x))^{1/r} + C_2 t^{\alpha-\frac{Q}{p}} \|b\|_{BMO_\gamma} \|f\|_{L_{p,\gamma}}.$$

Minimizing with respect to t, at $t = \left[(M_\gamma(|f|^r)(x))^{-1/r} \|f\|_{L_{p,\gamma}}\right]^{p/Q}$ we get

$$|[b,I_{\alpha,\gamma}]f(x)| \le C_3 \|b\|_{BMO_\gamma} \left((M_\gamma(|f|^r)(x))^{1/r}\right)^{p/q} \|f\|_{L_{p,\gamma}}^{1-p/q},$$

where $C_3 = C_2^{1-\frac{p}{q}} \left(\frac{Q}{\alpha q C_1}\right)^{-\frac{p}{q}} \left(\frac{Q}{\alpha q}+1\right)$.

Hence, by Theorem 4, we have

$$\int_{\mathbb{R}^n_{k,+}} |[b,I_{\alpha,\gamma}]f(y)|^q (y')^\gamma dy \le C_3 \|b\|^q_{BMO_\gamma} \|f\|^{q-p}_{L_{p,\gamma}} \int_{\mathbb{R}^n_{k,+}} (M_\gamma(|f|^r)(y))^{p/r} (y')^\gamma dy$$
$$\le C_3 C_{p,\gamma} \|b\|^q_{BMO_\gamma} \|f\|^{q-p}_{L_{p,\gamma}} \||f|^r\|^p_{L_{p/r,\gamma}} = C_4 \|b\|^q_{BMO_\gamma} \|f\|^{q-p}_{L_{p,\gamma}} \|f\|^p_{L_{p,\gamma}}$$
$$= C_4 \|b\|^q_{BMO_\gamma} \|f\|^q_{L_{p,\gamma}},$$

where $C_4 = C_3 \cdot C_{p,\gamma}$.

Therefore $[b,I_{\alpha,\gamma}]f \in L_{q,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|[b,I_{\alpha,\gamma}]f\|_{L_{q,\gamma}} \le C_4 \|b\|_{BMO_\gamma} \|f\|_{L_{p,\gamma}}.$$

Necessity: Let $[b,I_{\alpha,\gamma}]$ be bounded from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$, $1 \le p < \frac{Q}{\alpha}$.

Suppose $s(x) = \chi_{E(0,t)}(x) sgn(b(x) - b_{E(0,t)}(0))$, then

$$|E(0,t)|_\gamma \left| b(x) - \frac{1}{|E(0,t)|_\gamma} \int_{E(0,t)} b(y)(y')^\gamma dy \right| \chi_{E(0,t)}(x)$$

$$= s(x) \left(|E(0,t)|_\gamma b(x) - \int_{E(0,t)} b(y)(y')^\gamma dy \right)$$

$$= s(x)b(x) \int_{\mathbb{R}^n_{k,+}} \chi_{E(0,t)}(y)(y')^\gamma dy - \int_{\mathbb{R}^n_{k,+}} \chi_{E(0,t)} b(y)(y')^\gamma dy$$

$$= s(x)b(x) \int_{\mathbb{R}^n_{k,+}} \chi_{E(0,t)}(y) T^y|x|^{\alpha-Q} \left(T^y|x|^{\alpha-Q}\right)^{-1} (y')^\gamma dy$$

$$- \int_{\mathbb{R}^n_{k,+}} \chi_{E(0,t)} b(y) T^y|x|^{\alpha-Q} \left(T^y|x|^{\alpha-Q}\right)^{-1} (y')^\gamma dy$$

$$\leq s(x) \int_{\mathbb{R}^n_{k,+}} [b(x) - b(y)] \chi_{E(0,t)}(y) T^y|x|^{\alpha-Q} |x+y|^{Q-\alpha} (y')^\gamma dy$$

$$= \sum_{|\mu|+|\nu|=Q-\alpha} x^\mu s(x) [b, I_{\alpha,\gamma}](y^\nu \chi_{E(0,t)})(x).$$

