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The Eigenfunction Expansion of Singular Sturm-Liouville Problem with Sign-Valued Weight

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Abstract. In the present paper, we investigate an eigenfunction expansion associated to a singular Sturm-Liouville problem in finite interval with the sign valued weight by two ways, first by standard method, second by proving its convergence in some metric space $L_2(0, \pi; \rho(x))$.

1. Introduction

The subject of eigenfunction expansions is as old as operator theory which is important in quantum mechanics. The idea of expanding an arbitrary function in terms of the solution of a second order differential equation goes back to the time of Sturm-Liouville, more than a hundred years ago. Later a general theory of the singular cases was given by Weyl who based it on the theory of integral equation see [1], [2], subsequently generalized by Stone, Kodaria, Titchmarsh and others see [3]-[7]. E.C. Titchmarsh adopted another method which becomes the basic rule in the development of the spectral theory of singular differential operator which depends on contour integration and the calculus of residues were given by Cauchy. This method has been used by several authors in mathematics, geophysics ,and electromagnetic field see [8],[9]. Levitan in [10]gave anew method of getting the eigenfunction expansion in infinite interval by taking limit in regular case Also this method used by A.N.Techanov, M.G.Krien, M.G.Gasymov and their schools as they deal with sign valued weight coefficient. The purpose of this paper to calculate the eigenfunction expansion formula of (1)-(2)with sign valued function in the form (3)

$$-y'' + q(x)\, y = \lambda\, \rho(x)\, y \quad 0 \le x \le \pi \tag{1}$$

$$y(0) = 0 \quad , \quad y'(\pi) + H\, y(\pi) = 0, \tag{2}$$

where the non-negative real function $q(x)$ has a second piecewise integrable derivatives on $(0, \pi)$, H is positive number, λ is a spectral parameter and the weighted function or the explosive factor $\rho(x)$ is of the form

$$\rho(x) = \begin{cases} 1 \;; & 0 \le x \le a < \pi \\ -1 \;; & a < x \le \pi. \end{cases} \tag{3}$$

The form of $\rho(x)$ as ± 1 causes many analytical difficulties,see [11],[12] because we will deal with two separated problems and hence the eigenfunction expansion is written as two sums see [10] which is different from the formula of the eigenfunction expansion in [13] where the authors had taken different weight function which led to that the spectrum was both continuous and discreet so that its formula of eigenfunction expansion is written as a summation and integration.

2. Preliminaries

As a key ingredients of our work we mention some basic mathematical properties and definition which will be a great help for our work. As the author in [12] discussed the following basic mathematical background in two steps
First: let the functions $\varphi(x,\lambda)$, $\psi(x,\lambda)$ are solutions of (1) under the initial conditions

$$\varphi(0,\lambda) \ = \ 0, \quad \varphi'(0,\lambda) \ = \ 1 \tag{4}$$

$$\psi(\pi,\lambda) \ = \ 1, \quad \psi'(\pi,\lambda) \ = \ -H. \tag{5}$$

Second: the Wronskain of the two solutions $\varphi(x,\lambda)$, $\psi(x,\lambda)$ of (1)-(2) is defined by

$$W(\lambda) \ = \ \varphi(x,\lambda)\ \psi'(x,\lambda) \ - \ \varphi'(x,\lambda)\ \psi(x,\lambda). \tag{6}$$

And the eigenvalues of problem (1)-(2) coincide with the roots of the function $W(\lambda) = 0$ which can be written for $\lambda = \lambda_n$, $n = 0, 1, 2, ...$ as the following form

$$\varphi(a,\lambda_n)\ \psi'(a,\lambda_n) = \varphi'(a,\lambda_n)\ \psi(a,\lambda_n) \tag{7}$$

which implies that

$$\varphi'(a,\lambda_n) = \frac{\varphi(a,\lambda_n)\psi'(a,\lambda_n)}{\psi(a,\lambda_n)}, \tag{8}$$

$$\varphi'(a,\lambda_n) = \frac{\varphi'(a,\lambda_n)\psi(a,\lambda_n)}{\varphi(a,\lambda_n)}. \tag{9}$$

In the following lemma we discuss the reality of the eigenfunctions $\varphi(x,\lambda_n^{\pm}), \psi(x,\lambda_n^{\pm})$.
In [12] the author proved that the eigenvalues $\lambda_n^{\pm}$, $n = 0, 1, 2, ...$ of the problem (1)-(2) are real and corresponding eigenfunctions $\varphi(x,\lambda_n^{\pm})$, $\psi(x,\lambda_n^{\pm})$ are orthogonal with weight $\rho(x)$.
Lemma 2.1 Let $q(x)$ be real function, the eigenfunctions $\varphi(x,\lambda_n^{\pm})$, $\psi(x,\lambda_n^{\pm})$ of problem (1)-(2) are real.
Proof:
Let $\varphi(x,\lambda)$ be a solution of Sturm-Liouville problem (1), $x \in (0,a)$ which admit the initial condition (4), then

$$-\varphi''(x,\lambda) + q(x)\varphi(x,\lambda) = \lambda\ \varphi(x,\lambda). \tag{10}$$

Affecting by the complex conjugate for both sides of (10), we have

$$-\overline{\varphi''(x,\lambda)} \ + \ q(x)\ \overline{\varphi(x,\lambda)} \ = \ \lambda\overline{\varphi(x,\lambda)}. \tag{11}$$

Applying the uniqueness theorem, we obtain $\varphi(x,\lambda)=\overline{\varphi(x,\lambda)}$. By the same methodology we can proved that $\psi(x,\lambda)=\overline{\psi(x,\lambda)}$ where $\psi(x,\lambda)$ is the solution of (1), $x\in(a,\pi)$ which admit the initial condition (5).

