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Formula for Second Regularized Trace of the Sturm-Liouville Equation with Spectral Parameter Dependent Boundary Condition

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Abstract. In this paper we consider the problem generated by Sturm-Liouville operator equation with spectral parameter-λ dependent boundary condition. The asymptotic formula for this problem is obtained by B.A.Aliyev and the formula for the first regularized trace is calculated by N.M.Aslanova. The main goal of this paper is to derive a formula for the second regularized trace of that operator. In section 2 was given the problem statement. In section 3 were noted the auxiliary facts which we have used. Finally in section 4 was derived formula for the second regularized trace of this problem. For this at first was proved that the series formed the trace formula is absolutely convergent.

Key Words and Phrases: Hilbert space, regularized trace, trace class operator

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1. Introduction

A formula for the regularized trace was first obtained by I.M.Gelfand and B.M.Levitan [1] for Sturm-Liouville differential operators. After this work many papers were devoted to study of regularized traces various differential operators. A common tool for the study of partial differential equations, integro- differential equations and infinite systems of ordinary differential-operators is the theory of operator-differential equations with unbounded operator-coefficient. The one of the main tasks in this theory is to determine the behavior of the eigenvalues and calculation of traces of the associated differential operators. A formula for the first regularized trace of the Sturm-Liouville operator with unbounded operator-coefficient was first calculated by F.Q. Maksudov, M.Bayramoglu and A.A Adigozelov [2].Higher order regularized traces investigated for example, in [3-7]. The trace formulas for differential operators with unbounded operator coefficient are obtained in the works [2],[5-15].

In the paper [16],we considered the following spectral problem

$$-y''(t) + Ay(t) + q(t)y(t) = \lambda y(t)$$

$$y'(0) = 0, ay(1) + by'(1) = \lambda(cy(1) - dy'(1))$$

in the Hilbert space $L_2(H, (0, 1))$. The self-adjointness and symmetry of the operator associated with this problem were established .The asymptotic formula for the eigenvalues of this problem was derived. Trace formula for the special case of this problem was calculated in [9] .

In the present paper we consider an operator which different from operators in [6,7] by boundary condition only. The main goal of this paper is to calculate the second regularized trace of that operator. The asymptotic formula for this problem is obtained in [17] and the formula for the first regularized trace is calculated in [18].

2. Problem statement

Assume $L_2 = L_2(H, (0, 1)) \bigoplus H$ (H is a separable Hilbert space). Let's denote the scalar product by $(\cdot, \cdot)$ and the norm by $\| \cdot \|$ in the space H .The space of the vector functions $y(t)$ for which $\int_0^1 \|y(t)\|^2 dt < \infty$ is denoted by $L_2(H, (0, 1))$. The class of compact operators in separable Hilbert space H , whose singular values form a convergent series is denoted by $\sigma_1(H)$ and is called a trace class (see [19, p.88]).

Consider the following boundary value problem

$$l\,[y] \equiv -y''(t) + Ay(t) + q(t)y(t) = \lambda y(t) \tag{2.1}$$

$$y(0) = 0 \tag{2.2}$$

$$y(1) = -\lambda y'(1) \tag{2.3}$$

in the Hilbert space $L_2(H, (0, 1))$,where $A = A^* > E$ (E is an identity operator in H) with a compact inverse. For each t $q(t)$ is a selfadjoint operator-valued function in the space H . Assume operator-valued function $q(t)$ is weakly measurable. Suppose that the following conditions hold:

1) There exist fourth order weak derivatives on $[0, 1]$ denoted by $q^{(k)}(t)$ which is from $\sigma_1(H)$ .For each $t \in [0, 1]$ $\left\|q^{(k)}(t)\right\|_{\sigma_1(H)}$ $(k = \overline{0,4})$ are bounded. Also $Aq^{(k)}(t) \in \sigma_1(H)$ and $\left\|Aq^{(k)}(t)\right\|_{\sigma_1(H)} \leq const$ for $k = 0,\ 1,\ 2.$

