Journal of Contemporary Applied Mathematics
V. 6, No 2, 2016, December
ISSN 2222-5498

Some Common Fixed Point Theorems for Generalized Contraction Involving Rational Expressions in b-metric Spaces

G. S. Saluja

Abstract. In this paper, we establish some common fixed point theorems via generalized contraction involving rational expressions for a pair of mappings in the setting of b-metric spaces. Also, as a consequence, some results of integral type for such class of mappings is obtained. Our results extend and generalize several known results from the existing literature.
Key Words and Phrases: Common fixed point, generalized contraction involving rational expressions, b-metric space.
2010 Mathematics Subject Classifications: 47H10, 54H25.

1. Introduction and Preliminaries

Fixed point theory has the diverse applications in different branch of mathematics, statistics, engineering, and economics in dealing with the problem arising in approximation theory, potential theory, game theory, theory of differential equations, theory of integral equations and others. The Banach contraction mapping [2] is one of the pivotal results of analysis. It is a very popular tool in solving existence problems in different fields of mathematics. This famous theorem can be stated as follows.

Theorem 1.1. *([2]) Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying:*

$$d(Tx, Ty) \quad \leq \quad k\, d(x, y), \ \forall\, x, y \in X \tag{1.1}$$

where k is a constant in $[0, 1)$. Then T has a fixed point $x^* \in X$.

There are many generalizations of the Banach contraction principle in literature (see, [3, 4, 5, 11, 14, 15]).

In [1], Bakhtin introduced b-metric spaces as a generalization of metric spaces. He proved the contraction mapping principle in b-metric spaces that generalized the famous contraction principle in metric spaces. Czerwik used the concept of b-metric space and generalized the renowned Banach fixed point theorem in b-metric spaces (see, [7, 8]).

In this paper, we study common fixed point in b-metric spaces satisfying generalized contraction involving rational expressions. Our results extend many known results from the existing literature.

Definition 1.1. *Let X be a nonempty set and let $d\colon X \times X \to R_+$ be a function satisfying the conditions:*

(d_1) $0 \le d(x,y)$ *with* $x \ne y$ *and* $d(x,y) = 0 \Leftrightarrow x = y$;

(d_2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;

(d_3) $d(x,y) \le d(x,z) + d(z,y)$ *for all* $x, y, z \in X$.

The d is called a metric on X and the pair (X,d) is called a metric space.

Definition 1.2. *([1]) Let X be a nonempty set and $s \ge 1$ be a given real number. A mapping $d\colon X \times X \to R_+$ is called a b-metric if for all x, y, $z \in X$, the following conditions are satisfied:*

$(b1M)$ $d(x,y) = 0 \Leftrightarrow x = y$;

$(b2M)$ $d(x,y) = d(y,x)$;

$(b3M)$ $d(x,y) \le s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a b-metric space.

It is clear from the definition of b-metric space that every metric space is a b-metric space for $s = 1$. Therefore, the class of b-metric spaces is larger than the class of metric spaces.

Some known examples of b-metric, which shows that a b-metric space is a generalization of a metric space, are the following.

Example 1. *([9]) The set of real numbers together with the functional $d(x,y) = |x-y|^2$ for all $x, y \in \mathbb{R}$ is a b-metric space with constant $s = 2$. Also, we obtain that d is not a metric on $\mathbb{R}$.*

Example 2. *([10]) Let $X = \ell^p$ with $0 < p < 1$, where $\ell^p = \{\{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty\}$. Let $d\colon X \times X \to \mathbb{R}$ defined by $d(x,y) = \Big(\sum_{n=1}^{\infty} |x_n - y_n|^p\Big)^{\frac{1}{p}}$, where $x = \{x_n\}$, $y = \{y_n\} \in \ell^p$. Then (X,d) is a b-metric space with the coefficient $s = 2^{1/p} > 1$, but not a metric space.*

Example 3. *([13]) Let p be a given real number in the interval $(0,1)$. The space $L_p[0,1]$ of all real functions $x(t), y(t) \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < 1$, together with the functional*

$$d(x,y) = \Big(\int_0^1 |x(t) - y(t)|^p dt\Big)^{1/p}$$

for each $x, y \in L_p[0,1]$ is a b-metric space with the coefficient $s = 2^{1/p} > 1$.