In the following theorem we prove the simplicity of the eigenvalues which means that $\frac{d\ W(\lambda)}{d\ \lambda}=\ \dot{W}(\lambda)\neq\ 0.$

Theorem 2.1 The eigenvalues of problem (1)-(2) are simple under the following assumptions:

1. The prior conditions of problem(1)-(2) which stated in the introduction
2. q(x) is real.

Proof:

let $\varphi(x,\lambda)$ be a solution of the problem

$$\begin{aligned} -y''\ +\ q(x)\ y &= \lambda\ y \quad\ 0\leq x\leq a \\ y(0)\ =\ 0\quad , &\quad\ y'(0)\ =\ 1, \end{aligned} \tag{12}$$

and suppose that $\psi(x,\lambda)$ be the solution of the problem

$$\begin{aligned} -y''\ +\ q(x)\ y &= \ -\ \lambda\ y \quad\ a\leq x\leq\pi \\ y(\pi)\ =\ 1\quad , &\quad\ y'(\pi)\ =\ -H. \end{aligned} \tag{13}$$

Using(12), we get

$$-\varphi''(x,\lambda)+\ q(x)\varphi(x,\lambda)\ =\ \lambda\ \varphi(x,\lambda). \tag{14}$$

Differentiating (14) with respect to λ, we get

$$-\ \dot{\varphi}''(x,\lambda)\ +\ q(x)\ \dot{\varphi}(x,\lambda)\ =\ \lambda\ \dot{\varphi}(x,\lambda)\ +\ \varphi(x,\lambda) \tag{15}$$

multiplying (15)by $\varphi(x,\lambda)$ and (14) by $\dot{\varphi}(x,\lambda)$, and subtracting the two results we have

$$\frac{d}{d\ x}\left[\dot{\varphi}(x,\lambda)\ \varphi'(x,\lambda)\ -\ \dot{\varphi}'(x,\lambda)\ \varphi(x,\lambda)\right]\ =\varphi^2(x,\lambda). \tag{16}$$

Applying the initial conditions of (12) to the result which we get after integrating (16) with respect to x from 0 to a, we get

$$\varphi'(a,\lambda_n)\ \dot{\varphi}(a,\lambda_n)-\dot{\varphi}'(a,\lambda_n)\ \varphi(a,\lambda_n)=\int_0^a\ \varphi^2(x,\lambda_n)\ dx. \tag{17}$$

By similar methodology, from (13) we obtain

$$-\ \psi'(a,\lambda_n)\ \dot{\psi}(a,\lambda_n)+\dot{\psi}'(a,\lambda_n)\ \psi(a,\lambda_n)\ =\ -\int_a^\pi\ \psi^2(x,\lambda_n)\ dx. \tag{18}$$

Using (8) to substituting $\varphi'(a,\lambda_n)$ into (17),and also using (9)to substituting $\psi'(a,\lambda_n)$ into (18), hence adding the two results, we obtain

$$\dot{W}(\lambda_n)\ =\ C_n\ a_n, \tag{19}$$

where

$$C_n = \frac{\psi(a,\lambda_n)}{\varphi(a,\lambda_n)}, \qquad a_n = \int_0^a \varphi^2(x,\lambda_n)\, dx - \frac{1}{C_n^2} \int_a^\pi \psi^2(x,\lambda_n)\, dx, \tag{20}$$

where $a_n \neq 0$ are the normalization numbers of the eigenfunctions of the problem (1)-(2) which calculated previously in [12], and $C_n \neq 0$.

3. Properties Of Eigenfunctions

The author in [12] proved that the eigenfunctions of the Sturm-Liouville (1)-(2) are orthogonal with the weight $\rho(x)$, by the aid of which we will study some helping properties for our discussion of the eigenfunction expansion.

definition 1 consider the Green's function nonhomogeneous Sturm-Liouville problem

$$\begin{aligned} -y'' + q(x)\, y &= -\lambda\, \rho(x)\, y + \rho(x)\, f(x) \quad 0 \le x \le \pi, \\ y(0) &= y'(\pi) + H\, y(\pi) = 0, \end{aligned} \tag{21}$$

the Green function is defined by

$$G(x,t,\lambda) = \frac{-1}{W(\lambda)} \begin{cases} \varphi(x,\lambda)\, \psi(t,\lambda) & x \le t, \\ \varphi(t,\lambda)\, \psi(x,\lambda) & x \ge t, \end{cases} \tag{22}$$

where $\rho(x)$defined by (3) and $f(x) \in L_1(0,\pi)$, more over $G(x,t,\lambda)$ is called as the kernel of the resolvent $G_\lambda = (L - \lambda I)^{-1}$, where

$L \equiv -(\frac{d^2}{dx^2}) + q(x)$,

$D(L) = \{y(x) : \exists y'',\ y(0) = y'(\pi) + H\, y(\pi) = 0\}$.

Indeed: we will study some important properties of $G(x,t,\lambda)$ which will provide great help in our next study of the eigenfunction expansion of the problem (1)-(2) which can be summarized in next three lemmas .

Lemma 3.1 Denote by $y(x,\lambda)$ the solution of the nonhomogeneous Sturm-Liouville problem (21) which can be expressed as follow

$$y(x,\lambda) = \int_0^\pi G(x,t,\lambda)\, f(t)\, \rho(t)\, dt, \tag{23}$$

where $f(x) \in L_1(0,\pi)$.

Proof: We will claim this lemma in two steps.

First: the solution $y(x,\lambda)$ of problem (23) can be easily verified the boundary conditions of (21). Starting with equation (21) by using (12)and (13) respectively , we get

$$\begin{aligned} y(0) &= \frac{-1}{W(\lambda)} \int_0^\pi \varphi(0,\lambda)\, \psi(t,\lambda)\, \rho(t)\, f(t)\, dt = 0, \\ y'(\pi) + H\, y(\pi) &= \frac{-1}{W(\lambda)} \int_0^\pi \left[\psi'(\pi,\lambda) + H\, \psi(\pi,\lambda)\right] \varphi(t,\lambda)\, \rho(t)\, f(t)\, dt = 0. \end{aligned} \tag{24}$$