2) Let $q(1) = q'(0) = q'(1) = 0;$

3) $\int_0^1 (q(t)f, f)\, dt = 0$ is true for each $f \in H.$

In case $q(t) \equiv 0$ one can associate with problem (2.1)-(2.3) in space L_2 the operator L_0 defined as

$$D(L_0) = \{Y : Y = \{y(t),\, y_1\}/ - y''(t) + Ay(t) \in L_2(H, (0,1)),$$

$$y(0) = 0,\ y_1 = -y'(1)\}, \tag{2.4}$$

$$L_0 Y = \{-y''(t) + Ay(t),\ y(1)\}. \tag{2.5}$$

Denote the operator corresponding to the case $q(t) \not\equiv 0$ by $L = L_0 + Q$. Here Q : $Q\{y\,(t)\,,\ -y'\,(1)\} = \{q\,(t)\,y\,(t)\,,\ 0\}$. The scalar product in L_2 defined as

$$(Y,\ Z)_{L_2} = \int_0^1 (y(t),\ z(t))dt + (y_1,\ z_1) \tag{2.6}$$

where $Y = \{y(t),\ y_1\}$, $Z = \{z(t),\ z_1\}$, $y(t),\ z(t) \in L_2\,(H,\ (0,\ 1))\,,\ y_1,\ z_1 \in H$.

Obviousles, the operators L_0 and L have a discrete spectrum.Let the eigenvalues of these operators be $\mu_1 \le \mu_2 \le \ldots$ and $\lambda_1 \le \lambda_2 \le \ldots$, respectively.

3. Auxiliary facts

Let the eigenvalues and eigen-vectors of operator A be $\gamma_1 \le \gamma_2 \le \ldots$ and $\varphi_1, \varphi_2, \ldots$, respectively.

Denote the resolvent of the operator L_0^2 by R_λ^0. Obviousles , $R_\lambda^0 \in \sigma_1\,(H)$ (It follows from the asymptotics of μ_k).Suppose that the operator A_0 is a self-adjoint and positive discrete operator in separable Hilbert space H. $\{\lambda_n\}$ be its eigen-values arranged in ascending order. Assume B is a perturbation operator and the eigenvalues of $A_0 + B$ is denoted by $\{\mu_n\}$. Also, suppose that $A_0^{-1} \in \sigma_1\,(H)$. For operators A_0 and B the following theorem is true (see [20 ,p.133]) .

Theorem 3.1. *Let $D\,(A_0) \subset D\,(B)$. Assume that there exist a number $\delta \in [0;\,1)$ such that $BA_0^{-\delta}$ is continuable to bounded operator and some number $\omega \in [0;\,1)\,,\ \omega + \delta < 1$, such that $A_0^{-(1-\delta-\omega)}$ is a trace class operator. Then there exist subsequence of natural numbers $\{n_m\}_{m=1}^{\infty}$ and sequence of closed contours $\Gamma_m \in \mathbb{C}$ such that for $N \ge \frac{\delta}{\omega}$*

$$\lim_{m\to\infty}\left(\sum_{j=1}^{n_m}(\mu_j - \lambda_j) + \frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k}tr\,(BR_0\,(\lambda))^k\,d\lambda\right) = 0$$

where $R_0\,(\lambda)$ *is a resolvent of* A_0.

Note that the conditions of the above theorem are satisfied for L_0^2 and L^2. That is, if we take $A_0 = L_0^2$, $B = L_0 Q + Q L_0 + Q^2$, $\left(L^2 = A_0 + B\right)$ and $\delta = \frac{1}{2}$, provided $L_0 Q L_0^{-1}$ and BA_0^{-1} are bounded . For $\omega \in [0;\,1)\,,\ \omega < \frac{1}{2} - \frac{2+\alpha}{4\alpha}$, we will have that $A_0^{-(1-\delta-\omega)} = L_0^{-2(1-\delta-\omega)}$ is a trace class operator. Thus by statement of Theorem 3.1 for $N > \frac{1}{2\omega}$,we have (see [6,p.637])

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2 - \mu_n^2\right) + \frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=1}^{N}\frac{(-1)^{k-1}}{k}\times\right.$$

$$\left.\times tr\left[\left(L_0 Q + Q L_0 + Q^2\right) R_0\,(\lambda)\right]^k d\lambda\right) = 0. \tag{3.1}$$

4. Second regularized trace of L

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\int_0^1 trq^2(t)\,dt\right)+\right.$$

$$\left.+\frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]^k d\lambda\right) \qquad (4.1)$$

is called the second regularized trace of the operator L . Denote it by $\sum_{n=1}^{\infty}\left(\lambda_n^{(2)}-\mu_n^{(2)}\right)$. In this paper we will show later that, its value is independent of the choice of the subsequence $\{n_m\}$.

By virtue of lemma 3 in [20, p.132] the number of eigenvalues of L_0^2 and L^2 inside the contour Γ_m is the same for great m values and equals to n_m.

