Journal of Contemporary Applied Mathematics
V. 7, No 2, 2017, December
ISSN 2222-5498

Approximation of analytic functions in multiply connected domains by linear operators

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Abstract. We consider the space of analytic functions in multiply connected domains with the topology of compact convergence, find criterion for convergence of sequences in this space and prove the theorems on the approximation and statistical approximation of functions in this space by the sequences of linear and, in particular, linear k-positive operators.

Key Words and Phrases: space of analytical functions; multiply connected domain; Korovkin type theorem; linear k-positive operators; statistical convergence.

2010 Mathematics Subject Classifications: 41A35, 47A58

1. Introduction

It is well known that the sequences of linear positive operators play an important role in the theory of approximation of functions (see [2, 15]). Some authors explored the problem of approximation of analytic functions by the sequences of linear operators with the use of k-positivity properties of linear operators. The concept of k-positivity was first introduced in [7] to obtain Korovkin-type approximation theorems in the space of analytic functions. Using this definition of k-positivity, some results on approximation of analytic functions by means of k-positive linear operators in the unit disk were obtained in [7] and [8]. Later, various problems of approximation by linear k-positive operators in the space of analytic functions on simply connected domains have been studied extensively in [1, 3, 5, 9, 11, 12]. The approximation of analytic functions by linear operators without k-positivity has been considered in [10] and [14].

This paper is dedicated to the approximation of analytic functions in the multiply connected domains by linear operators. The paper is structured as follows. In Section 2, we introduce the space of analytic functions in the multiply connected domains and prove a theorem about the necessary and sufficient conditions for convergence to zero of the sequence of functions in this space. We also prove theorems on the approximation by linear operators, in particular, by k-positive operators. In Section 3, we present similar results for statistical convergence.

2. Approximation by linear operators in the space of analytic functions on multiply connected domains.

Let $D_0 = \{z \in C: |z-a_0| < R\}$, $a_1,...,a_m \in D_0$ and $D = \left\{z \in D_0: |z-a_i| > r_i,\ i=\overline{1,m}\right\}$, where $0 < r_i < R - |a_0 - a_i|$, $i=\overline{1,m}$, $r_i + r_j < |a_i - a_j|$, $i,j = \overline{1,m}$, $i \neq j$. We will denote by $A(D)$ the space of all analytic functions in $m+1$ connected domain D with the topology of compact convergence. This means that, by a convergence in this space we will mean the uniform convergence in any compact of D. For $\max\limits_{i=\overline{1,m}} [|a_0 - a_i| + r_i] < R' < R$, $r_i < r_i' < R' - |a_0 - a_i|$, $i=\overline{1,m}$, $r_i' + r_j' < |a_i - a_j|$, $i,j=\overline{1,m}$, $i \neq j$ we will consider the seminorms

$$\|f\|_{A(D), r_1',...,r_m',R'} = \max\left\{ |f(z)| \ : \ z \in \bar{D}_{r_1',...,\,r_m',\,R'} \right\},$$

that convert $A(D)$ into a Frechet–type space, where

$$\bar{D}_{r_1',...,r_m',R'} = \left\{z \in C: |z-a_0| \leq R',\ |z-a_i| \geq r_i',\ i=\overline{1,m}\right\}.$$

It is known [see, for example, 4] that the system of functions 1, $(z-a_0)^k$, $(z-a_1)^{-k}$,..., $(z-a_m)^{-k}$, $k \in N$, forms a basis for $A(D)$, i.e. every function $f \in A(D)$ can be uniquely represented in the form

$$f(z) = \sum_{i=0}^{m}\sum_{k=0}^{\infty} f_k^{(i)} (z-a_i)^{\omega_i(k)}, \tag{1}$$

where $\omega_0(k) = k$, $k \in Z_+$, $\omega_i(k) = -k-1$ for $i=\overline{1,\,m}$, $k \in Z_+$, and the coefficients $f_k^{(i)}$, $k \in Z_+$, $i=\overline{0,\,m}$ are defined by the formula

$$f_k^{(i)} = \frac{1}{2\pi i}\int_{\Gamma_i} f(z)(z-a_i)^{-\omega_i(k)-1}\,dz, k \in Z_+, i=\overline{0,\,m}$$

and Γ_i, $i=\overline{0,m}$ are any circles centered at the points a_i and belonging to the domain D.

Denote $\alpha_0 = R^{-1}$ and $\alpha_i = r_i$ for $i=\overline{1,\,m}$.

First we prove the following theorem on the convergence to zero in $A(D)$.