Second:applying the method of variation of parameter to the problem (21) to obtain it's solution. Let the solution of nonhomogeneous problem (21) in the next form

$$y(x,\lambda) \;=\; B_1\;\varphi(x,\lambda)\;+\;B_2\;\psi(x,\lambda). \tag{25}$$

the asymptotic representation of $\varphi(x,\lambda)$, $\psi(x,\lambda)$ are obtained in [12]. The constants B_1 and B_2 can be obtained by the standard method

$$\begin{aligned} B_1 &= \frac{-1}{w(\lambda)}\int_x^\pi f(t)\;\rho(t)\;\psi(t,\lambda)\;dt,\\ B_2 &= \frac{-1}{w(\lambda)}\int_0^x f(t)\;\rho(t)\;\varphi(t,\lambda)\;dt. \end{aligned} \tag{26}$$

We get our required form (23) by substituting from (26) into (25).
Lemma 3.2 The function $G(x,t,\lambda)$ admit the relation(27)under the conditions of lemma (3.1) where $G_1(x,t,\lambda)$ is regular in the neighborhood of $\lambda=\lambda_n$ and $a_n \;=\; \int_0^\pi\;\rho(t)\;\varphi^2(x,\lambda_n)\;dx.$

$$G(x,t,\lambda) \;=\; \frac{-1}{\lambda-\lambda_n}\;\frac{\varphi(x,\lambda_n)\;\varphi(t,\lambda_n)}{a_n}\;+\;G_1(x,t,\lambda). \tag{27}$$

Proof: The formula (27) can be proved as follow, from the simplicity of the roots of function $W(\lambda)$ which are claimed in lemma (2.1),we can see that the poles of the function $G(x,t,\lambda)$ are also simple, then we can represent the function $G(x,t,\lambda)$ as two parts one is principal part and the other is regular part which denoted by $G_1(x,t,\lambda)$ as follow :

$$G(x,t,\lambda) \;=\; \frac{Res[G(x,t,\lambda)]}{\lambda-\lambda_n}\;+\;G_1(x,t,\lambda), \tag{28}$$

to calculate the term $Res\;[G(x,t,\lambda)]$ using residue theorem and (22); for $x\le t$,we obtain

$$Res_{\lambda=\lambda_n}\;[G(x,t,\lambda)] \;=\; -\,\frac{\varphi(x,\lambda_n)\;\psi(t,\lambda_n)}{\dot W(\lambda_n)}, \tag{29}$$

where dot represents the differentiation with respect to λ,using (19) and (20),
the equation (29) becomes

$$Res_{\lambda=\lambda_n}\;[G(x,t,\lambda)] \;=\; -\,\frac{\varphi(x,\lambda_n)\;\varphi(t,\lambda_n)}{a_n}. \tag{30}$$

Substituting from (30) into (28), we have the required formula (27). And by same methodology we can deduce the the formula (27) for $t\;\le\;x$ to complete our picture.
Lemma 3.3 The following integral representation

$$\int_0^\pi G(x,t,\lambda)\rho(t)f(t)dt = -\frac{f(x)}{\lambda}+\frac{1}{\lambda}\int_0^\pi G(x,t,\lambda)\left[-f''(t)+q(t)f(t)\right]dt, \tag{31}$$

holds true under the following assumptions :
i- $f(x) \in [0,\pi]$ has a second order integrable derivatives,
ii- $f(0) = f'(\pi) + H\ f(\pi) = 0$.
Where $G(x,t,\lambda)$ is the kernel of the inverse operator for nonhomogeneous Sturm-Liouville problem (21).
Proof: According to definition of the function $G(x,t,\lambda)$ and lemma (3.1),by the help of (22), we have

$$\int_0^\pi G(x,t,\lambda)\rho(t)f(t)dt = \frac{-1}{W(\lambda)}\left\{\psi(x,\lambda)\int_0^x \varphi(t,\lambda)\rho(t)f(t)dt + \phi(x,\lambda)\int_x^\pi \psi(t,\lambda)\rho(t)f(t)dt\right\}. \tag{32}$$

Since $\varphi(x,\lambda),\psi(x,\lambda)$ are the solutions of (1)-(2), then we can simplify the equation (32) as follow

$$\int_0^\pi G(x,t,\lambda)\rho(t)f(t)dt = \frac{-1}{W(\lambda)}\left\{\frac{\psi(x,\lambda)}{\lambda}\int_0^x \left[-\varphi''(t,\lambda)+q(t)\varphi(t,\lambda)\right] f(t)\ dt + \frac{\varphi(x,\lambda)}{\lambda}\int_x^\pi \left[-\psi''(t,\lambda)+q(t)\psi(t,\lambda)\right] f(t)\ dt\right\}, \tag{33}$$

simplify again we get

$$\int_0^\pi G(x,t,\lambda)\rho(t)f(t)dt = \frac{1}{\lambda W(\lambda)}\left\{\psi(x,\lambda)\int_0^x \varphi''(t,\lambda)\ f(t)dt + \varphi(x,\lambda)\int_x^\pi \psi''(t,\lambda)f(t)\ dt\right\} + \frac{1}{\lambda}\int_0^\pi G(x,t,\lambda)\ q(t)\ f(t)\ dt, \tag{34}$$

to evaluate the two integrals $\int_0^x \varphi''(t,\lambda)f(t)dt$, and $\int_x^\pi \psi''(t,\lambda)f(t)dt$, by integration by parts twice and using the boundary conditions $f(0) = \varphi(0,\lambda) = 0$, and $f'(\pi) + Hf(\pi) = \psi'(\pi,\lambda) + H\psi(\pi,\lambda) = 0$, respectively we obtain

$$\left\{\psi(x,\lambda)\int_0^x \varphi''(t,\lambda)\ f(t)dt + \varphi(x,\lambda)\int_x^\pi \psi''(t,\lambda)f(t)\ dt\right\} = -W(\lambda)\left[f(x)+\int_0^\pi G(x,t,\lambda)\ f(t)\ dt\right]. \tag{35}$$

Using (35) into (34), we obtain the final formula (31)which complete our picture.