From (3.1),we get

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\int_0^1 trq^2(t)\,dt\right)+\right.$$

$$\left.+\frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]^k d\lambda\right)=$$

$$=\lim_{m\to\infty}\left(-\frac{1}{2\pi i}\int_{\Gamma_m}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]d\lambda-\sum_{n=1}^{n_m}\int_0^1 trq^2(t)\,dt\right). \qquad (4.2)$$

Let ψ_1, ψ_2, ... be the eigenvectors of the operator L_0 . The operator $L_0QL_0^{-1}$ is bounded and $\left(L_0Q+QL_0+Q^2\right)R_\lambda^0$ is trace class operator . Eigenvectors of the operator L_0 form a basis in the space L_2. From (4.2) follows that (see [6,pp.638])

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_m}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]\,d\lambda=$$

$$-\frac{1}{2\pi i}\int\limits_{\Gamma_m}\sum_{n=1}^{\infty}\left((L_0Q+QL_0+Q^2)R_0(\lambda)\psi_n\psi_n\right)_{L_2}d\lambda=$$

$$=-\frac{1}{2\pi i}\sum_{n=1}^{\infty}\int\limits_{\Gamma_m}\frac{1}{\mu_n^2-\lambda}\left((L_0Q+QL_0+Q^2)R_0(\lambda)\psi_n\psi_n\right)_{L_2}d\lambda=$$

$$\sum_{n=1}^{n_m}\left([L_0Q+QL_0+Q^2]\psi_n\psi_n\right)_{L_2}. \qquad (4.3)$$

In [18 , p.29] the orthonormal eigenvectors of the operator L is obtained and are of the form:

$$\psi_n = \sqrt{\frac{4x_{k,n}}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3 cos^2\, x_{k,n}}} \left\{ sin\left(x_{k,n} t\right) \varphi_k, x_{k,n} \cos x_{k,n}\, \varphi_k \right\}. \tag{4.4}$$

Here $x_{k,\,n}$ are the roots of the following equation

$$\sin z + \left(z^2 + \gamma_k\right) z \cos z = 0, z = \sqrt{\lambda - \gamma_k}. \tag{4.5}$$

Note that in [17 ,p.12] this problem is studied and the following asymptotic formula for the eigenvalues of L_0 is obtained:

$$\lambda_{k,n} \sim \gamma_k + \pi^2 \left(n - \frac{1}{2}\right)^2, n = \overline{1, \infty}.$$

Thus we get

$$\sum_{n=1}^{n_m} \left([L_0 Q + Q L_0 + Q^2] \psi_n, \psi_n \right)_{L_2} =$$

$$= 2 \sum_{n=1}^{n_m} \frac{4x_{k,n}(x_{k,n}^2 + \gamma_k)}{2x_{k,n} - \sin 2x_{k,n} + 4x_{k,n}^3 \cos^2 x_{k,n}} \int\limits_0^1 \sin^2(x_{k,n} t)(q(t)\varphi_k, \varphi_k) dt +$$

$$+ \sum_{n=1}^{n_m} \frac{4x_{k,n}}{2x_{k,n} - \sin 2x_{k,n} + 4x_{k,n}^3 \cos^2 x_{k,n}} \int\limits_0^1 \sin^2(x_{k,n} t)(q^2(t)\varphi_k, \varphi_k) dt. \tag{4.6}$$

We can prove the following lemma about the above series.

Lemma 4.1. *Provided that hold the conditions 1,2 and* $\gamma_k \sim g k^\alpha, 0 < g < \infty, 2 < \alpha < \infty$*, then*

$$\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left| \left(x_{k,n}^2 + \gamma_k\right) \frac{2x_{k,n} \int_0^1 \cos\left(2x_{k,n} t\right) f_k(t) dt}{2x_{k,n} - \sin 2x_{k,n} + 4x_{k,n}^3 \cos^2 x_{k,n}} \right| +$$

$$+ \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \left| \frac{4x_{k,n} \int_0^1 \sin^2\left(x_{k,n} t\right) g_k(t) dt}{2x_{k,n} - \sin 2x_{k,n} + 4x_{k,n}^3 \cos^2 x_{k,n}} - \int_0^1 g_k(t) dt \right| < \infty. \tag{4.7}$$

where $f_k\left(t\right) = \left(q\left(t\right) \varphi_k,\, \varphi_k\right),\, g_k\left(t\right) = \left(q^2\left(t\right) \varphi_k,\, \varphi_k\right).$