Theorem 1. The sequence $f_n(z)$ convergence to zero in $A(D)$ if and only if the coefficients of expansion

$$f_n(z) = \sum_{i=0}^{m}\sum_{k=0}^{\infty} f_{n,k}^{(i)} (z-a_i)^{\omega_i(k)}$$

satisfy the conditions

$$\left|f_{n,k}^{(i)}\right| < \varepsilon_n (1+\delta_n)^k \alpha_i^k, k \in Z_+, i=\overline{0,m} \tag{2}$$

for any $n \in N$, where

$$\lim_{n\to\infty} \varepsilon_n = \lim_{n\to\infty} \delta_n = 0. \tag{3}$$

Proof. The sufficiency follows from the fact that for each $\max\limits_{i=\overline{1,m}} [|a_0 - a_i| + r_i] < R' < R$, $r_i < r_i' < R' - |a_0 - a_i|$, $i = \overline{1,m}$, $r_i' + r_j' < |a_i - a_j|$, $i, j = \overline{1,m}$, $i \neq j$ we have

$$\|f_n\|_{A(D), r_1',\dots,r_m',R'} \leq \sum_{k=0}^{\infty} \varepsilon_n (1+\delta_n)^k \left(\frac{R'}{R}\right)^k + \sum_{i=1}^{m}\sum_{k=1}^{\infty} \varepsilon_n (1+\delta_n)^k \left(\frac{r_i}{r_i'}\right)^k \cdot \frac{1}{r_i'} =$$

$$= \left[\frac{R}{R - (1+\delta_n) R'} + \sum_{i=1}^{m} \frac{(1+\delta_n) r_i}{r_i' - (1+\delta_n) r_i} \cdot \frac{1}{r_i'}\right] \varepsilon_n \tag{4}$$

and the right-hand side of the inequality (4) tends to zero as $n \to \infty$.

To prove the necessity, we choose $\delta_n \to 0$ such that

$$\varepsilon_n = \max\left\{ |f_n(z)| \;:\; z \in \bar{D}_{r_1(1+\delta_n),\dots,\, r_m(1+\delta_n),\, R(1+\delta_n)^{-1}} \right\}$$

tends to zero as $n \to \infty$. Then for $k \in Z_+$ the equality

$$f_{n,k}^{(0)} = \frac{1}{2\pi i} \int_{|z-a_0|=R(1+\delta_n)^{-1}} f_n(z)(z-a_0)^{-k-1}\, dz$$

implies the estimate

$$\left| f_{n,k}^{(0)} \right| \leq \varepsilon_n (1+\delta_n)^k R^{-k},$$

and for $k \in Z_+$, $i = \overline{1,m}$ the equality

$$f_{n,k}^{(i)} = \frac{1}{2\pi i} \int_{|z-a_i|=r_i(1+\delta_n)} f_n(z)(z-a_i)^k\, dz$$

implies the estimate

$$\left| f_{n,k}^{(i)} \right| \leq \varepsilon_n (1+\delta_n)^{k+1} r_i^{k+1}.$$

This completes the Proof of the Theorem 1.

Now we consider the linear operators in $A(D)$. It follows from (1) that for any linear operator $T : A(D) \to A(D)$ the expansion

$$(Tf)(z) = \sum_{i=0}^{m}\sum_{k=0}^{\infty} \left(\sum_{j=0}^{m}\sum_{p=0}^{\infty} T_{k,p,(i,j)} f_p^{(j)} \right) (z-a_i)^{\omega_i(k)}$$

is valid, where $f(z) = \sum_{i=0}^{m}\sum_{k=0}^{\infty} f_k^{(i)} (z-a_i)^{\omega_i(k)}$ and

$$T\left((z-a_j)^{\omega_j(p)}\right) = \sum_{i=0}^{m}\sum_{k=0}^{\infty} T_{k,p,(i,j)} (z-a_i)^{\omega_i(k)}, p \in Z_+, j = \overline{0,m}.$$

Let the system of sequences of positive numbers $g = \left\{ \left\{ g_k^{(i)} \right\}_{k=0}^{\infty} \; : \; i = \overline{0,m} \right\}$ satisfies the conditions:

$$\forall i = \overline{0,m}, \forall k \in Z_+ : \Delta_k^{(i)}(g) =$$
$$= \inf \left\{ \left| \sqrt{g_k^{(i)}} - \sqrt{g_p^{(j)}} \right| \; : \; p \in Z_+, \; j = \overline{0,m}, \; (j,p) \neq (i,k) \right\} > 0, \tag{5}$$

$$\forall i = \overline{0,m} : \lim_{k\to\infty} \left(\Delta_k^{(i)}(g) \right)^{\frac{1}{k}} = 1, \lim_{k\to\infty} \left(g_k^{(i)} \right)^{\frac{1}{k}} = 1. \tag{6}$$

Definition 1. By $A_g(D)$ we denote the set of analytic functions

$$f(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} f_k^{(i)} (z - a_i)^{\omega_i(k)} \in A(D)$$

whose coefficients satisfy the following conditions:

$$\left| f_k^{(i)} \right| \leq M_f \, g_k^{(i)} \, \alpha_i^k, k \in Z_+, i = \overline{0,m}, \tag{7}$$

where M_f is a constant independent of k.