Example 4. *([9]) Let $X = \{0,1,2\}$. Define $d\colon X \times X \to \mathbb{R}_+$ as follows $d(0,0) = d(1,1) = d(2,2) = 0$, $d(1,2) = d(2,1) = d(0,1) = d(1,0) = 1$, $d(2,0) = d(0,2) = p \ge 2$ for $s = \frac{p}{2}$ where $p \ge 2$, the function defined as above is a b-metric space but not a metric space for $p > 2$.*

Example 5. *([9]) Let $X = \{1,2,3,4\}$ and $E = \mathbb{R}^2$. Define $d\colon X \times X \to \mathbb{R}$ by*

$$d(x,y) = \begin{cases} (|x-y|^{-1}, |x-y|^{-1}) & \text{if } x \ne y, \\ 0, & \text{if } x = y. \end{cases}$$

Then (X,d) is a b-metric space with the coefficient $s = \frac{6}{5} > 1$. But it is not a metric space since the triangle inequality is not satisfied,

$$d(1,2) > d(1,4) + d(4,2), \quad d(3,4) > d(3,1) + d(1,4).$$

In our main result we will use the following definitions which can be found in [9] and [13].

Definition 1.3. *Let (X,d) be a b-metric space, $x \in X$ and $\{x_n\}$ be a sequence in X. Then*

- *$\{x_n\}$ is a Cauchy sequence whenever, if for $\varepsilon > 0$, there exists a positive integer N such that for all $n, m \geq N$, $d(x_n, x_m) < \varepsilon$;*
- *$\{x_n\}$ is called convergent if for $\varepsilon > 0$ and $n \geq N$, we have $d(x_n, x) < \varepsilon$, where x is called the limit point of the sequence $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$;*
- *(X,d) is said to be a complete b-metric space if every Cauchy sequence in X converges to a point in x.*

Remark 1.1. *In a b-metric space (X,d), the following assertions hold:*
(i) a convergent sequence has a unique limit;
(ii) each convergent sequence is Cauchy;
(iii) in general, a b-metric is not continuous.

2. Main Results

In this section we shall prove some common fixed point theorems for generalized contraction involving rational expressions in the framework of b-metric spaces.

Theorem 2.2. *Let (X,d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mappings $S, T\colon X \to X$ satisfy:*

$$\begin{aligned} d(Sx,Ty) \leq\ & a_1\, d(x,y) + \frac{a_2[1 + d(x,Sx)]d(y,Ty)}{1 + d(x,y)} \\ & + a_3\,[d(x,Sx) + d(y,Ty)] \\ & + a_4\,[d(x,Ty) + d(y,Sx)] \end{aligned} \tag{2.1}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1 + a_2 + (s+1)a_3 + s(s+1)a_4 < 1$. Then S and T have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X and define

$$x_{2k+1} = Sx_{2k},\ x_{2k+2} = Tx_{2k+1},\ k = 0, 1, 2, \ldots.$$

Then from (2.1), we have

$$d(x_{2k+1}, x_{2k+2}) = d(Sx_{2k}, Tx_{2k+1})$$

$$
\begin{aligned}
&\leq a_1\, d(x_{2k}, x_{2k+1}) + \frac{a_2[1 + d(x_{2k}, Sx_{2k})]d(x_{2k+1}, Tx_{2k+1})}{1 + d(x_{2k}, x_{2k+1})} \\
&\quad + a_3\,[d(x_{2k}, Sx_{2k}) + d(x_{2k+1}, Tx_{2k+1})] \\
&\quad + a_4\,[d(x_{2k}, Tx_{2k+1}) + d(x_{2k+1}, Sx_{2k})] \\
&= a_1\, d(x_{2k}, x_{2k+1}) + \frac{a_2[1 + d(x_{2k}, x_{2k+1})]d(x_{2k+1}, x_{2k+2})}{1 + d(x_{2k}, x_{2k+1})} \\
&\quad + a_3\,[d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] \\
&\quad + a_4\,[d(x_{2k}, x_{2k+2}) + d(x_{2k+1}, x_{2k+1})] \\
&= (a_1 + a_3)\, d(x_{2k}, x_{2k+1}) + (a_2 + a_3)\, d(x_{2k+1}, x_{2k+2}) \\
&\quad + a_4\, d(x_{2k}, x_{2k+2}) \\
&\leq (a_1 + a_3)\, d(x_{2k}, x_{2k+1}) + (a_2 + a_3)\, d(x_{2k+1}, x_{2k+2}) \\
&\quad + sa_4\,[d(x_{2k}, x_{2k+1}) + d(x_{2k+1}, x_{2k+2})] \\
&= (a_1 + a_3 + sa_4)\, d(x_{2k}, x_{2k+1}) \\
&\quad + (a_2 + a_3 + sa_4)\, d(x_{2k+1}, x_{2k+2}). \qquad (2.2)
\end{aligned}
$$