4. The extended asymptotic formulas for the eigenfunctions and the function $G(x,t,\lambda)$

To study the eigenfunction expansion of Sturm-Liouville problem (1)-(2) and it's convergence, we need to extend the asymptotic formulas of eigenfunctions $\varphi(x,\lambda)$, $\psi(x,\lambda)$ all over the interval $[0,\pi]$ because the author in [12] calculate the asymptotic formula for these eigenfunctions $\varphi(x,\lambda)$ where $x \in [0,a]$ and $\psi(x,\lambda)$

where$x \in\ (a,\pi]$, and this study will extend a helping hand to write an equality for the function $G(x,t,\lambda)$. In the following lemma we write the asymptotic formula for these eigenfunctions all over the interval $[0,\pi]$.

Lemma 4.1 The asymptotic formula (36)-(37)admit for the solutions $\varphi(x,\lambda)$, and $\psi(x,\lambda)$ of Sturm-Liouville problem (1)-(2) all over the interval $[0,\pi]$.

$$\varphi(x,\lambda) = \begin{cases} \frac{\sin sx}{s} + O\left(\frac{e^{|Ims|x)}}{|s^2|}\right), & 0 \le x \le a, \\ \frac{1}{s}\left[\sin sa\ \cosh s(a-x) - \cos sa\ \sinh s(a-x)\right] \\ + O\left(\frac{e^{|Ims|a+|Res|(a-x))}}{|s^2|}\right), & a < x \le \pi, \end{cases} \tag{36}$$

$$\psi(x,\lambda) = \begin{cases} \frac{u(x)}{u(a)}\left[\cos s(x-a)\ \cosh s(\pi-a) - \sin s(x-a)\ \sinh s(\pi-a)\right] \\ + O\left(\frac{e^{|Ims|(x-a)+|Res|(\pi-a))}}{|s|}\right), & 0 \le x \le a, \\ \cosh s(\pi-x) + O\left(\frac{e^{|Res|(\pi-x))}}{|s|}\right), & a < x \le \pi, \end{cases} \tag{37}$$

where

$$u(x) = \frac{1}{2}\int_0^x q(t)\ dt, \qquad v(x) = \frac{1}{2}\int_x^\pi q(t)\ dt, \quad \lambda = s^2. \tag{38}$$

Proof: Following the methodology of the author in [14], we can deduce the formula (36) and
(37) as follow: consider the following equation

$$-y'' + q(x)y = s^2\ y \qquad 0 \le\ x\ \le\ a \tag{39}$$

the solution $y(x,s)$ of (39) which satisfied the condition $y(0,s) = 0$ has the following representation

$$y(x,s) = e^{isx}\left[\frac{u(x)}{s} + O\left(\frac{1}{s^2}\right)\right], \qquad y'(x,s) = e^{isx}\left[i\ u(x) + O\left(\frac{1}{s}\right)\right], \tag{40}$$

also, consider the equation

$$-y'' + q(x)y = -s^2\ y \qquad a \le\ x\ \le\ \pi \tag{41}$$

the solution $z(x,s)$ of (41) which satisfied the condition $z(\pi,s) = 1$ has the following representation

$$z(x,s) = e^{s(\pi-x)}\left[1 + \frac{v(x)}{s} + O\left(\frac{1}{s^2}\right)\right], \quad z'(x,s) = e^{s(\pi-x)}\left[-s - v(x) + O\left(\frac{1}{s}\right)\right], \tag{42}$$

where $u(x)$, and $v(x)$ are defined in (38). Now we start with evaluation the extended asymptotic formula for $\varphi(x,\lambda)$, where $x \in (a,\pi]$ in terms of two linearly independent solutions $z(x,s), z(x,-s)$ as

$$\varphi(x,\lambda) \;=\; c_1\; z(x,s) \;+\; c_2\; z(x,-s). \tag{43}$$

The author in [12] obtained the solution $\varphi(x,\lambda)$ for Sturm-Liouville problem (1)-(2) as follow

$$\varphi(x,\lambda) \;=\; \frac{\sin sx}{s} \;+\; O\left(\frac{e^{|Ims|x)}}{|s^2|}\right), \qquad\qquad 0 \le x \le a, \tag{44}$$

then to calculate the constants c_1, and c_2 in (43) applying the continuity of $\varphi(x,\lambda)$ at $x = a$ for the equations (44)and (42), we get

$$\begin{aligned} c_1 \;&=\; \tfrac{e^{s(a-\pi)}}{2}\left[\tfrac{\sin sa \;-\; \cos sa}{s} \;+\; O\left(\tfrac{e^{|Ims|a)}}{|s^2|}\right)\right], \\ c_2 \;&=\; \tfrac{e^{s(\pi-a))}}{2}\left[\tfrac{\sin sa \;+\; \cos sa}{s} \;+\; O\left(\tfrac{e^{|Ims|a)}}{|s^2|}\right)\right]. \end{aligned} \tag{45}$$

Substituting from(45)into(43), we get $\varphi(x,\lambda)$ for $x \in \;(a,\pi]$

$$\varphi(x,\lambda) \;=\; \frac{1}{s}\left[\sin sa\;\cosh s(a-x) \;-\; \cos sa\;\sinh s(a-x)\right] \;+\; O\left(\frac{e^{|Ims|a+|Res|(a-x))}}{|s^2|}\right). \tag{46}$$

So that the relation (36)is obtained when we put equation (46) together with (44). By the same manner we can deduce the formula (37), following [12] the asymptotic formula of $\psi(x,\lambda)$ for $x \in (a,\pi]$ is given by

$$\psi(x,\lambda) \;=\; \cosh s(\pi-x) \;+\; O\left(\frac{e^{|Res|(\pi-x))}}{|s|}\right). \tag{47}$$