Theorem 2. Let $T_n : A(D) \to A(D)$ be a sequence of linear operators

$$(T_n f)(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{m} \sum_{p=0}^{\infty} T_{k,p,\,(i,j)}^{(n)} f_p^{(j)} \right) (z - a_i)^{\omega_i(k)}, \tag{8}$$

where $f(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} f_k^{(i)} (z - a_i)^{\omega_i(k)} \in A(D)$. If there exist sequences ε_n and δ_n satisfying (3) such that the inequalities

$$\left| \sum_{j=0}^{m} \sum_{p=0}^{\infty} T_{k,p,\,(i,j)}^{(n)} \alpha_j^p - \alpha_i^k \right| < \varepsilon_n (1 + \delta_n)^k \alpha_i^k, i = \overline{0,m}, k \in Z_+ \tag{9}$$

$$\left| \sum_{j=0}^{m} \sum_{p=0}^{\infty} \left| T_{k,p,\,(i,j)}^{(n)} \right| \alpha_j^p - \alpha_i^k \right| < \varepsilon_n (1 + \delta_n)^k \alpha_i^k, i = \overline{0,m}, k \in Z_+, \tag{10}$$

$$\left| \sum_{j=0}^{m} \sum_{p=0}^{\infty} \left| T_{k,p,\,(i,j)}^{(n)} \right| \sqrt{g_p^{(j)}} \alpha_j^p - \sqrt{g_k^{(i)}} \alpha_i^k \right| < \varepsilon_n (1 + \delta_n)^k \alpha_i^k, i = \overline{0,m}, k \in Z_+ \tag{11}$$

$$\left| \sum_{j=0}^{m} \sum_{p=0}^{\infty} \left| T_{k,p,\,(i,j)}^{(n)} \right| g_p^{(j)} \alpha_j^p - g_k^{(i)} \alpha_i^k \right| < \varepsilon_n (1 + \delta_n)^k \alpha_i^k, i = \overline{0,m}, k \in Z_+ \tag{12}$$

holds, then for any function $f \in A_g(D)$ and for every $\max\limits_{i=\overline{1,m}} [|a_0 - a_i| + r_i] < R' < R$, $r_i < r_i' < R' - |a_0 - a_i|$, $i = \overline{1,m}$, $r_i' + r_j' < |a_i - a_j|$, $i, j = \overline{1,m}$, $i \neq j$ we have

$$\lim_{n\to\infty} \|T_n f - f\|_{A(D),\, r_1', \ldots, r_m', R'} = 0.$$

Proof. From the equations (10) - (12) we have

$$\sum_{j=0}^{m}\sum_{p=0}^{\infty}\left|T_{k,p,\,(i,j)}^{(n)}\right|\left(\sqrt{g_p^{(j)}}-\sqrt{g_k^{(i)}}\right)^2\alpha_j^p$$

$$\leq \varepsilon_n\left(1+\delta_n\right)^k\alpha_i^k\left(1+\sqrt{g_k^{(i)}}\right)^2, i=\overline{0,m}, k\in Z_+ \tag{13}$$

and this implies

$$\sum_{j=0}^{m}\sum_{\substack{p=0\\(j,p)\neq(i,k)}}^{\infty}\left|T_{k,p,\,(0,j)}^{(n)}\right|\alpha_j^p\leq$$

$$\leq \varepsilon_n\left(1+\delta_n\right)^k\alpha_i^k\left(1+\sqrt{g_k^{(i)}}\right)^2\left(\Delta_k^{(i)}\left(g\right)\right)^{-2}, i=\overline{0,m}, k\in Z_+. \tag{14}$$

For each $f\left(z\right)=\sum_{i=0}^{m}\sum_{k=0}^{\infty}f_k^{(i)}\left(z-a_i\right)^{\omega_i(k)}\in A\left(D\right)$ we have

$$T_nf\left(z\right)-f\left(z\right)=\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{p=0}^{\infty}T_{k,p,\,(i,j)}^{(n)}f_p^{(j)}-f_k^{(i)}\right\}\left(z-a_i\right)^{\omega_i(k)}=$$

$$=\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{p=0}^{\infty}T_{k,p,\,(i,j)}^{(n)}\alpha_j^p-\alpha_i^k\right\}f_k^{(i)}\alpha_i^{-k}\left(z-a_i\right)^{\omega_i(k)}+$$

$$+\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{p=0}^{\infty}T_{k,p,\,(i,j)}^{(n)}\left(f_p^{(j)}\alpha_j^{-p}-f_k^{(i)}\alpha_i^{-k}\right)\alpha_j^p\right\}\left(z-a_i\right)^{\omega_i(k)}=J_n^{(1)}\left(z\right)+J_n^{(2)}\left(z\right). \tag{15}$$