This implies that

$$
d(x_{2k+1}, x_{2k+2}) \leq \left(\frac{a_1 + a_3 + sa_4}{1 - a_2 - a_3 - sa_4}\right) d(x_{2k}, x_{2k+1}). \qquad (2.3)
$$

Similarly, we have

$$
\begin{aligned}
d(x_{2k+2}, x_{2k+3}) &= d(Sx_{2k+1}, Tx_{2k+2}) \\
&\leq a_1\, d(x_{2k+1}, x_{2k+2}) + \frac{a_2[1 + d(x_{2k+1}, Sx_{2k+1})]d(x_{2k+2}, Tx_{2k+2})}{1 + d(x_{2k+1}, x_{2k+2})} \\
&\quad + a_3\,[d(x_{2k+1}, Sx_{2k+1}) + d(x_{2k+2}, Tx_{2k+2})] \\
&\quad + a_4\,[d(x_{2k+1}, Tx_{2k+2}) + d(x_{2k+2}, Sx_{2k+1})] \\
&= a_1\, d(x_{2k+1}, x_{2k+2}) + \frac{a_2[1 + d(x_{2k+1}, x_{2k+2})]d(x_{2k+2}, x_{2k+3})}{1 + d(x_{2k+1}, x_{2k+2})} \\
&\quad + a_3\,[d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})] \\
&\quad + a_4\,[d(x_{2k+1}, x_{2k+3}) + d(x_{2k+2}, x_{2k+2})] \\
&= (a_1 + a_3)\, d(x_{2k+1}, x_{2k+2}) + (a_2 + a_3)\, d(x_{2k+2}, x_{2k+3}) \\
&\quad + a_4\, d(x_{2k+1}, x_{2k+3}) \\
&\leq (a_1 + a_3)\, d(x_{2k+1}, x_{2k+2}) + (a_2 + a_3)\, d(x_{2k+2}, x_{2k+3}) \\
&\quad + sa_4\,[d(x_{2k+1}, x_{2k+2}) + d(x_{2k+2}, x_{2k+3})] \\
&= (a_1 + a_3 + sa_4)\, d(x_{2k+1}, x_{2k+2}) \\
&\quad + (a_2 + a_3 + sa_4)\, d(x_{2k+2}, x_{2k+3}). \qquad (2.4)
\end{aligned}
$$

This implies that

$$
d(x_{2k+2}, x_{2k+3}) \leq \left(\frac{a_1 + a_3 + sa_4}{1 - a_2 - a_3 - sa_4}\right) d(x_{2k+1}, x_{2k+2}). \qquad (2.5)
$$

Putting

$$\theta = \Big(\frac{a_1 + a_3 + sa_4}{1 - a_2 - a_3 - sa_4}\Big).$$

AS $a_1 + a_2 + (s+1)a_3 + s(s+1)a_4 < 1$, it follows that $0 < \theta < 1/s$.

By induction, we have

$$\begin{aligned} d(x_{n+1}, x_n) &\leq \theta\, d(x_n, x_{n-1}) \leq \theta^2\, d(x_{n-1}, x_{n-2}) \leq \dots \\ &\quad \theta^n d(x_1, x_0). \end{aligned} \tag{2.6}$$

Let $B_n = d(x_{n+1}, x_n)$, $B_0 = d(x_1, x_0)$, then (2.6), reduces to

$$B_n \leq \theta^n B_0. \tag{2.7}$$

Hence for any $m, n \geq 1$ and $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_m)] \\ &= sd(x_n, x_{n+1}) + sd(x_{n+1}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)] \\ &= sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m) \\ &\leq sd(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) \\ &\quad + \dots + s^{n+m-1} d(x_{n+m-1}, x_m) \\ &\leq [s\theta^n + s^2\theta^{n+1} + s^3\theta^{n+2} + \dots + s^m\theta^{n+m-1}]B_0 \\ &= s\theta^n[1 + s\theta + s^2\theta^2 + s^3\theta^3 + \dots + (s\theta)^{m-1}]B_0 \\ &\leq \Big[\frac{s\theta^n}{1 - s\theta}\Big]B_0. \end{aligned}$$