Thus

$$|E(0,t)|_\gamma \int_{E(0,t)} |T^x b(y) - b_{E(0,t)}(x)| (y')^\gamma dy$$

$$\leq C_7 \sum_{|\mu|+|\nu|=Q-\alpha} \int_{E(0,t)} |x|^\mu s(x) |T^x [b, I_{\alpha,\gamma}](y^\nu \chi_{E(0,t)})(y)| (y')^\gamma dy$$

$$\leq C_8 \sum_{|\mu|+|\nu|=Q-\alpha} |E(0,t)|_\gamma^{\frac{|\mu|}{Q}} \|s\|_{L_{q',\gamma}} \|[b, I_{\alpha,\gamma}](y^\nu \chi_{E(0,t)})\|_{L_{q,\gamma}}$$

$$\leq C_9 \sum_{|\mu|+|\nu|=Q-\alpha} |E(0,t)|_\gamma^{\frac{|\mu|}{Q} + \frac{1}{q'} + \frac{|\nu|}{Q} + \frac{1}{q} + \frac{\alpha}{Q}} \leq C_9 |E(0,t)|_\gamma^2.$$

Hence we get

$$|E(0,t)|_\gamma^{-1} \int_{E(0,t)} |T^x b(y) - b_{E(0,t)}(x)| (y')^\gamma dy \leq C_9.$$

This shows that $b \in BMO_\gamma(\mathbb{R}^n_{k,+})$.
2) *Sufficiency* : Let $f \in L_{1,\gamma}(\mathbb{R}^n_{k,+})$. Then we have

$$\left| \left\{ x \in \mathbb{R}^n_{k,+} \ : \ |[b, I_{\alpha,\gamma}] f(x)| > 2\beta \right\} \right|_\gamma \leq | \left\{ x \in \mathbb{R}^n_{k,+} \ : \ |F_1(x,t)| > \beta \right\} |_\gamma$$

$$+|\left\{x\in\mathbb{R}^n_{k,+}\ :\ |F_2(x,t)|>\beta\right\}|_\gamma,$$

where $F_1(x,t)$ and $F_2(x,t)$ are given by the equality (1) .

Taking into account inequality (2) and Theorem 4 we get

$$|\left\{x\in\mathbb{R}^n_{k,+}\ :\ |F_1(x,t)|>\beta\right\}|_\gamma$$
$$\leq\left|\left\{x\in\mathbb{R}^n_{k,+}\ :\ (M_\gamma(|f|^r)(x))^{1/r}>\frac{\beta}{C_1t^\alpha\,\|b\|_{BMO_\gamma}}\right\}\right|_\gamma$$
$$\leq\frac{C_5t^\alpha}{\beta}\cdot\|b\|_{BMO_\gamma}\,\|f\|_{L_{1,\gamma}}\,,$$

where $C_5=C_1\cdot C_{1,\gamma}$. If $C_2t^{-\frac{Q}{q}}\,\|b\|_{BMO_\gamma}\,\|f\|_{L_{1,\gamma}}=\beta$, then $|F_2\,(x,t)|\leq\beta$ and therefore $|\left\{x\in\mathbb{R}^n_{k,+}\ :\ |F_2(x,t)|>\beta\right\}|_\gamma=0$.

Hence we get the following inequality

$$|\left\{x\in\mathbb{R}^n_{k,+}\ :\ |[b,I_{\alpha,\gamma}]f(x)|>2\beta\right\}|_\gamma\leq\frac{C_5}{\beta}t^\alpha\,\|b\|_{BMO_\gamma}\,\|f\|_{L_{1,\gamma}}=C_6\left(\frac{\|b\|_{BMO_\gamma}\,\|f\|_{L_{1,\gamma}}}{\beta}\right)^q,$$

where $C_6=C_5\cdot C_2^{q-1}$.

Necessity: The proof of necessity part is similar to the proof in the first part of this theorem.