If $f\in A_g\left(D\right)$, then from (7), (9), (13) and (14) we have:

$$\left|J_n^{(1)}\left(z\right)\right|\leq M_f\varepsilon_n\cdot\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left(1+\delta_n\right)^k\alpha_i^kg_k^{(i)}\left|z-a_i\right|^{\omega_i(k)};$$

$$\left|J_n^{(2)}\left(z\right)\right|\leq M_f\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{\substack{p=0\\(j,p)\neq(i,k)}}^{\infty}\left|T_{k,p,\,(i,j)}^{(n)}\right|\left[g_p^{(j)}+g_k^{(i)}\right]\alpha_j^p\right\}\left|z-a_i\right|^{\omega_i(k)}\leq$$

$$\leq M_f\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{\substack{p=0\\(j,p)\neq(i,k)}}^{\infty}\left|T_{k,p,\,(i,j)}^{(n)}\right|\left[2\left(\sqrt{g_p^{(j)}}-\sqrt{g_k^{(i)}}\right)^2+3g_k^{(i)}\right]\alpha_j^p\right\}$$

$$|z-a_i|^{\omega_i(k)} \le$$

$$\le M_f \varepsilon_n \cdot \sum_{i=0}^{m}\sum_{k=0}^{\infty}\left(2+3g_k^{(i)}\left(\Delta_k^{(i)}(g)\right)^{-2}\right)(1+\delta_n)^k \alpha_i^k \left(1+\sqrt{g_k^{(i)}}\right)^2 |z-a_i|^{\omega_i(k)}.$$

Hence, by virtue of (15), we obtain that

$$\lim_{n\to\infty} \|T_n f - f\|_{A(D), r_1', \ldots, r_m', R'} = 0$$

for each $\max_{i=\overline{1,m}} [|a_0-a_i|+r_i] < R' < R$, $r_i < r_i' < R' - |a_0-a_i|$, $i=\overline{1,m}$, $r_i'+r_j' < |a_i-a_j|$, $i,j=\overline{1,m}$, $i\neq j$. This completes the Proof of the Theorem 2.

Now, we will present the following general result on approximation in $A(D)$.

Theorem 3. Let the sequences of positive numbers $b=\left\{\left\{b_k^{(i)}\right\}_{k=0}^{\infty} : i=\overline{0,m}\right\}$ and $g=\left\{\left\{g_k^{(i)}\right\}_{k=0}^{\infty} : i=\overline{0,m}\right\}$ satisfy the conditions (5), (6) and $T_n : A(D)\to A(D)$ be a linear operators defined as (8). If there exist sequences ε_n and δ_n satisfying (3) such that the inequalities

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty} T_{k,p,(i,j)}^{(n)} g_p^{(j)} \alpha_j^p - g_k^{(i)}\alpha_i^k\right| < \varepsilon_n (1+\delta_n)^k \alpha_i^k, i=\overline{0,m}, k\in Z_+, \tag{16}$$

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty} \left|T_{k,p,(i,j)}^{(n)}\right| g_p^{(j)} \alpha_j^p - g_k^{(i)}\alpha_i^k\right| < \varepsilon_n (1+\delta_n)^k \alpha_i^k, i=\overline{0,m}, k\in Z_+, \tag{17}$$

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty} \left|T_{k,p,(i,j)}^{(n)}\right| \sqrt{b_p^{(j)}} g_p^{(j)} \alpha_j^p - \sqrt{b_k^{(i)}} g_k^{(i)}\alpha_i^k\right| < \varepsilon_n (1+\delta_n)^k \alpha_i^k, i=\overline{0,m}, k\in Z_+, 18)$$

$$\left|\sum_{j=0}^{m}\sum_{p=0}^{\infty} \left|T_{k,p,(i,j)}^{(n)}\right| b_p^{(j)} g_p^{(j)} \alpha_j^p - b_k^{(i)} g_k^{(i)}\alpha_i^k\right| < \varepsilon_n (1+\delta_n)^k \alpha_i^k, i=\overline{0,m}, k\in Z_+ \tag{18}$$

holds, then for any function $f\in A_g(D)$ and for every $\max_{i=\overline{1,m}} [|a_0-a_i|+r_i] < R' < R$, $r_i < r_i' < R' - |a_0-a_i|$, $i=\overline{1,m}$, $r_i'+r_j' < |a_i-a_j|$, $i,j=\overline{1,m}$, $i\neq j$ we have

$$\lim_{n\to\infty} \|T_n f - f\|_{A(D), r_1', \ldots, r_m', R'} = 0.$$

Proof. From (17) - (18) we have

$$\sum_{j=0}^{m}\sum_{p=0}^{\infty} \left|T_{k,p,(i,j)}^{(n)}\right| \left(\sqrt{b_p^{(j)}} - \sqrt{b_k^{(i)}}\right)^2 g_p^{(j)} \alpha_j^p \le$$

$$\le \varepsilon_n (1+\delta_n)^k \alpha_i^k \left(1+\sqrt{b_k^{(i)}}\right)^2, i=\overline{0,m}, k\in Z_+, \tag{19}$$

and this implies

$$\sum_{j=0}^{m} \sum_{\substack{p=0 \\ (j,p)\neq(i,k)}}^{\infty} \left|T_{k,p,(i,j)}^{(n)}\right| g_p^{(j)}\alpha_j^p \leq$$

$$\leq \varepsilon_n (1+\delta_n)^k \alpha_i^k \left(1+\sqrt{b_k^{(i)}}\right)^2 \left(\Delta_k^{(i)}(b)\right)^{-2}, i=\overline{0,m}, k\in Z_+. \qquad (20)$$