Proof. For simplicity, denote the sums on the left of (4.7) by d_1, d_2 according to their order. Integrating by parts at first twice, then four times and using the condition 2, we get

$$\int_0^1 \cos\, 2x_{k,n} t f_k\left(t\right) dt \; = -\frac{1}{\left(2x_{k,n}\right)^2} \int_0^1 \cos\, 2x_{k,n} t f_k''\left(t\right) dt. \tag{4.8}$$

$$\int_0^1 \cos\, 2x_{k,n} t f_k\left(t\right) dt = -\frac{1}{\left(2x_{k,n}\right)^3} f_k''\left(1\right) \sin\, 2x_{k,n} -$$

$$-\frac{1}{\left(2x_{k,n}\right)^4} \cos\, 2x_{k,n} t f_k'''\left(t\right)\, |_0^1 + \frac{1}{\left(2x_{k,n}\right)^4} \int_0^1 \cos\, 2x_{k,n} t f_k^{(IV)}\left(t\right) dt. \tag{4.9}$$

Take into consideration

$$\frac{2x_{k,n}}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3 \cos^2\, x_{k,n}} < \frac{1}{1 - \frac{\sin 2x_{k,n}}{2x_{k,n}}} = 1 + O\left(\frac{1}{x_{k,n}}\right) \tag{4.10}$$

and by using (4.9) , asymptotics $x_{k,n} = \pi n + \frac{\pi}{2} + O\left(\frac{1}{n^3}\right)$ (see [17 ,p.13]) ,property $\left\|Aq''\left(t\right)\right\|_{\sigma_1(H)} \leq const$ and (4.8) ,we have

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2x_{k,n}\gamma_k \int_0^1 \cos\, 2x_{k,n} t f_k\left(t\right) dt}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3 \cos^2\, x_{k,n}}\right| =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \gamma_k \left(1 + O\left(\frac{1}{n}\right)\right) O\left(\frac{1}{n^2}\right) \int_0^1 \left|f_k''\left(t\right)\right| dt =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} O\left(\frac{1}{n^2}\right) \int_0^1 \left|\left(Aq''\left(t\right)\varphi_k,\, \varphi_k\right)\right| dt < const \tag{4.11}$$

Since $\left\|q^{(k)}\left(t\right)\right\|_{\sigma_1(H)} \leq const\, (k = 2, 3, 4)$, in virtue of asymptotics $x_{k,n}$ and (4.9), we get

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2x_{k,n}^3 \int_0^1 \cos\, 2x_{k,n} t f_k\left(t\right) dt}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3 \cos^2\, x_{k,n}}\right| =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(1 + O\left(\frac{1}{n}\right)\right)\left[\frac{1}{\left(2x_{k,n}\right)^2}\left(\left|f_k'''\left(1\right)\right| + \right.\right.$$

$$\left.\left. + \left|f_k'''\left(0\right)\right|\right) + \frac{1}{\left(2x_{k,n}\right)^2} \int_0^1 \left|f_k^{(IV)}\left(t\right)\right| dt\right] < \infty \tag{4.12}$$

Above we have used $\sin\, 2x_{k,n} \sim \frac{c}{n^3}, n \to \infty$.
From (4.11) and (4.12) we have that d_1 is convergent.
Therefore

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{4x_{k,n} \int\limits_0^1 \sin^2\left(x_{k,n} t\right) g_k\left(t\right) dt}{2x_{k,n} - \sin 2x_{k,n} + 4x_{k,n}^3 \cos^2\, x_{k,n}} - \int\limits_0^1 g_k\left(t\right) dt\right| =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\frac{2x_{k,n}\int_0^1 g_k(t)\,dt}{2x_{k,n}-\sin 2x_{k,n}+4x_{k,n}^3\cos^2 x_{k,n}}-\frac{2x_{k,n}\int_0^1 \cos(2x_{k,n}t)\,g_k(t)\,dt}{2x_{k,n}-\sin 2x_{k,n}+4x_{k,n}^3\cos^2\, x_{k,n}}-\right.$$

$$\left.-\int_0^1 g_k(t)\,dt\right| =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left|\left(1+O\left(\frac{1}{n}\right)\right)\int_0^1 g_k(t)\,dt-\left(1+O\left(\frac{1}{n}\right)\right)\int_0^1 \cos 2x_{k,n}t\ g_k(t)\,dt-\int_0^1 g_k(t)\,dt\right|$$

$$\leq \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(1+O\left(\frac{1}{n}\right)\right)\int_0^1 |\cos 2x_{k,n}t\ g_k(t)|\,dt+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}O\left(\frac{1}{n}\right)\int_0^1 |g_k(t)|\,dt.$$

The last equality in virtue of (4.8) and properties $g_k(t)\in\sigma_1(H)$, $g_k''(t)\in\sigma_1(H)$ follows that series denoted by d_2 converges.