3) Let $f\in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $1<p=\frac{Q}{\alpha}$. For given $t>0$ we denote

$$f_1(z)=f(z)\chi_{E(0,2t)}(z),\quad f_2(z)=f(z)-f_1(z),$$

where $\chi_{E(0,2t)}$ is the characteristic function of the set $E(0,2t)$. Then

$$[b,\widetilde{I}_{\alpha,\gamma}]f(z)=[b,\widetilde{I}_{\alpha,\gamma}]f_1(z)+[b,\widetilde{I}_{\alpha,\gamma}]f_2(z)=F_1(z)+F_2(z),$$

where

$$F_1(z)=\int\limits_{E(0,2t)}[b(z)-b(y)]f(y)\left(T^y|z|^{\alpha-Q}-|y|^{\alpha-Q}\chi_{E^*(0,1)}(y)\right)(y')^\gamma dy,$$

$$F_2(z)=\int\limits_{\mathbb{R}^n_{k,+}\setminus E(0,2t)}[b(z)-b(y)]f(y)\left(T^y|z|^{\alpha-Q}-|y|^{\alpha-Q}\chi_{E^*(0,1)}(y)\right)(y')^\gamma dy.$$

Note that the function f_1 has a compact (bounded) support and thus

$$a_1(z)=-\int\limits_{E(0,2t)\setminus E(0,\min\{1,2t\})}[b(z)-b(y)]f(y)|y|^{\alpha-Q}(y')^\gamma dy$$

is finite.

Note that

$$F_1(z)-a_1(z)=\int\limits_{E(0,2t)}[b(z)-b(y)]f(y)T^y|z|^{\alpha-Q}(y')^\gamma dy$$

$$-\int\limits_{E(0,2t)\setminus E(0,\min\{1,2t\})}[b(z)-b(y)]|y|^{\alpha-Q}f(y)(y')^\gamma dy$$

$$+\int\limits_{E(0,2t)\setminus E(0,\min\{1,2t\})}[b(z)-b(y)]|y|^{\alpha-Q}f(y)(y')^\gamma dy$$

$$=\int\limits_{\mathbb{R}^n_{k,+}}[b(z)-b(y)]f_1(y)T^y|z|^{\alpha-Q}(y')^\gamma dy.$$

Therefore

$$|F_1(z)-a_1|\le\int\limits_{\mathbb{R}^n_{k,+}}|y|^{\alpha-Q}T^y\left|[b-b(z)]f_1(z)\right|(y')^\gamma dy$$

$$=\int\limits_{\{y\in\mathbb{R}^n_{k,+}:T^y|z|<2t\}}|y|^{\alpha-Q}T^y\left|[b-b(z)]f(z)\right|(y')^\gamma dy.$$

Further, for $z\in E(0,t)$, $T^y|z|<2t$ we have

$$|y|\le|z|+|z-y|\le|z|+T^y|z|<3t.$$

Consequently

$$|F_1(z)-a_1|\le\int\limits_{E(0,3t)}|y|^{\alpha-Q}T^y\left|[b-b(z)]f(z)\right|(y')^\gamma dy,$$

$$\le\sum_{j=-\infty}^{-1}\left(2^jt\right)^{\alpha-Q}\int_{E(0,2^{j+1}t)\setminus E(0,2^jt)}T^y|[b-b(z)]f(z)|(y')^\gamma dy$$

$$\le\sum_{j=-\infty}^{-1}\left(2^jt\right)^{\alpha-Q}\left(\int_{E(0,2^{j+1}t)\setminus E(0,2^jt)}|T^yb(z)-b|^{r'}(y')^\gamma dy\right)^{1/r'}$$

$$\times\left(\int_{E(0,2^{j+1}t)\setminus E(0,2^jt)}T^y\left|f(z)\right|^r(y')^\gamma dy\right)^{1/r}$$

$$\le Ct^\alpha\|b\|_{BMO_\gamma}\left(M_\gamma(|f|^r)(z)\right)^{1/r}. \tag{4}$$

if $z\in E(0,t)$.

By the Theorem 4 and the inequality (4) for $\alpha p = Q$

$$|E(0,t)|_\gamma^{-1} \int\limits_{E(0,t)} |T^x F_1(z) - a_1| \, (z')^\gamma dz$$

$$\leq \frac{2^{Q-\alpha} 3^\alpha}{2^\alpha - 1} t^{\alpha - Q} \, \|b\|_{BMO_\gamma} \int\limits_{E(0,t)} T^x((M_\gamma(|f|^r)(z))^{1/r})(z')^\gamma dz$$

$$\leq \frac{2^{Q-\alpha} 3^\alpha}{2^\alpha - 1} t^{\alpha - Q} t^{Q/p'} \, \|b\|_{BMO_\gamma} \left(\int\limits_{E(0,t)} T^x((M_\gamma(|f|^r)(z))^{p/r} \, (z')^\gamma dz \right)^{1/p}$$

$$\leq \frac{2^{Q-\alpha} 3^\alpha}{2^\alpha - 1} t^{\alpha - Q/p} \, \|b\|_{BMO_\gamma} \, \||f|^r\|_{L_{\frac{p}{r},\gamma}}^{\frac{1}{r}} \leq C_{11} \, \|b\|_{BMO_\gamma} \, \|f\|_{L_{p,\gamma}}. \tag{5}$$