For each $f(z)=\sum_{i=0}^{m}\sum_{k=0}^{\infty} f_k^{(i)}(z-a_i)^{\omega_i(k)} \in A(D)$ we have

$$T_n f(z) - f(z) = \sum_{i=0}^{m}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{p=0}^{\infty} T_{k,p,(i,j)}^{(n)} f_p^{(j)} - f_k^{(i)}\right\}(z-a_i)^{\omega_i(k)} =$$

$$= \sum_{i=0}^{m}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{p=0}^{\infty} T_{k,p,(i,j)}^{(n)} g_p^{(j)}\alpha_j^p - g_k^{(i)}\alpha_i^k\right\}\frac{f_k^{(i)}}{\alpha_i^k g_k^{(i)}}(z-a_i)^{\omega_i(k)} +$$

$$+\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{p=0}^{\infty} T_{k,p,(i,j)}^{(n)}\left(\frac{f_p^{(j)}}{\alpha_j^p g_p^{(j)}} - \frac{f_k^{(i)}}{\alpha_i^k g_k^{(i)}}\right) g_p^{(j)}\alpha_j^p\right\}(z-a_i)^{\omega_i(k)} = J_n^{(3)}(z) + J_n^{(4)}(z). \qquad (21)$$

If $f\in A_g(D)$, then from (7), (16), (19) and (20) we have:

$$\left|J_n^{(3)}(z)\right| \leq M_f\varepsilon_n\left[\sum_{i=0}^{m}\sum_{k=0}^{\infty}(1+\delta_n)^k\alpha_i^k\left|z-a_i\right|^{\omega_i(k)}\right];$$

$$\left|J_n^{(4)}(z)\right| \leq 2M_f\cdot\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left\{\sum_{j=0}^{m}\sum_{\substack{p=0 \\ (j,p)\neq(i,k)}}^{\infty}\left|T_{k,p,(i,j)}^{(n)}\right| g_p^{(j)}\alpha_j^p\right\}\left|z-a_i\right|^{\omega_i(k)} \leq$$

$$\leq 2M_f\varepsilon_n\cdot\sum_{i=0}^{m}\sum_{k=0}^{\infty}(1+\delta_n)^k\alpha_i^k\left(1+\sqrt{b_k^{(i)}}\right)^2\left(\Delta_k^{(i)}(b)\right)^{-2}\left|z-a_i\right|^{\omega_i(k)}.$$

Hence, by virtue of (21), we obtain that

$$\lim_{n\to\infty}\left\|T_n f - f\right\|_{A(D),\, r_1',\dots,r_m',R'} = 0$$

for each $\max\limits_{i=\overline{1,m}}\left[|a_0-a_i|+r_i\right] < R' < R$, $r_i < r_i' < R' - |a_0-a_i|$, $i=\overline{1,m}$, $r_i'+r_j' < |a_i-a_j|$, $i,j=\overline{1,m}$, $i\neq j$. This completes the Proof of the Theorem 3.

Similarly to [6], we define the k-positive operator in space $A(D)$.

Definition 2. Linear operator $T : A(D) \to A(D)$ is called k-positive if it preserve the subclass of analytic functions with positive coefficients in the expansion of functions in series (1).

It is obvious that the k-positiveness of the operator

$$(Tf)(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} \left(\sum_{j=0}^{m} \sum_{p=0}^{\infty} T_{k,p,(i,j)} f_p^{(j)} \right) (z - a_i)^{\omega_i(k)},$$

where $f(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} f_k^{(i)} (z - a_i)^{\omega_i(k)} \in A(D)$, is equivalent to the non-negativeness of the coefficients $T_{k,p,(i,j)}$, $i, j = \overline{0, m}$, $k, p \in Z_+$.

Let the sequences of positive numbers $g = \left\{ \left\{ g_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0, m} \right\}$ satisfy the conditions (5), (6). Denote

$$h_\nu(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} \left(g_k^{(i)} \right)^{\frac{\nu}{2}} \alpha_i^k (z - a_i)^{\omega_i(k)}, \nu = 0, 1, 2. \tag{22}$$

Theorem 4. Let $T_n : A(D) \to A(D)$ be a sequence of linear k-positive operators. The sequence $T_n f(z)$ tends to $f(z)$ in $A(D)$ for each function $f \in A_g(D)$ if and only if

$$\lim_{n \to \infty} T_n h_\nu(z) = h_\nu(z)$$

in $A(D)$ for $\nu = 0, 1, 2$.

Proof. By Theorem 1, the convergence to zero of the sequence $T_n h_\nu - h_\nu$, $\nu = 0, 1, 2$ in $A(D)$ and the positiveness of the operators $T_n : A(D) \to A(D)$ imply the inequalities (9) – (12). From here and from theorem 2 we receive the theorem. This completes the Proof of the Theorem 4.