The lemma is proved . □

By using lemma 4.1, prove the following theorem.

Theorem 4.1 *Let $L_0^{-1}QL_0$ be bounded operator in L_2 and $\gamma_k\sim gk^{\alpha}\, g>0,\quad \alpha>2$. Provided operator-valued function $q(t)$ satisfies conditions 1-3, then the formula*

$$\sum_{n=1}^{\infty}\left(\lambda_n^{(2)}-\mu_n^{(2)}\right) = -\frac{trq^2(0)}{4}-\frac{tr\,Aq(0)-tr\ Aq(1)}{2}+$$

$$+\frac{trq''(0)-trq''(1)}{8} \tag{4.13}$$

is true.

Proof. From Lemma 4.1 and relations (4.2) and (4.3) ,we get

$$\lim_{m\to\infty}\left(\sum_{n=1}^{n_m}\left(\lambda_n^2-\mu_n^2-\int_0^1 trq^2(t)\,dt\right)+\right.$$

$$\left.+\frac{1}{2\pi i}\int_{\Gamma_m}\sum_{k=2}^{N}\frac{(-1)^{k-1}}{k}tr\left[\left(L_0Q+QL_0+Q^2\right)R_0(\lambda)\right]^k d\lambda\right) =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(x_{k,n}^2+\gamma_k\right)\frac{4x_{k,n}\int_0^1(1-\cos\, 2x_{k,n}t)\,f_k(t)\,dt}{2x_{k,n}-\sin\, 2x_{k,n}+4x_{k,n}^3\cos^2\, x_{k,n}}+$$

$$+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{4x_{k,n}\int_0^1\frac{1-\cos 2x_{k,n}t}{2}g_k(t)dt}{2x_{k,n}-\sin 2x_{k,n}+4x_{k,n}^3\cos^2 x_{k,n}}-\int_0^1 g_k(t)dt\right] =$$

$$= -\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(x_{k,n}^2+\gamma_k\right)\frac{4x_{k,n}\int_0^1 \cos\, 2x_{k,n}tf_k\left(t\right)dt}{2x_{k,n}-\sin\, 2x_{k,n}+4x_{k,n}^3\cos^2\, x_{k,n}}+$$

$$+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{2x_{k,n}\int_0^1\left(1-\cos 2x_{k,n}t\right)g_k\left(t\right)dt}{2x_{k,n}-\sin 2x_{k,n}+4x_{k,n}^3\cos^2 x_{k,n}}-\int_0^1 g_k\left(t\right)dt\right]. \tag{4.14}$$

At first we derive a formula for the following sum

$$\sum_{n=1}^{\infty}\left[\frac{2x_{k,n}}{2x_{k,n}-\sin\, 2x_{k,n}+4x_{k,n}^3\cos^2\, x_{k,n}}-1\right]=$$

$$=\lim_{N\to\infty}\sum_{n=1}^{N}\left[\frac{2x_{k,n}}{2x_{k,n}-\sin\, 2x_{k,n}+4x_{k,n}^3\cos^2\, x_{k,n}}-1\right]. \tag{4.15}$$

For this we will investigate the asymptotics behavior of the following function at $N\to\infty$

$$S_N\left(t\right)=\sum_{n=1}^{N}\left[\frac{2x_{k,n}}{2x_{k,n}-\sin\, 2x_{k,n}+4x_{k,n}^3\cos^2\, x_{k,n}}-1\right].$$

Express the $k-$th term of sum $S_N\left(t\right)$ as a residue at the pole $x_{k,n}$ of some function of complex variable z:

$$P\left(z\right)=\frac{z}{\left(\frac{tg\, z}{z}+z^2+\gamma_k\right)z^2\cos^2\, z}. \tag{4.16}$$