Since $s\theta < 1$, therefore taking limit $m, n \to \infty$, we have

$$\lim_{m,n\to\infty} d(x_m, x_n) = 0.$$

Hence $\{x_n\}$ is a Cauchy sequence in complete b-metric space X. Since X is complete, so there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$. Now, we have to show that u is a common fixed point of S and T. For this consider

$$\begin{aligned} d(x_{2n+1}, Tu) &= d(Sx_{2n}, Tu) \\ &\leq a_1\, d(x_{2n}, u) + \frac{a_2[1 + d(x_{2n}, Sx_{2n})]d(u, Tu)}{1 + d(x_{2n}, u)} \\ &\quad + a_3\,[d(x_{2n}, Sx_{2n}) + d(u, Tu)] \\ &\quad + a_4\,[d(x_{2n}, Tu) + d(u, Sx_{2n})] \\ &\leq a_1\, d(x_{2n}, u) + \frac{a_2[1 + d(x_{2n}, x_{2n+1})]d(u, Tu)}{1 + d(x_{2n}, u)} \\ &\quad + a_3\,[d(x_{2n}, x_{2n+1}) + d(u, Tu)] \end{aligned}$$

$$+a_4\left[d(x_{2n},Tu)+d(u,x_{2n+1})\right]$$

Taking limit $n\to\infty$, we have

$$d(u,Tu)\leq(a_1+a_3+a_4)d(u,Tu).$$

The above inequality is possible only if $d(u,Tu)=0$ and so $Tu=u$. Thus u is a fixed point of T.

In an exactly the same fashion we can prove that $Su=u$. Hence $Su=Tu=u$. This shows that u is a common fixed point of S and T.

Uniqueness

Let v be another common fixed point of S and T, that is, $Sv=Tv=v$ such that $u\neq v$. Then from (2.1), we have

$$\begin{aligned}
d(u,v) &= d(Su,Tv)\\
&\leq a_1\,d(u,v)+\frac{a_2[1+d(u,Su)]d(v,Tv)}{1+d(u,v)}\\
&\quad +a_3\left[d(u,Su)+d(v,Tv)\right]\\
&\quad +a_4\left[d(u,Tv)+d(v,Su)\right]\\
&= a_1\,d(u,v)+\frac{a_2[1+d(u,u)]d(v,v)}{1+d(u,v)}\\
&\quad +a_3\left[d(u,u)+d(v,v)\right]\\
&\quad +a_4\left[d(u,v)+d(v,u)\right]\\
&= (a_1+2a_4)d(u,v).
\end{aligned}$$

The above inequality is possible only if $d(u,v)=0$ and so $u=v$.Thus u is a unique common fixed point of S and T. This completes the proof. □

Putting $S=T$ in Theorem 2.2, then we have the following result.

Corollary 2.1. *Let (X,d) be a complete b-metric space (CbMS) with the coefficient $s\geq 1$. Suppose that the mapping $T\colon X\to X$ satisfies:*

$$\begin{aligned}
d(Tx,Ty) &\leq a_1\,d(x,y)+\frac{a_2[1+d(x,Tx)]d(y,Ty)}{1+d(x,y)}\\
&\quad +a_3\left[d(x,Tx)+d(y,Ty)\right]\\
&\quad +a_4\left[d(x,Ty)+d(y,Tx)\right] \qquad (2.8)
\end{aligned}$$

for all $x,y\in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1+a_2+(s+1)a_3+s(s+1)a_4<1$. Then T has a unique fixed point in X.