Let the sequences of positive numbers $b = \left\{ \left\{ b_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0, m} \right\}$ and $g = \left\{ \left\{ g_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0, m} \right\}$ satisfy the conditions (5), (6). Denote

$$H_\nu(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} \left(b_k^{(i)} \right)^{\frac{\nu}{2}} g_k^{(i)} \alpha_i^k (z - a_i)^{\omega_i(k)}, \nu = 0, 1, 2. \tag{23}$$

Theorem 5. Let $T_n : A(D) \to A(D)$ be a sequence of linear k-positive operators. The sequence $T_n f(z)$ tends to $f(z)$ in $A(D)$ for each function $f \in A_g(D)$ if and only if

$$\lim_{n \to \infty} T_n H_\nu(z) = H_\nu(z)$$

in $A(D)$ for $\nu = 0, 1, 2$.

Proof. By Theorem 1, the convergence to zero of the sequence $T_n H_\nu - H_\nu$, $\nu = 0, 1, 2$ in $A(D)$ and the positiveness of the operators $T_n : A(D) \to A(D)$ imply the inequalities (16) – (18). From here and by theorem 3 we receive the theorem. This completes the Proof of the Theorem 5.

Following the terminology of the theory of approximation of functions of real variable by positive linear operators (see [14]), we can assert that the system of three functions $\{h_0(z), h_1(z), h_2(z)\}$ (or $\{H_0(z), H_1(z), H_2(z)\}$) by theorems 4 and 5 forms a Korovkin system in $A(D)$. The next theorem shows that the system of functions $(z-a_i)^{\omega_i(k)}$, $i=\overline{0,m}$, $k\in Z_+$ does not form a Korovkin system in $A(D)$.

Theorem 6. For every sequence of positive numbers $g=\left\{\left\{g_k^{(i)}\right\}_{k=0}^{\infty} : i=\overline{0,m}\right\}$ satisfying the conditions (5), (6) there exists a sequence of k-positive operators $W_n : A(D)\to A(D)$ such that the sequence of functions $W_n\left((z-a_i)^{\omega_i(k)}\right)-(z-a_i)^{\omega_i(k)}$ converges to zero in $A(D)$ for every $i=\overline{0,m}$, $k\in Z_+$, and there exists a function $f^*\in A_g(D)$ such that

$$\lim_{n\to\infty}\|W_n f^*(z)-f^*(z)\|_{A(D),r_1',\dots,r_m',R'}=$$

$$=\left\|\frac{R}{R-(z-a_0)}+\sum_{i=1}^{m}\frac{(z-a_i)}{(z-a_i)-r_i}\right\|_{A(D),r_1',\dots,r_m',R'}>0 \tag{24}$$

for every $\max\limits_{i=\overline{1,m}}[|a_0-a_i|+r_i]<R'<R$, $r_i<r_i'<R'-|a_0-a_i|$, $i=\overline{1,m}$, $r_i'+r_j'<|a_i-a_j|$, $i,j=\overline{1,m}$, $i\neq j$.

Proof. Let

$$W_{k,p,(0,j)}^{(n)}=\delta_{0,j}\cdot\delta_{k,p}+\frac{1}{n^{k+1}}\begin{pmatrix}p\\k\end{pmatrix}\left(\frac{n}{n+1}\right)^p\frac{1}{g_p^{(i)}\alpha_j^{p-k}}\left[\delta_{0,j}(1-\delta_{0,k})+\delta_{0,k}\right],$$

$$j=\overline{0,m},k,p\in Z_+,$$

$$W_{k,p,(i,j)}^{(n)}=\delta_{i,j}\left[\delta_{k,p}+\frac{1}{n^{k+2}}\begin{pmatrix}p\\k+1\end{pmatrix}\left(\frac{n}{n+1}\right)^p\frac{1}{g_p^{(j)}\alpha_j^{p-k-1}}\right], i=\overline{1,m}, j=\overline{0,m}, k,p\in Z_+,$$

where $\delta_{k,p}$ is the Kronecker symbol and $\begin{pmatrix}p\\k\end{pmatrix}=\frac{p!}{k!\cdot(p-k)!}$ if $p\geq k$, $\begin{pmatrix}p\\k\end{pmatrix}=0$ if $p<k$. The sequence of operators $W_n : A(D)\to A(D)$ we define as follows:

$$(W_n f)(z)=\sum_{i=0}^{m}\sum_{k=0}^{\infty}\left(\sum_{j=0}^{m}\sum_{p=0}^{\infty}W_{k,p,(i,j)}^{(n)}f_p^{(j)}\right)(z-a_i)^{\omega_i(k)},$$

where $f(z)=\sum_{i=0}^{m}\sum_{k=0}^{\infty}f_k^{(i)}(z-a_i)^{\omega_i(k)}\in A(D)$. Then we have

$$(W_n f)(z)=f(z)+\frac{1}{n}\left[\sum_{k=0}^{\infty}\left(\frac{n+R^{-1}(z-a_0)}{n+1}\right)^k\frac{f_k^{(0)}R^k}{g_k^{(0)}}+\sum_{i=1}^{m}\sum_{k=1}^{\infty}\left(\frac{n+(z-a_i)^{-1}}{n+1}\right)^k\frac{f_k^{(i)}}{g_k^{(i)}r_i^k}\right].$$