$P\left(z\right)$ has simple poles at the points $x_{k,n}$, $(n+\frac{1}{2})\pi$ and $z=0$.
Find the residue at the point $x_{k,n}$:

$$\underset{z=x_{k,n}}{res}\, P\left(z\right)=\frac{x_{k,n}}{x_{k,n}^2\cos^2 x_{k,n}\left(\frac{tgz}{z}+z^2+\gamma_k\right)'_{z=x_{k,n}}}=$$

$$=\frac{2x_{k,n}}{2x_{k,n}-\sin 2x_{k,n}+4x_{k,n}^3\cos^2 x_{k,n}}.$$

We have

$$\underset{z=\left(n+\frac{1}{2}\right)\pi}{res}\, P\left(z\right)=\underset{z=\left(n+\frac{1}{2}\right)\pi}{res}\,\frac{z}{\left(\frac{tgz}{z}+z^2+\gamma_k\right)z^2\cos^2 z}=$$

$$=\underset{z=\left(n+\frac{1}{2}\right)\pi}{res}\,\frac{z}{\left(\frac{\sin z}{z}+\left(z^2+\gamma_k\right)\cos z\right)z^2\cos z}=$$

$$=\frac{\pi n+\frac{\pi}{2}}{\frac{\sin(\pi n+\frac{\pi}{2})}{\pi n+\frac{\pi}{2}}\left(\pi n+\frac{\pi}{2}\right)^2\left(-\sin(\pi n+\frac{\pi}{2})\right)}=-1.$$

For the contour of integration let take the rectangle with vertices at the points $\pm iB$, $A_N \pm iB$, .Then this contour bypass the origin on the right hand side of imaginary axis. B will go to infinity and take $A_N = \pi N$. In this case for A_N we get $x_{k,N} < A_N < x_{k,N+1}$.

Obviously, the function $P(z)$ is an odd function of z. That is why the integral along the contour on imaginary axis is equal to zero.

Let be $z = x + iy$. Then for $x \geq 0$ and for large y the function $P(z)$ is of order $O\left(\frac{e^{2|x|(t-1)}}{|y|^3}\right)$. In this case for the given value of A_N the integrals along lower and upper sides of the contour also vanish when $B \to \infty$.
Therefore ,we have the following

$$S_N(t) = \frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N-iB}^{A_N+iB} P(z)\,dz + \frac{1}{2\pi i}\lim_{r\to 0}\int_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}} P(z)\,dz. \tag{4.17}$$

As $N \to \infty$

$$\frac{1}{2\pi i}\lim_{B\to\infty}\int_{A_N-iB}^{A_N+iB} P(z)\,dz \sim \frac{1}{2\pi i}\int_{A_N-i\infty}^{A_N+i\infty}\frac{dz}{z^3\cos^2 z} =$$

$$= \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dy}{(A_N+iy)^3\left(1+\cos\left(2A_N+2iy\right)\right)} = \frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{dy}{(A_N+iy)^3\left(1+\cos 2iy\right)} =$$

$$= \frac{1}{\pi}\int\limits_{-\infty}^{+\infty}\frac{dy}{(A_N+iy)^3(1+\ ch2y)} \equiv L \tag{4.18}$$

Then,

$$|L| < \frac{2}{\pi}\int\limits_{0}^{+\infty}\frac{dy}{\sqrt{\left(A_N^2+y^2\right)^3}\left(1+\ ch2y\right)} < \frac{2}{\pi A_N}\int\limits_{0}^{+\infty}\frac{dy}{\sqrt{\left(A_N^2+y^2\right)^3}\left(1+\ ch2y\right)} \equiv$$

$$= \frac{const}{A_N}. \tag{4.19}$$

Therefore,

$$\int_0^1 S_N(t)\,g_k(t)\,dt = \frac{1}{2\pi i}\int_0^1 g_k(t)\,dt\int\limits_{A_N-i\infty}^{A_N+i\infty}\frac{z}{\left(\frac{tg\,z}{z}+z^2+\gamma_k\right)z^2\cos^2 z}dz+$$

$$+\frac{1}{2\pi i}\lim_{r\to 0}\int_0^1 g_k(t)\,dt\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{z}{\left(\frac{tg\,z}{z}+z^2+\gamma_k\right)z^2\cos^2 z}dz. \tag{4.20}$$

We get

$$\frac{1}{2\pi i}\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{zdz}{\left(\frac{tg\,z}{z}+z^2+\gamma_k\right)z^2\cos^2 z}=$$

$$=\frac{1}{2\pi i}\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{dz}{\frac{1}{2}\sin\,2z+\left(z^2+\gamma_k\right)z\cos^2 z}=$$

$$=\frac{1}{2\pi i}\int\limits_{\substack{|z|=r\\ -\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}\frac{dz}{\frac{1}{2}\sin 2z+\left(z^2+\gamma_k\right)z(1-\sin^2 z)}\sim$$

$$\sim\frac{1}{2\pi i}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{ire^{i\varphi}d\varphi}{re^{i\varphi}+\left(\left(re^{i\varphi}\right)^2+\gamma_k\right)re^{i\varphi}\left(1-(re^{i\varphi})^2\right)}=$$

$$=\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\varphi}{1+\left(\left(re^{i\varphi}\right)^2+\gamma_k\right)\left(1-(re^{i\varphi})^2\right)}\underset{r\to 0}{\longrightarrow}\frac{1}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{d\varphi}{1+\gamma_k}=\frac{1}{2(1+\gamma_k)} \tag{4.21}$$