Proof. The proof of corollary 2.1 is immediately follows from Theorem 1.1 by taking $S=T$. This completes the proof. □

Corollary 2.2. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies (for fixed n):*

$$\begin{aligned} d(T^n x, T^n y) &\leq a_1\, d(x,y) + \frac{a_2[1 + d(x, T^n x)]d(y, T^n y)}{1 + d(x,y)} \\ &\quad + a_3\,[d(x, T^n x) + d(y, T^n y)] \\ &\quad + a_4\,[d(x, T^n y) + d(y, T^n x)] \end{aligned} \tag{2.9}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1 + a_2 + (s+1)a_3 + s(s+1)a_4 < 1$. Then T has a unique fixed point in X.

Proof. By Corollary 2.1, there exists $u \in X$ such that $T^n u = u$. Then

$$\begin{aligned} d(Tu, u) &= d(TT^n u, T^n u) = d(T^n Tu, T^n u) \\ &\leq a_1\, d(Tu, u) + \frac{a_2[1 + d(Tu, T^n Tu)]d(u, T^n u)}{1 + d(Tu, u)} \\ &\quad + a_3\,[d(Tu, T^n Tu) + d(u, T^n u)] \\ &\quad + a_4\,[d(Tu, T^n u) + d(u, T^n Tu)] \\ &= a_1\, d(Tu, u) + \frac{a_2[1 + d(Tu, TT^n u)]d(u, T^n u)}{1 + d(Tu, u)} \\ &\quad + a_3\,[d(Tu, TT^n u) + d(u, T^n u)] \\ &\quad + a_4\,[d(Tu, T^n u) + d(u, TT^n u)] \\ &= a_1\, d(Tu, u) + \frac{a_2[1 + d(Tu, Tu)]d(u, u)}{1 + d(Tu, u)} \\ &\quad + a_3\,[d(Tu, Tu) + d(u, u)] \\ &\quad + a_4\,[d(Tu, u) + d(u, Tu)] \\ &= (a_1 + 2a_4)\, d(Tu, u). \end{aligned}$$

The above inequality is possible only if $d(Tu, u) = 0$ and so $Tu = u$. This shows that T has a unique fixed point in X. This completes the proof. □

Putting $a_1 = k$, $a_2 = a_3 = a_4 = 0$ in Corollary 2.1, then we have the following result.

Corollary 2.3. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$d(Tx, Ty) \leq k\, d(x, y)$$

for all $x, y \in X$, where $k \in (0, 1)$ is a constant with $sk < 1$. Then T has a unique fixed point in X.

Remark 2.2. *Corollary 2.3 extends well known Banach contraction principle from complete metric space to that setting of complete b-metric space considered in this paper.*

Putting $a_3 = b$, $a_1 = a_2 = a_4 = 0$ in Corollary 2.1, then we have the following result.

Corollary 2.4. *Let (X,d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$d(Tx,Ty) \leq b\,[d(x,Tx)+d(y,Ty)]$$

for all $x,y \in X$, where $b \in (0,\frac{1}{2})$ is a constant with $(s+1)b<1$. Then T has a unique fixed point in X.

Remark 2.3. *Corollary 2.4 extends Kannan [12] contraction from complete metric space to that setting of complete b-metric space considered in this paper.*

Putting $a_4 = c$, $a_1 = a_2 = a_3 = 0$ in Corollary 2.1, then we have the following result.

Corollary 2.5. *Let (X,d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$d(Tx,Ty) \leq c\,[d(x,Ty)+d(y,Tx)]$$

for all $x,y \in X$, where $c \in (0,\frac{1}{2})$ is a constant with $s(s+1)c<1$. Then T has a unique fixed point in X.

Remark 2.4. *Corollary 2.5 extends Chatterjea [6] contraction from complete metric space to that setting of complete b-metric space considered in this paper.*

Other consequences of our results for the mappings involving contractions of integral type are the following.

Denote Λ the set of functions $\varphi\colon [0,\infty) \to [0,\infty)$ satisfying the following hypothesis:
(h1) φ is a Lebesgue-integrable mapping on each compact subset of $[0,\infty)$;
(h2) for any $\varepsilon > 0$ we have $\int_0^{\varepsilon} \varphi(t)dt > 0$.