This shows that the operators W_n are linear k-positive operators from $A(D)$ to $A(D)$. As

$$(W_n)\left((z-a_0)^k\right) = (z-a_0)^k + \frac{1}{n}\left(\frac{n+R^{-1}(z-a_0)}{n+1}\right)^k \frac{R^k}{g_k^{(0)}}, k \in Z_+,$$

$$(W_n)\left((z-a_i)^{-k}\right) = (z-a_i)^{-k} + \frac{1}{n}\left(\frac{n+r_i(z-a_i)^{-1}}{n+1}\right)^{k-1} \frac{1}{g_{k-1}^{(i)} r_i^{k-1}}, i = \overline{1,m}, k \in N,$$

therefore,

$$\left\|W_n\left((z-a_0)^k\right) - (z-a_0)^k\right\|_{A(D), r_1', \ldots, r_m', R'} \le \frac{1}{n} \cdot \frac{R^k}{g_k^{(0)}}, k \in Z_+,$$

$$\left\|W_n\left((z-a_i)^{-k}\right) - (z-a_i)^{-k}\right\|_{A(D), r_1', \ldots, r_m', R'} \le \frac{1}{n} \cdot \frac{1}{g_{k-1}^{(i)} r_i^{k-1}}, i = \overline{1,m}, k \in N,$$

and the sequence of functions $W_n\left((z-a_i)^{\omega_i(k)}\right) - (z-a_i)^{\omega_i(k)}$ converges to zero in $A(D)$ for every $i = \overline{0,m}$, $k \in Z_+$.

Consider the function

$$f^*(z) = \sum_{i=0}^{m} \sum_{k=0}^{\infty} \alpha_i^k g_k^{(i)} (z-a_i)^{\omega_i(k)} \in A(D).$$

For this function we have

$$(W_n f^*)(z) = f^*(z) + \frac{1}{n}\left(\sum_{k=0}^{\infty}\left(\frac{n+R^{-1}(z-a_0)}{n+1}\right)^k + \sum_{i=1}^{m}\sum_{k=0}^{\infty}\left(\frac{n+r_i(z-a_i)^{-1}}{n+1}\right)^k\right) =$$

$$= f^*(z) + \frac{n+1}{n}\left[\frac{R}{R-(z-a_0)} + \sum_{i=1}^{m}\frac{z-a_i}{(z-a_i)-r_i}\right].$$

The latter implies (24). This completes the Proof of the Theorem 6.

3. Statistical approximation by linear operators in the space of analytic functions.

Using the methods of the work [13], it is not difficult to obtain statistical analogues of above theorems. Let us first recall

Definition 3 [6]. A sequence x_n is said to be statistically convergent to a number x if for every $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{|\{k \le n : |x_k - x| > \varepsilon\}|}{n} = 0,$$

where $|\{k \le n : |x_k - x| > \varepsilon\}|$ be the number of all $k \le n$, for which $|x_k - x| > \varepsilon$. In this case we write $st. - \lim\limits_{n\to\infty} x_n = x$.

Theorem 7. The sequence $f_n(z)$ statistically convergence to zero in $A(D)$ if and only if the coefficients of expansion

$$f_n(z) = \sum_{i=0}^{m}\sum_{k=0}^{\infty} f_{n,k}^{(i)} (z - a_i)^{\omega_i(k)}$$

satisfy the conditions

$$\left|f_{n,k}^{(i)}\right| < \varepsilon_n (1 + \delta_n)^k \alpha_i^k$$

for any $i = \overline{1, m}$, $k \in Z_+$ and $n \in N$, where

$$st. - \lim_{n\to\infty} \varepsilon_n = 0, \lim_{n\to\infty} \delta_n = 0. \tag{25}$$

Proof. The sufficiency follows from inequality (4). We prove the necessity. Under the terms of the theorem follows that for any $\delta > 0$ and $\varepsilon > 0$ the equation

$$\lim_{n\to\infty} \frac{1}{n}\left|\left\{k \le n \,:\, \|f_k\|_{A(D_0),\, r_1(1+\delta),\dots,\, r_m(1+\delta),\, \frac{R}{1+\delta}} > \varepsilon\right\}\right| = 0. \tag{26}$$

holds. Take $\delta = \varepsilon = \frac{1}{p}$, $p = 1, 2, 3, ...$, from (26) we will receive that, there exists a sequence $n_1 < n_2 < n_3 < .... < n_p < ...$ that, if $n \ge n_p$, then

$$\frac{1}{n}\left|\left\{k \le n \,:\, \|f_k\|_{A(D),\, r_1\left(1+\frac{1}{p}\right),\dots,\, r_m\left(1+\frac{1}{p}\right),\, \frac{pR}{1+p}} > \frac{1}{p}\right\}\right| < \frac{1}{p}. \tag{27}$$