Indeed:we can write the extended asymptotic formula of $\psi(x,\lambda)$ for $x \in [0,a]$ as linear combination of two linearly independent solutions $y(x,s)$, and $y(x,-s)$ which has the next representation

$$\psi(x,\lambda) \;=\; H_1\; y(x,s) + H_2\; y(x,-s), \tag{48}$$

where $y(x,s)$ defined in (40). Applying similar method with the constants H_1, and H_2 are given by

$$\begin{aligned} H_1 \;&=\; e^{-isa}\;\tfrac{s}{2u(a)}\;\left[\cosh s(\pi-a) + i\sinh s(\pi-a)\right] \;+\; O\left(e^{|Res|(\pi-a))}\right), \\ H_2 \;&=\; e^{isa}\;\tfrac{s}{2u(a)}\;\left[i\sinh s(\pi-a) - \cosh s(\pi-a)\right] \;+\; O\left(e^{|Res|(\pi-a))}\right). \end{aligned} \tag{49}$$

From the continuity of $\psi(x,\lambda)$ at $x = a$, substituting from (49) into (48) and using asymptotic relation (47) we get the final asymptotic formula for $\psi(x,\lambda)$ for $x \in [0,a]$ as follow

$$\begin{aligned} \psi(x,\lambda) = \tfrac{u(x)}{u(a)}\;&\left[\cos s(x-a)\cosh s(\pi-a) - \sin s(x-a)\sinh s(\pi-a)\right] \\ &+\; O\left(e^{|Ims|(x-a)+|Res|(\pi-a))}\right). \end{aligned} \tag{50}$$

The extended formula (37) can be obtained from the relation (50) together with (47), and this complete the proof.
In the next lemma we discuss an inequality satisfied by $G(x,t,\lambda)$.
Lemma 4.2 Under the conditions of lemma (4.1) The function $G(x,t,\lambda)$ satisfies the asymptotic relations

$$G(x,t,\lambda) = \begin{cases} O\left(\frac{e^{-|Im\ s||2a-x-t|}}{|s|}\right), & x,t \in [0,a] \\ O\left(\frac{e^{-|Re\ s||x+t-2a|}}{|s|}\right), & x,t \in (a,\pi] \\ O\left(\frac{e^{-|Im\ s|(a-x)-|Re\ s|(t-a))}}{|s|}\right), & 0 \le x \le a < t \le \pi, \\ O\left(\frac{e^{-|Im\ s|(a-t)-|Re\ s|(x-a)}}{|s|}\right), & 0 \le t \le a < x \le \pi, \end{cases} \tag{51}$$

Proof: According to definition of the resolvent $G(x,t,\lambda)$ in (22) we need to discuss the behavior of functions $W(\lambda)$, $\varphi(x,\lambda)$, and $\psi(x,\lambda)$ to get the extended asymptotic formula for $G(x,t,\lambda)$ all over $[0,\pi]$. Starting with $W(\lambda)$, following [12] the quadratic contour Γ_n, as follow

$$\Gamma_n = \left\{|Re\ s| < \frac{\pi}{a}(n-\frac{1}{4}) + \frac{\pi}{2a}\ ,\ |Im\ s| \le \frac{\pi}{\pi-a}(n-\frac{1}{4}) + \frac{\pi}{2(\pi-a)}\right\}, \tag{52}$$

and by simple calculations the author obtained for $s \in \Gamma_n$

$$W(\lambda) \ \ge\ C\ O\left(e^{|Ims|a+|Res|(\pi-a)}\right). \tag{53}$$

Also from (36), and (37)we can write $\varphi(x,\lambda)$, and $\psi(x,\lambda)$ as follow

$$\varphi(x,\lambda) = \begin{cases} O\left(\frac{e^{-|Im\ s|x)}}{|s|}\right), & 0 \le x \le a, \\ O\left(\frac{e^{-|Im\ s|a-|Re\ s|(a-x)}}{|s|}\right), & a < x \le \pi, \end{cases} \tag{54}$$

$$\psi(x,\lambda) = \begin{cases} O\left(e^{-|Im\ s|(x-a)-|Re\ s|(\pi-a)}\right), & 0 \le x \le a, \\ O\left(e^{-|Re\ s|(\pi-x)}\right), & a < x \le a. \end{cases} \tag{55}$$

Moreover, we have two cases contains six possibilities from (22), the first case is $x \le t$ and the second case when $t \le x$. now we begin with the first case $x \le t$ which have three possibilities as follow:(1) $0 \le x \le t \le a$, (2) $a < x \le t \le \pi$, and (3) $0 \le x \le a \le t \le$

π. Using the direct substitutions from (54), (55), and (53) into the first branch of (22) in the above three cases, we obtain

$$(1)\ G(x,t,\lambda) = O\left(\frac{e^{-|Im\ s||2a-x-t|)}}{|s|}\right), \qquad 0 \le x \le t \le a, \tag{56}$$

$$(2)\ G(x,t,\lambda) = O\left(\frac{e^{-|Re\ s||x+t-2a|)}}{|s|}\right), \qquad a \le x \le t \le \pi, \tag{57}$$

$$(3)\ G(x,t,\lambda) = O\left(\frac{e^{-|Im\ s|(a-x)-|Re\ s|(t-a)}}{|s|}\right), \qquad 0 \le x \le a \le t \le \pi. \tag{58}$$

Now, in second case we deal with $t \le x$, then by same methodology we have three other probabilities as follow: (4) $0 \le t \le x \le a$, (5) $a < t \le x \le \pi$, and (6) $0 \le t \le a \le x \le \pi$. Using the direct substitutions from (54), (55), and (53) into the second branch of (22), we obtain

$$(4)\ G(x,t,\lambda) = O\left(\frac{e^{-|Im\ s||2a-x-t|)}}{|s|}\right), \qquad 0 \le t \le x \le a, \tag{59}$$

$$(5)\ G(x,t,\lambda) = O\left(\frac{e^{-|Re\ s||x+t-2a|)}}{|s|}\right), \qquad a \le t \le x \le \pi, \tag{60}$$

$$(6)\ G(x,t,\lambda) = O\left(\frac{e^{-|Im\ s|(a-t)-|Re\ s|(x-a)}}{|s|}\right), \qquad 0 \le t \le a \le x \le \pi. \tag{61}$$

To get the formula of (51),we notice that from (56) and (59) we get

$$G(x,t,\lambda) = O\left(\frac{e^{-|Im\ s||2a-x-t|)}}{|s|}\right), \qquad x,t \in [0,a], \tag{62}$$

also from (57) and (60),we have

$$G(x,t,\lambda) = O\left(\frac{e^{-|Re\ s||x+t-2a|)}}{|s|}\right), \qquad x,t \in (a,\pi], \tag{63}$$

when we collect together formulas (62) with (63), (58), and(61) we reach to the required inequality in (51).