Theorem 2.3. *Let (X,d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mappings S, $T\colon X \to X$ satisfy:*

$$\begin{aligned}\int_0^{d(Sx,Ty)} \psi(t)dt \leq\ & a_1 \int_0^{d(x,y)} \psi(t)dt + a_2 \int_0^{\frac{[1+d(x,Sx)]d(y,Ty)}{1+d(x,y)}} \psi(t)dt \\ & + a_3 \int_0^{[d(x,Sx)+d(y,Ty)]} \psi(t)dt \\ & + a_4 \int_0^{[d(x,Ty)+d(y,Sx)]} \psi(t)dt\end{aligned}$$

for all $x,y \in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1+a_2+(s+1)a_3+s(s+1)a_4<1$ and $\psi \in \Lambda$. Then S and T have a unique common fixed point in X.

Putting $S = T$ in Theorem 2.3, we have the following result.

Theorem 2.4. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\begin{aligned}\int_0^{d(Tx,Ty)} \psi(t)dt &\leq a_1 \int_0^{d(x,y)} \psi(t)dt + a_2 \int_0^{\frac{[1+d(x,Tx)]d(y,Ty)}{1+d(x,y)}} \psi(t)dt \\ &+ a_3 \int_0^{[d(x,Tx)+d(y,Ty)]} \psi(t)dt \\ &+ a_4 \int_0^{[d(x,Ty)+d(y,Tx)]} \psi(t)dt\end{aligned}$$

for all $x, y \in X$, where a_1, a_2, a_3, a_4 are nonnegative reals with $a_1 + a_2 + (s+1)a_3 + s(s+1)a_4 < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Putting $S = T$, $a_1 = k$ and $a_2 = a_3 = a_4 = 0$ in Theorem 2.3, we have the following result.

Theorem 2.5. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \leq k \int_0^{d(x,y)} \psi(t)dt$$

for all $x, y \in X$, where k is a nonnegative constant with $0 < sk < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Remark 2.5. *Theorem 2.5 extends Theorem 2.1 of Branciari [3] from complete metric space to that setting of complete b-metric space considered in this paper.*

Corollary 2.6. *([3], Theorem 2.1) Let (X, d) be a complete metric space (CMS). Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \leq k \int_0^{d(x,y)} \psi(t)dt$$

for all $x, y \in X$, where k is a nonnegative constant with $0 < k < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Putting $S = T$, $a_3 = \mu$ and $a_1 = a_2 = a_4 = 0$ in Theorem 2.3, we have the following result.

Theorem 2.6. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \leq \mu \int_0^{[d(x,Tx)+d(y,Ty)]} \psi(t)dt$$

for all $x, y \in X$, where μ is a nonnegative constant with $0 < \mu < 1/(s+1)$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Remark 2.6. *Theorem 2.6 extends Kannan [12] contraction from complete metric space to that setting of complete b-metric space for integral type contraction considered in this paper.*

Putting $S = T$, $a_4 = \delta$ and $a_1 = a_2 = a_3 = 0$ in Theorem 2.3, we have the following result.

Theorem 2.7. *Let (X, d) be a complete b-metric space (CbMS) with the coefficient $s \geq 1$. Suppose that the mapping $T\colon X \to X$ satisfies:*

$$\int_0^{d(Tx,Ty)} \psi(t)dt \quad \leq \quad \delta \int_0^{[d(x,Ty)+d(y,Tx)]} \psi(t)dt$$

for all $x, y \in X$, where δ is a nonnegative constant with $0 < s(s+1)\delta < 1$ and $\psi \in \Lambda$. Then T has a unique fixed point in X.

Remark 2.7. *Theorem 2.7 extends Chatterjae [6] contraction from complete metric space to that setting of complete b-metric space for integral type contraction considered in this paper.*

Now, we give some examples in support of our results.

Example 6. *Let $X = [0, 1]$ and $d(x, y) = |x - y|^2$. Then (X, d) is a b-metric space with the coefficient $s = 2$. We consider the mapping $T\colon X \to X$ defined by $T(x) = \frac{x}{3}$. Hence*

$$\begin{aligned} d(Tx, Ty) &= \left|\frac{x}{3} - \frac{y}{3}\right|^2 \\ &= \frac{1}{9}|x - y|^2 \\ &\leq \frac{1}{2}|x - y|^2 \\ &= \frac{1}{2}\, d(x, y). \end{aligned}$$

Clearly $0 \in X$ is the unique fixed point of T.