Denote $\delta_n = \frac{1}{p}$, $\varepsilon_n = \|f_n\|_{A(D_0),\, r_1(1+\delta_k),\dots,\, r_m(1+\delta_k),\, \frac{R}{1+\delta_k}}$ at $n \in \overline{n_p, n_{p+1} - 1}$, $p = 1, 2, 3, ...$ From inequality (27) it follows that $st. - \lim\limits_{n\to\infty} \varepsilon_n = 0$ and $\lim\limits_{n\to\infty} \delta_n = 0$. Then for $k \in Z_+$ the equality

$$f_{n,k}^{(0)} = \frac{1}{2\pi i}\int_{|z-a_0|=R(1+\delta_n)^{-1}} f_n(z)(z - a_0)^{-k-1} dz$$

implies the estimate

$$\left|f_{n,k}^{(0)}\right| = \frac{1}{2\pi}\left|\int_{|z-a_0|=R(1+\delta_n)^{-1}} f_n(z)(z - a_0)^{-k-1} dz\right| \le \varepsilon_n (1 + \delta_n)^k R^{-k},$$

and for $k \in Z_+$, $i = \overline{1, m}$ the equality

$$f_{n,k}^{(i)} = \frac{1}{2\pi i}\int_{|z-a_i|=r_i(1+\delta_n)} f_n(z)(z - a_i)^k dz$$

implies the estimate

$$\left|f_{n,k}^{(i)}\right| = \frac{1}{2\pi}\left|\int_{|z-a_i|=r_i(1+\delta_n)} f_n(z)(z - a_i)^k dz\right| \le \varepsilon_n (1 + \delta_n)^{k+1} r_i^{k+1}.$$

This completes the Proof of the Theorem 7.

Theorem 8. Let the sequences of positive numbers $g = \left\{ \left\{ g_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0,m} \right\}$ satisfy the conditions (5), (6) and $T_n : A(D) \to A(D)$ be a linear operators defined as (8). If there exist sequences ε_n and δ_n satisfying (25) such that the inequalities (9) – (12) holds, then for any function $f \in A_g(D)$ and for every $\max\limits_{i=\overline{1,m}} [|a_0 - a_i| + r_i] < R' < R$, $r_i < r_i' < R' - |a_0 - a_i|$, $i = \overline{1,m}$, $r_i' + r_j' < |a_i - a_j|$, $i, j = \overline{1,m}$, $i \neq j$ we have

$$st. - \lim_{n\to\infty} \|T_n f - f\|_{A(D),\, r_1',\ldots,r_m',R'} = 0.$$

The proof is similar to the proof of the theorem 2 with use the theorem 7.

Theorem 9. Let the sequences of positive numbers $b = \left\{ \left\{ b_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0,m} \right\}$ and $g = \left\{ \left\{ g_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0,m} \right\}$ satisfy the conditions (5), (6) and $T_n : A(D) \to A(D)$ be a linear operators defined as (8). If there exist sequences ε_n and δ_n satisfying (25) such that the inequalities (16) – (18) holds, then for any function $f \in A_g(D)$ and for every $\max\limits_{i=\overline{1,m}} [|a_0 - a_i| + r_i] < R' < R$, $r_i < r_i' < R' - |a_0 - a_i|$, $i = \overline{1,m}$, $r_i' + r_j' < |a_i - a_j|$, $i, j = \overline{1,m}$, $i \neq j$ we have

$$st. - \lim_{n\to\infty} \|T_n f - f\|_{A(D),\, r_1',\ldots,r_m',R'} = 0.$$

The proof is similar to the proof of the theorem 3 with use the theorem 7.

Theorem 10. Let the sequences of positive numbers $g = \left\{ \left\{ g_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0,m} \right\}$ satisfy the conditions (5), (6) and $T_n : A(D) \to A(D)$ be a sequence of linear k-positive operators. The sequence $T_n f(z)$ statistically tends to $f(z)$ in $A(D)$ for each function $f \in A_g(D)$ if and only if

$$st. - \lim_{n\to\infty} T_n h_\nu(z) = h_\nu(z)$$

in $A(D)$ for $\nu = 0, 1, 2$, where $h_\nu(z)$ defined in (22).

The proof is similar to the proof of the theorem 4 with use the theorem 7 and 8.

Theorem 11. Let the sequences of positive numbers $b = \left\{ \left\{ b_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0,m} \right\}$ and $g = \left\{ \left\{ g_k^{(i)} \right\}_{k=0}^{\infty} : i = \overline{0,m} \right\}$ satisfy the conditions (5), (6) and $T_n : A(D) \to A(D)$ be a sequence of linear k-positive operators. The sequence $T_n f(z)$ statistically tends to $f(z)$ in $A(D)$ for each function $f \in A_g(D)$ if and only if

$$st. - \lim_{n\to\infty} T_n H_\nu(z) = H_\nu(z)$$

in $A(D)$ for $\nu = 0, 1, 2$, where $H_\nu(z)$ defined in (23).

The proof is similar to the proof of the theorem 5 with use the theorems 7 and 9.

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Received
Accepted