Denote

$$a_2(z) = \int\limits_{E(0,\max\{1,2t\}) \setminus E(0,2t)} [b(z) - b(y)] f(y) |y|^{\alpha - Q} (y')^\gamma dy$$

and estimate $|F_2(z) - a_2(z)|$ for $z \in E(0,t)$.

$$|F_2(z) - a_2(z)| \leq \int\limits_{\mathbb{R}^n_{k,+} \setminus E(0,2t)} |b(z) - b(y)||f(y)| \left| T^y |z|^{\alpha - Q} - |y|^{\alpha - Q} \right| (y')^\gamma dy.$$

Applying Lemma 4 and Hölder's inequality we have

$$|F_2(z) - a_2(z)| \leq 2^{Q-\alpha+1} |z| \int\limits_{\mathbb{R}^n_{k,+} \setminus E(0,2t)} |b(z) - b(y)||f(y)||y|^{\alpha - Q - 1} (y')^\gamma dy$$

$$\leq 2^{Q-\alpha+1} t^{a-Q-1} |z| \sum_{j=1}^{\infty} (2^{j+1})^{a-Q-1} \left(\int_{2^j t \leq |y| < 2^{j+1} t} |b(z) - b(y)|^{p'} (y')^\gamma dy \right)^{1/p'}$$

$$\times \left(\int_{2^j t \leq |y| < 2^{j+1} t} |f(y)|^p (y')^\gamma dy \right)^{1/p} \leq C \|b\|_{BMO_\gamma} \|f\|_{L_{p,\gamma}}.$$

Thus for $\alpha p = Q$ and for all $x \in \mathbb{R}^n_{k,+}$, $z \in E(0,t)$ we obtain

$$|T^x F_2(z) - a_2| \leq T^x \, |F_2(z) - a_2| \leq C_{13} \|b\|_{BMO_\gamma} \|f\|_{L_{p,\gamma}}. \tag{6}$$

Denote

$$a_f(z) = a_1(z) + a_2(z) = \int\limits_{E(0,\max\{1,2t\})} [b(z) - b(y)] |y|^{\alpha - Q} f(y) (y')^\gamma dy.$$

Finally, from (5) and (6) we have

$$\sup_{x,t}|E(0,t)|_\gamma^{-1}\int\limits_{E(0,t)}\left|T^x[b,\widetilde{I}_{\alpha,\gamma}]f(z)-a_f\right|(z')^\gamma dz\le (C_{11}+C_{13})\|b\|_{BMO_\gamma}\|f\|_{L_{p,\gamma}}.$$

Thus

$$\left\|[b,\widetilde{I}_{\alpha,\gamma}]f\right\|_{BMO_\gamma}\le 2\sup_{x,t}|E(0,t)|_\gamma^{-1}\int\limits_{E(0,t)}\left|T^z[b,\widetilde{I}_{\alpha,\gamma}]f(x)-a_f\right|(z')^\gamma dz$$

$$\le C\|b\|_{BMO_\gamma}\|f\|_{L_{p,\gamma}}.$$

Therefore the proof of the theorem is completed. □

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Gulnara A. Abasova
Azerbaijan State Economic University, Baku, Azerbaijan
E-mail: g.abasov2015@mail.ru

Lale R. Aliyeva
Institute of Mathematics And Mechanics, Baku, Azerbaijan
E-mail: laleliyeva@box.az

Javanshir J. Hasanov
Azerbaijan State Oil and Industry University, Baku, Azerbaijan
Institute of Mathematics And Mechanics, Baku, Azerbaijan
E-mail: hasanovjavanshir@yahoo.com.tr

Esmira S. Shirinova
Agjabady branch of Azerbaijan State Pedagogical University, Agjabady, Azerbaijan
E-mail: esmirashirinova@gmail.com

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