5. The Formula Of The Eigenfunctions Expansion for Sturm-Liouville

To construct and prove the eigenfunction expansion formula for Sturm-Liouville problem (1)-(2), we need to introduce an orthonormal system for this problem which will be helpful in following theorem, following [12] we have the following

- suppose that the nonnegative eigenvalues λ_n^+ and the negative eigenvalues λ_n^- of Sturm-Liouville (1)-(2), n=0,1,2,......

- the normalization numbers which corresponding to eigenfunctions $\varphi(x,\lambda_n^{\pm})$as

$$a_n^{\pm} \ = \ \int_0^{\pi} \ \rho(x) \ \varphi^2(x,\lambda_n^{\pm}) \ dx. \tag{64}$$

- Consider an orthonormal system $\{u_k^{\pm}(x)\}_{k=0}^{\infty}$ of the eigenfunctions of Sturm-Liouville problem (1)-(2) which can written as follow

$$u_k^{\pm} \ = \ \frac{\varphi(x,\lambda_k^{\pm})}{\sqrt{a_k^{\pm}}}. \tag{65}$$

Theorem 5.1 Let the eigenfunction expansions formula is of the form

$$f(x) \ = \ \sum_{k=0}^{\infty} \ d_k^+ \ u_k^+ \ + \ \sum_{k=0}^{\infty} \ d_k^- \ u_k^- \ , \tag{66}$$

where

$$d_k^{\pm} \ = \ \int_0^{\pi} \ u_k^{\pm}(t) \ f(t) \ \rho(t) \ dt,$$

and the series is uniformly convergence to $f(x)$, $x \in \ [0,\pi]$ or equivalently

$$\begin{aligned} f(x) &= \textstyle\sum_{k=0}^{\infty} \ \frac{1}{a_k^+} \ \varphi(x,\lambda_k^+) \int_0^{\pi} \ \varphi(t,\lambda_k^+) \ f(t) \ \rho(t)dt \\ &+ \sum_{k=0}^{\infty} \ \frac{1}{a_k^-} \ \varphi(x,\lambda_k^-) \int_0^{\pi} \ \varphi(t,\lambda_k^-) \ f(t) \ \rho(t)dt, \end{aligned} \tag{67}$$

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} \ \frac{1}{(c_k)^2 \ a_k^+} \ \psi(x,\lambda_k^+) \int_0^{\pi} \ \psi(t,\lambda_k^+) \ f(t) \ \rho(t)dt \\ &+ \sum_{k=0}^{\infty} \ \frac{1}{(c_k)^2 \ a_k^-} \ \psi(x,\lambda_k^-) \ \int_0^{\pi} \ \psi(t,\lambda_k^-) \ f(t) \ \rho(t)dt, \\ & where \ \ c_k \ = \ \tfrac{\psi(x,\lambda_k^{\pm})}{\varphi(x,\lambda_k^{\pm})}, \quad 0 \le x \le \pi. \end{aligned} \tag{68}$$

Proof: Let

$$g(x,\lambda) \ = \ \frac{1}{\lambda} \int_0^{\pi} \ G(x,t,\lambda) \left[-f''(t) + q(t)f(t)\right] dt. \tag{69}$$

Now the equation (31)from lemma (3.1)can be written as follow

$$\int_0^{\pi} \ G(x,t,\lambda)\rho(t)f(t)dt \ = \ \frac{-f(x)}{\lambda} + g(x,\lambda). \tag{70}$$

Suppose that $\lambda = s^2$, denote Γ_n^+ the upper half of the contour Γ_n which is defined [12], and given by (52). Define γ_n as the image of the contour Γ_n^+ under the transformation $\lambda = s^2$, moreover the poles of the function $G(x,t,\lambda)$ as a function of λ, lie only $\lambda_o^\pm, \lambda_1^\pm, \lambda_2^\pm, ..., \lambda_n^\pm$ inside, we can use the residues theorem after multiply both sides of (70) by $\frac{1}{2\pi i}$ and integrating with respect to λ on the contour γ_n, we have

$$\begin{aligned}\frac{1}{2\pi i}\oint_{\gamma_n} \{G(x,t,\lambda)f(t)\rho(t)dt\} &= \sum_{k=0}^{n} Res_{\lambda=\lambda_k^\pm}\left\{\int_0^\pi G(x,t,\lambda)f(t)\rho(t)dt\right\} \\ &= \frac{-f(x)}{2\pi i}\oint_{\gamma_n}\frac{d\lambda}{\lambda} + \frac{1}{2\pi i}\oint_{\gamma_n} g(x,\lambda)\, d\lambda.\end{aligned} \tag{71}$$

By the aid of residues formula (30), we get

$$\begin{aligned}&\sum_{k=0}^{n} Res_{\lambda=\lambda_k^\pm}\left\{\int_0^\pi G(x,t,\lambda)f(t)\rho(t)dt\right\} = \\ &-\sum_{k=0}^{n}\frac{1}{a_k^+}\varphi(x,\lambda_k^+)\int_0^\pi \varphi(t,\lambda_k^+)\, f(t)\,\rho(t)dt \\ &-\sum_{k=0}^{n}\frac{1}{a_k^-}\varphi(x,\lambda_k^-)\int_0^\pi \varphi(t,\lambda_k^-)\, f(t)\,\rho(t)dt,\end{aligned} \tag{72}$$

also

$$\frac{-f(x)}{2\pi i}\oint_{\gamma_n}\frac{d\lambda}{\lambda} = -f(x). \tag{73}$$