Take into consideration (4.17), (4.18), (4.19) and (4.21) in (4.20) we have

$$\lim_{N\to\infty}\int_0^1 S_N(t)\,g_k(t)\,dt=\frac{1}{2(1+\gamma_k)}\int_0^1 g_k(t)\,dt. \tag{4.22}$$

For the fixed k investigate the asymptotic behavior of the function

$$T_N(t)=\sum_{n=1}^{N}\frac{-2x_{k,n}\cos\,2x_{k,n}t}{2x_{k,n}-\sin\,2x_{k,n}+4x_{k,n}^3\cos^2\,x_{k,n}}.$$

For this consider the function of complex variable

$$F(z)=\frac{-z\cos 2zt}{\left(\frac{tgz}{z}+z^2+\gamma_k\right)z^2\cos^2 z}.$$

This function also has simple poles at the points $x_{k,n}$, $\pi n+\frac{\pi}{2}$ and $z=0$:

$$\operatorname*{res}_{z=x_{k,n}}F(z)=\frac{-2x_{k,n}\cos 2x_{k,n}t}{2x_{k,n}-\sin 2x_{k,n}+4x_{k,n}^3\cos^2 x_{k,n}}$$

and

$$\operatorname*{res}_{z=\left(n+\frac{1}{2}\right)\pi n}F(z)=\cos\left(2n+1\right)\pi t.$$

Again take as a contour of integration the above considered contour. One can show that as $N \to \infty$ (see [18 ,p.32])

$$\frac{1}{2\pi i}\int_{A_N-i\infty}^{A_N+i\infty}\frac{-z\cos 2zt\,dz}{\left(\frac{tgz}{z}+z^2+\gamma_k\right)z^2\cos^2 z}\sim\frac{const}{A_N} \tag{4.23}$$

from which we get

$$\lim_{N\to\infty}\int_0^1 g_k(t)\int_{A_N-i\infty}^{A_N+i\infty}F(z)\,dzdt=0. \tag{4.24}$$

From (4.24),we have the following

$$\lim_{N\to\infty}\int_0^1 T_N(t)\,g_k(t)\,dt=-\lim_{N\to\infty}\int_0^1 M_N(t)\,g_k(t)\,dt+$$

$$+\frac{1}{2\pi i}\lim_{r\to 0}\int_0^1 g_k(t)\,dt\int\limits_{\substack{|z|=r\\-\frac{\pi}{2}<\varphi<\frac{\pi}{2}}}F(z)\,dz \tag{4.25}$$

where

$$M_N(t)=\sum_{n=1}^{N}\cos(2n+1)\,\pi t.$$

The first term in (4.25) is calculated in [18 ,p.33] and equal to

$$\lim_{N\to\infty}\int_0^1 M_N(t)\,g_k(t)\,dt=\frac{g_k(0)-g_k(1)}{4}$$

and the second term in (4.25) as $r\to 0$ goes to $-\frac{1}{2(1+\gamma_k)}\int_0^1 g_k(t)\,dt$.

Other words

$$\lim_{N\to\infty}\int_0^1 T_N(t)\,g_k(t)\,dt=-\frac{g_k(0)-g_k(1)}{4}-\frac{1}{2(1+\gamma_k)}\int_0^1 g_k(t)\,dt. \tag{4.26}$$

From (4.22) and (4.26), we have

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{2x_{k,n}\int_0^1(1-\cos 2x_{k,n}t)\,g_k(t)\,dt}{2x_{k,n}-\sin 2x_{k,n}+4x_{k,n}^3\cos^2 x_{k,n}}-\int_0^1 g_k(t)\,dt\right)=$$

$$=-\sum_{k=1}^{\infty}\frac{g_k(0)-g_k(1)}{4}-\sum_{k=1}^{\infty}\int_0^1\left(-\frac{g_k(t)}{2(1+\gamma_k)}+\frac{g_k(t)}{2(1+\gamma_k)}\right)dt=$$

$$=-\sum_{k=1}^{\infty}\frac{g_k(0)-g_k(1)}{4}=-\sum_{k=1}^{\infty}\frac{g_k(0)}{4}=-\frac{trq^2(0)}{4}. \tag{4.27}$$