Example 7. *Let $X = [0, 1]$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$ be a b-metric space with the coefficient $s = 2$. We define the mapping $T\colon X \to X$ defined by*

$$T(x) = \frac{2}{3} \text{ if } x \in [0, 1) \text{ and } T(1) = 0$$

then T satisfies all the conditions of Corollary 2.1 for $a_1 = \frac{4}{9}$ and $a_2 = a_3 = a_4 = 0$ with $sa_1 + a_2 + (s+1)a_3 + s(s+1)a_4 < 1$ having $x = \frac{2}{3}$ is its unique fixed point in X.

Example 8. *Let $X = \{0, 1, 2, 3, 4\}$ and $d(x, y) = |x - y|^2$ for all $x, y \in X$ be a b-metric space with the coefficient $s = 2$, we define the mapping T by*

$$T(x) = \begin{cases} 3, & \text{if } x = 0, \\ 2, & \text{otherwise.} \end{cases}$$

Let us take $x = 0$, $y = 1$ and $s = 2$. Then from condition (2.8) we have

$$1 = d(Tx, Ty) \leq a_1 + 5a_2 + 10a_3 + 8a_4.$$

The above inequality is satisfied for $a_1 = a_2 = \frac{2}{11}$ and $a_3 = a_4 = 0$ with $sa_1 + a_2 + (s+1)a_3 + s(s+1)a_4 < 1$. Then all the conditions of corollary 2.1 are satisfied and 2 is of course a unique fixed point of T.

3. Conclusion

In this paper, we establish some common fixed point theorems for generalized contraction involving rational expressions in the setting of b-metric spaces. Also, as a consequence, we obtain some results of integral type contraction for such class of mappings. Our results extend and generalize several results from the existing literature.

References

[1] I.A. Bakhtin. The contraction mapping principle in almost metric spaces. *Funct. Anal. Gos. Ped. Inst. Unianowsk*, 30:26–37, 1989.

[2] S. Banach. Surles operation dans les ensembles abstraits et leur application aux equation integrals. *Fund. Math.*, 3:133–181, 1922.

[3] A. Branciari. A fixed point theorem for mapping satisfying a general contractive condition of integral type. *Int. J. Math. Math. Sci.*, 29:531–536, 2002.

[4] S. Chandok. Some fixed point theorems for generalized nonlinear contractive mappings. *Comput. Math. Appl.*, 62:3692–3699, 2011.

[5] S. Chandok. Some fixed points for generalized nonlinear contractive mappings in metric spaces. *Math. Vesnik*, 65:29–34, 2013.

[6] S.K. Chatterjae. Fixed point theorems . *comptes Rend. Acad. Bulgare Sci.*, 25:727–730, 1972.

[7] S. Czerwik. Contraction mappings in b-metric spaces. *Acta Math. Inf. Univ. Ostraviensis*, 1:5–11, 1993.

[8] S. Czerwik. Nonlinear set-valued contraction mappings in b-metric spaces. *Atti. Semin. Mat. Fis. Univ. Modena*, 46:263–276, 1998.

[9] E. Karapinar H. Ayadi, M.-F. Bota and S. Mitrovic. A fixed point theorem for set valued quasi contraction in b-metric spaces. *Fixed Point Theory Appl.*, 2012(Article ID 88), 2012.

[10] H. Huang and S. Xu. Fixed point theorems of contractive mappings in cone b-metric spaces and applications. *Fixed Point Theory Appl.*, 2013(Article ID 112):1–10, 2013.

[11] D. S. Jaggi. Some unique fixed point theorems. *Indian J. Pure Appl. Maths.*, 8:223–230, 1977.

[12] R. Kannan. Some results on fixed point theorems. *Bull. Calcutta Math. Soc.*, 60:71–78, 1969.

[13] M. Kir and H. Kizitune. On some well known fixed point theorems in b-metric spaces. *Turkish J. Anal. Number Theory*, 1:13–16, 2013.

[14] V. Berinde M. A. Alghamdi and N. Shahazad. Fixed points non-self almost contractions. *Carpathian J. Math.*, **30**(1):7–11, 2014.

[15] D. O'Regan R. P. Agarwal and D. R. Sahu. *Fixed point theory for Lipschitzian type mappings with applications*. Springer, New York, 2009.

G. S. Saluja
Department of Mathematics, Govt. Nagarjuna P. G. College of Science, Raipur - 492010 (C.G.), India.
E-mail: saluja1963@gmail.com

Received 24 October 2016
Published 29 November 2016