Moreover, to evaluate $\frac{1}{2\pi i}\oint_{\gamma_n} g(x,\lambda)d\lambda$, using condition on $q(x)$ and by the aid of lemma (4.1), we can see that

$$|g(x,\lambda)| \leq \frac{k_o}{\lambda}, \qquad s\in\Gamma_n, \tag{74}$$

where k_o is constant which is independent of x, t, λ, then

$$\left|\frac{1}{2\pi i}\oint_{\gamma_n} g(x,\lambda)d\lambda\right| \leq \frac{k_o}{2\pi}\oint_{\gamma_n}\frac{|d\lambda|}{\lambda}, \tag{75}$$

from that apply the substitution $\lambda = s^2$, we get

$$\left|\frac{1}{2\pi i}\oint_{\gamma_n} g(x,\lambda)d\lambda\right| \leq \frac{k_o}{2\pi}\int_{\Gamma_n^+}\frac{|ds|}{s^2} \tag{76}$$

Using substitution from (72),(73),and(76) into (71), we have

$$\left|f(x) - \sum_{k=0}^{n}\left(d_k^+\, u_k^+ + d_k^-\, u_k^-\right)\right| \leq \frac{constant}{n}, \quad x\in[0,\pi]\;\forall\, n, \tag{77}$$

where

$$d_k^{\pm} \;=\; \int_0^{\pi} \frac{\varphi(t,\lambda_k^{\pm})}{\sqrt{a_k^{\pm}}} f(t)\rho(t)dt, \quad u_k^{\pm} \;=\; \frac{\varphi(t,\lambda_k^{\pm})}{\sqrt{a_k^{\pm}}}, \tag{78}$$

so, we get

$$f(x) \;=\; \sum_{k=0}^{n} \left[d_k^{+}\, u_k^{+} \;+\; d_k^{-}\, u_k^{-} \right], \tag{79}$$

which complete the proof of the uniform convergence of the series
$\sum_{k=0}^{n} \left[d_k^{+}\, u_k^{+} \;+\; d_k^{-}\, u_k^{-} \right]$ to
$f(x)$, $x \in [0, \pi]$.

Indeed: we can prove that the series in (79) is also absolutely convergent and we will not talk about the sum of series as the proof of uniform convergence of the series which means we need to evaluate $|d_k^{\pm}\, u_k^{\pm}|$ when $k \to \infty$ by the aid of the asymptotic relations of $m_k^{\pm}$, $u_k^{\pm}$ for $k \to \infty$. following [12], we have

$$a_n^{+} \;=\; \frac{h_1}{n^2} \;+\; O\left(\frac{1}{n^3}\right), \quad a_n^{-} \;=\; -\,\frac{h_2^2}{8\pi}\, e^{\frac{n}{h_2}} e^{\frac{-1}{4h_2}} \left[\frac{h_3}{n^2} + \; O\left(\frac{1}{n^3}\right)\right], \tag{80}$$

where

$$h_1 = \frac{a^3}{2\pi^2}, \quad h_2 = \frac{\pi - a}{2\pi}, \quad h_3 = \frac{1}{2\pi}\left[H + \int_a^{\pi} \; q(t)dt \right],$$

so that we can easily see

$$\sqrt{a_n^{\pm}} \;=\; o\left(\frac{1}{n}\right). \tag{81}$$

From definition of $u_k^{\pm}$ in (65),using (81), and (36) we deduce that,

$$|u_k^{\pm}| \;\le\; D^{\pm}, \quad \forall x \in [0, \pi], andallk. \tag{82}$$

Where $D^{\pm}$ are some constants. Moreover according to the methodology of lemma (4.1) and $f(0) = f'(\pi) + Hf(\pi) = 0$,we obtain

$$d_k^{\pm} \;=\; \int_0^{\pi} \; u_k^{\pm}(x)\; f(x)\; \rho(x)\; dx \;=\; \frac{1}{\lambda_k^{\pm}} \int_0^{\pi} \; \left[f''(x) + q(x)f(x) \right] \; u_k^{\pm}(x)\; dx. \tag{83}$$

From [12], we get $\lambda_k^{\pm} \;=\; \pm\; k^2 \;+\; O\,(1)$, by substitution into(83), and using the result we obtained in (82), we have

$$|d_k^{\pm}\; u_k^{\pm}(x)| \le \frac{constant}{k^2}, \quad k \to \infty. \tag{84}$$

which complete our proof of the series is absolutely convergence and not its sum as we mention before.

According to the previous theorem we need to prove the Parsval's identity which insures the convergence of series (79) and this will done in the next lemma.

Lemma 5.1 The Parsval's identity in (85) admit under the conditions of theorem (5.1), where $f(x)$ satisfied this conditions.

$$\int_0^\pi \rho(x)\ |f(x)|^2\ dx \ = \ \sum_{k=0}^{\infty} \left(|d_k^+|^2 \ + \ |d_k^-|^2\right), \tag{85}$$

where

$$d_k^\pm \ = \ \int_0^\pi u_k^\pm\ (x)\ \rho(x)\ f(x)\ dx. \tag{86}$$

Proof:
Starting with the relation (79) in theorem (5.1),we have

$$f(x) \ = \ \sum_{k=0}^{n} \left[d_k^+\ u_k^+ \ + \ d_k^-\ u_k^-\right], \tag{87}$$

to prove the relation in (85) integrating both sides of (87) with respect to $x \in [0, \pi]$ after multiplied the equation by $\overline{f(x)}\rho(x)$, we obtain

$$\begin{aligned} &\int_0^\pi \rho(x)\ |f(x)|^2\ dx \ = \\ &\int_0^\pi \sum_{k=0}^{\infty} \left(d_k^+\ u_k^+\ (x) \ + \ d_k^-\ u_k^-\ (x)\right)\ \overline{f(x)}\ \rho(x)\ dx. \end{aligned} \tag{88}$$

which equivalent to (89) which obtained by the help of uniform convergence of the series (79), which means we can interchanged the integration and summation.

$$\begin{aligned} &\int_0^\pi \rho(x)|f(x)|^2\ dx = \\ &\sum_{k=0}^{\infty} d_k^+ \int_0^\pi u_k^+(x)\ \overline{f(x)}\ \rho(x)\ dx + \sum_{k=0}^{\infty} d_k^- \int_0^\pi u_k^-(x)\ \overline{f(x)}\rho(x)dx, \end{aligned} \tag{89}$$

moreover $\rho(x)$ and $u_k^\pm(x)$ are real, then we can written (89) as follow

$$\begin{aligned} &\int_0^\pi \rho(x)|f(x)|^2\ dx = \\ &\sum_{k=0}^{\infty} d_k^+ \int_0^\pi \overline{u_k^+(x)}\ \overline{f(x)}\ \overline{\rho(x)}\ dx + \sum_{k=0}^{\infty} d_k^- \int_0^\pi \overline{u_k^-(x)}\ \overline{f(x)}\ \overline{\rho(x)}\ dx. \end{aligned} \tag{90}$$

Which led to our relation in (85)and complete our proof.