Here we used the following relation

$$g_k(1) = \left(q^2(1)\varphi_k, \varphi_k\right) = \left(q(1)\varphi_k, q(1)\varphi_k\right) = 0.$$

Note that in [18 ,p.33] the following is calculated

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1 \frac{2x_{k,n}\cos\, 2x_{k,n}t f_k(t)\, dt}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3 \cos^2\, x_{k,n}} =$$

$$= \frac{1}{4}\sum_{k=1}^{\infty}\left[\sum_{n=0}^{\infty}\cos\, n\cdot 0\cdot \frac{2}{\pi}\int_0^{\pi}\cos\, nz f_k\left(\frac{z}{\pi}\right) dz - \right.$$

$$\left. - \sum_{n=0}^{\infty}\cos\, n\cdot\pi\cdot\frac{2}{\pi}\int_0^{\pi}\cos\, nz f_k\left(\frac{z}{\pi}\right) dz\right] = \sum_{k=1}^{\infty}\frac{f_k(0) - f_k(1)}{4}. \tag{4.28}$$

From (4.28) we get

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\gamma_k \frac{-4x_{k,n}\int_0^1 \cos\, 2x_{k,n}t f_k(t)\, dt}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3\cos^2\, x_{k,n}} =$$

$$= -\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}2\gamma_k \frac{2x_{k,n}\int_0^1 \cos\, 2x_{k,n}t f_k(t)\, dt}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3\cos^2\, x_{k,n}} =$$

$$= -\sum_{k=1}^{\infty}\gamma_k\frac{f_k(0) - f_k(1)}{2} = -\frac{1}{2}\sum_{k=1}^{\infty}\gamma_k[(q(0)\varphi_k\varphi_k) - (q(1)\varphi_k\varphi_k)] \tag{4.29}$$

Take into considiration the following

$$\sum_{k=1}^{\infty}\gamma_k(q(t)\varphi_k\varphi_k) = \sum_{k=1}^{\infty}(Aq(t)\varphi_k\varphi_k) = trAq(t),$$

from (4.29) we get

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\gamma_k\frac{-4x_{k,n}\int_0^1 \cos\, 2x_{k,n}t f_k(t)\, dt}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3\cos^2\, x_{k,n}} = -\frac{trAq(0) - trAq(1)}{2} \tag{4.30}$$

and by using condition 2) we have (Note that in this case we consider as the complex valued function $H(z) = \frac{-2z\cos 2zt}{\left(\frac{tgz}{z} + z^2 + \gamma_k\right)\cos^2 z}$. The residues of this function at the poles $x_{k,n}$ and $\pi n + \frac{\pi}{2}$ are equal to $\frac{-4x_{k,n}^3\cos 2x_{k,n}t}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3\cos^2 x_{k,n}}$ and $2\left(\pi n + \frac{\pi}{2}\right)^2\cos(2n+1)\pi t$, respectively.)

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{-4x_{k,n}^3\int_0^1 \cos\, 2x_{k,n}t\ f_k(t)\, dt}{2x_{k,n} - \sin\, 2x_{k,n} + 4x_{k,n}^3\cos^2\, x_{k,n}} =$$

$$= -\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1 2\left(\pi n+\frac{\pi}{2}\right)^2\cos(2n+1)\pi t\ f_k(t)\,dt =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\int_0^1 (\pi n+\frac{\pi}{2})\sin(2n+1)\pi t\ f_k'(t)\,dt = \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2}\int_0^1 \cos(2n+1)\pi t\ f_k''(t)\,dt =$$

$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{2\pi}\int_0^{\pi}\cos(2n+1)z\ f_k''\left(\frac{z}{\pi}\right)dz =$$

$$= \frac{1}{8}\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left[\cos n\cdot 0\cdot\frac{2}{\pi}\int_0^{\pi}\cos nz\ f_k''\left(\frac{z}{\pi}\right)dz -\right.$$

$$\left. -\cos n\cdot\pi\cdot\frac{2}{\pi}\int_0^{\pi}\cos nz\ f_k''\left(\frac{z}{\pi}\right)dz\right] = \sum_{k=1}^{\infty}\frac{f_k''(0)-f_k''(1)}{8} = \frac{trq''(0)-trq''(1)}{8}. \quad (4.31)$$

From (4.27), (4.30) and (4.31) we get the formula (4.13) is true.Theorem is proved.

□

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