6. The validity of eigenfunction expansion and the Parsval's identity in $L_2(0, \pi; \rho)$

The study of the eigenfunction expansion and the Parsval's identity and their validity can be extended to any function of $L_2(0, \pi; \rho)$, and in some weak sense, which is the metric sense of $L_2(0, \pi; \rho)$ we will show the convergence of the series (79) in the following theorem.
Theorem 6.1 The following statements holds true to any function $f(x) \in L_2(0, \pi; \rho)$:

1. The Parsval's identity in (85) admit.
2. The eigenfunction expansion in (79) admit and in the metric sense of the space $L_2(0,\pi;\rho)$ the series in (79) convergence to $f(x)$.

Proof:

1. suppose that $f(x)$ is any function belonging to $L_2(0,\pi;\rho)$ from the density property in $L_2(0,\pi;\rho)$ the set of infinitely differenntial functions which vanish at the neighborhood of the points $x=0$, and $x=\pi$ are dense in $L_2(0,\pi)$, then there exists $\{f_n(x)\}$ of finite smooth functions which admit the condition of the theorem and convergence to $f(x)$ in the metric sense of $L_2(0,\pi;\rho)$, and this can be written using mathematical expression as follow

$$\|f_n(x) - f(x)\|_{L_2} = \left(\int_0^\pi \rho(x)\, |f_n(x) - f(x)|^2\, dx\right)^{\frac{1}{2}} \to 0,\ as\ n\to\infty. \quad (91)$$

By the help of lemma (5.1) every function $f_n(x)$ satisfies Parsval's identity, then

$$\int_0^\pi \rho(x)\, |f_n(x)|^2\, dx = \sum_{k=0}^{\infty} \left(|d_k^+|^2 + |d_k^-|^2\right), \quad (92)$$

where $d_k^\pm = \int_0^\pi u_k^\pm(x)\, \rho(x)\, f_n(x)\, dx$. equation(91) equivalent to

$$\left\|f_n(x)\right\|_{L_2}^2 = \left\|d_k^{(n)+}\right\|_{L_2}^2 + \left\|d_k^{(n)-}\right\|_{L_2}^2. \quad (93)$$

First we show that $\{f_n(x)\}$ is a fundamental sequence, hence consider the following difference

$$\left\|f_n(x) - f_m(x)\right\|_{L_2}^2 = \left\|d_k^{(n)+} - d_k^{(m)+}\right\|_{L_2}^2 + \left\|d_k^{(n)-} - d_k^{(m)-}\right\|_{L_2}^2. \quad (94)$$

By the help of (91),consequently $\{f_n(x)\}$ is a fundamental sequence. Second we claim that $\{d_k^{(n)\pm}\}$ are fundamentals also , from the completeness of L_2, so that here exists a limiting d_k^+ such that $\|d_k^{(n)+} - d_k^+\|_{L_2}^2 \to 0$, and similar for d_k^- we get $\|d_k^{(n)-} - d_k^-\|_{L_2}^2 \to 0$, , hence for (93) using the continuity of the norm and passing the limit as $n\to\infty$,we have

$$\left\|f(x)\right\|_{L_2}^2 = \left\|d_k^+\right\|_{L_2}^2 + \left\|d_k^-\right\|_{L_2}^2, \quad (95)$$

which is Parsval's identity.

2. Now we prove the eigenfunction expansion for any n, we have

$$\begin{aligned}&\int_0^\pi \rho(x) \left|f(x)- \sum_{k=0}^{n} \left(d_k^+ \, u_k^+(x) \; + \; d_k^- \, u_k^-(x)\right)\right|^2 dx\\ &=\int_0^\pi \rho(x) \left\{\left[f(x)- \sum_{k=0}^{n} \left(d_k^+ \, u_k^+(x) \; + \; d_k^- \, u_k^(x)\right)\right]\right.\\ &\times \left.\left[\overline{f(x)} \; - \; \sum_{k=0}^{n} \left(\overline{d_k^+} \, u_k^+(x) \; + \; \overline{d_k^-} \, u_k^-(x)\right)\right]\right\} dx,\end{aligned} \tag{96}$$

after some simplifications, we obtain

$$\begin{aligned}&\int_0^\pi \rho(x) \left|f(x)- \sum_{k=0}^{n} \left(d_k^+ \, u_k^+(x) \; + \; d_k^- \, u_k^-(x)\right)\right|^2 dx\\ &=\int_0^\pi \rho(x) \, |f(x)|^2 \, dx - \sum_{k=0}^{n} \left(|d_k^+|^2 \; + \; |d_k^-|^2\right).\end{aligned} \tag{97}$$

Using Parsval's identity in (85) , we get

$$\lim_{n\to\infty}\int_0^\pi \rho(x) \left|f(x)- \sum_{k=0}^{n} \left(d_k^+ \, u_k^+(x) \; + \; d_k^- \, u_k^-(x)\right)\right|^2 dx \to 0. \tag{98}$$

From which we have , $\sum_{k=0}^{n} \left(d_k^+ \, u_k^+(x) \, + \, d_k^- \, u_k^-(x)\right) \; \to \; f(x)$ in the metric of $L_2(0, \pi; \rho)$, which end our vision of the proof.

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