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Necessary and sufficient conditions for the boundedness of B-Riesz potential in modified B-Morrey spaces

Javanshir J. Hasanov*†, Xayyam A. Badalov, Ayna E. Fleydanli

Abstract. We consider the generalized shift operator, associated with the Laplace-Bessel differential operator $\Delta_B = \sum\limits_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i} + \sum\limits_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$, $\gamma_i > 0$, $i = 1, \ldots, k$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$. The maximal operator M_γ (B-maximal operator), fractional maximal operator $M_{\alpha,\gamma}$ (B-fractional maximal operator) and the Riesz potential operator $I_{\alpha,\gamma}$ (B-Riesz potential operator), associated with the generalized shift operator are investigated. We prove that the B-maximal operator M_γ is bounded in the modified B-Morrey space $\widetilde{L}_{p,\lambda,\gamma}$ for all $1 < p < \infty$ and $0 \le \lambda < n+|\gamma|$. We study the B-Riesz potential and their modified version in the modified B-Morrey space and B-BMO space. We prove that the fractional maximal operator $M_{\alpha,\gamma}$ and the Riesz potential operator $I_{\alpha,\gamma}$, $0 < \alpha < n + |\gamma|$ are bounded from the modified Morrey space $\widetilde{L}_{1,\lambda,\gamma}$ to the weak modified Morrey space $W\widetilde{L}_{q,\lambda,\gamma}$ if and only if, $\alpha/(n + |\gamma|) \le 1 - 1/q \le \alpha/(n + |\gamma| - \lambda)$ and from $\widetilde{L}_{p,\lambda,\gamma}$ to $\widetilde{L}_{q,\lambda,\gamma}$ if and only if, $\alpha/(n + |\gamma|) \le 1/p - 1/q \le \alpha/(n + |\gamma| - \lambda)$.

Key Words and Phrases: B–fractional maximal operator, B–Riesz potential, B–Morrey space, modified B-Morrey space, Sobolev -Morrey type estimate, B-BMO space.

2000 Mathematics Subject Classifications: 42B20, 42B25, 42B35

Introduction

For $x \in \mathbb{R}^n$ and $t > 0$, let $B(x,t)$ denote the open ball centered at x of radius t and ${}^{\complement}B(x,t) = \mathbb{R}^n \setminus B(x,t)$.

One of the most important variants of the Hardy-Littlewood maximal function is the so-called fractional maximal function defined by the formula

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{-1+\alpha/n} \int_{B(x,t)} |f(y)|dy, \quad 0 \le \alpha < n,$$

where $|B(x,t)|$ is the Lebesgue measure of the ball $B(x,t)$.

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*Corresponding author.
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It coincides with the Hardy-Littlewood maximal function $Mf \equiv M_0 f$ and is intimately related to the Riesz potential operator

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)dy}{|x-y|^{n-\alpha}}, \qquad 0 < \alpha < n$$

(see, for example, [2] and [34]).

The operators M_α and I_α play important role in real and harmonic analysis (see, for example [41]).

In the theory of partial differential equations, together with weighted $L_{p,w}(\mathbb{R}^n)$ spaces, Morrey spaces $L_{p,\lambda}(\mathbb{R}^n)$ play an important role. Morrey spaces were introduced by C. B. Morrey in 1938 in connection with certain problems in elliptic partial differential equations and calculus of variations (see [32]). Later, Morrey spaces found important applications to Navier-Stokes ([42]) and Schrödinger ([35], [36], [37], [39]) equations, elliptic problems with discontinuous coefficients ([6], [21]), and potential theory ([2], [3]). An exposition of the Morrey spaces can be found in the book [26].

Definition 1. *Let $1 \le p < \infty$, $0 \le \lambda \le n$, $[t]_1 = \min\{1, t\}$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ Morrey space, and by $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms*

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left(t^{-\lambda} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p},$$

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} \left([t]_1^{-\lambda} \int_{B(x,t)} |f(y)|^p dy \right)^{1/p}$$

respectively.

Note that

$$\widetilde{L}_{p,0}(\mathbb{R}^n) = L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \subset\!\succ L_{p,\lambda}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \quad \text{and} \quad \max\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\} \le \|f\|_{\widetilde{L}_{p,\lambda}}$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Definition 2. *[10, 11, 14] Let $1 \le p < \infty$, $0 \le \lambda \le n$. We denote by $WL_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space and by $W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified weak Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$ with finite norms*

$$\|f\|_{WL_{p,\lambda}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \Big(t^{-\lambda} \; |\{y \in B(x,t) : \; |f(y)| > r\}| \Big)^{1/p},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}} = \sup_{r>0} r \sup_{x\in\mathbb{R}^n,\, t>0} \Big([t]_1^{-\lambda} \; |\{y \in B(x,t) : \; |f(y)| > r\}| \Big)^{1/p}$$

respectively.

Note that

$$W L_p(\mathbb{R}^n) = W L_{p,0}(\mathbb{R}^n) = W \widetilde{L}_{p,0}(\mathbb{R}^n),$$
$$L_{p,\lambda}(\mathbb{R}^n) \subset W L_{p,\lambda}(\mathbb{R}^n) \ \text{ and } \ \|f\|_{W L_{p,\lambda}} \le \|f\|_{L_{p,\lambda}},$$
$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \subset W \widetilde{L}_{p,\lambda}(\mathbb{R}^n) \ \text{ and } \ \|f\|_{W \widetilde{L}_{p,\lambda}} \le \|f\|_{\widetilde{L}_{p,\lambda}}.$$

The classical result by Hardy-Littlewood-Sobolev states that if $1 < p < q < \infty$, then I_α is bounded from $L_p(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(\frac{1}{p} - \frac{1}{q}\right)$ and for $p = 1 < q < \infty$, I_α is bounded from $L_1(\mathbb{R}^n)$ to $W L_q(\mathbb{R}^n)$ if and only if $\alpha = n\left(1 - \frac{1}{q}\right)$. D. R. Adams [2] studied the boundedness of the Riesz potential in Morrey spaces and proved the follows statement

Theorem A. *Let* $0 < \alpha < n$ *and* $0 \le \lambda < n$, $1 \le p < \frac{n-\lambda}{\alpha}$.

1) If $1 < p < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness* I_α *from* $L_{p,\lambda}(\mathbb{R}^n)$ *to* $L_{q,\lambda}(\mathbb{R}^n)$.

2) If $p = 1$, *then condition* $1 - \frac{1}{q} = \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness* I_α *from* $L_{1,\lambda}(\mathbb{R}^n)$ *to* $W L_{q,\lambda}(\mathbb{R}^n)$.

If $\alpha = \frac{n}{p} - \frac{n}{q}$, then $\lambda = 0$ and the statement of Theorem A reduces to the aforementioned result by Hardy-Littlewood-Sobolev.

Recall that, for $0 < \alpha < n$,

$$M_\alpha f(x) \le v_n^{\frac{\alpha}{n}-1} I_\alpha(|f|)(x),$$

hence Theorem A also implies the boundedness of the fractional maximal operator M_α, where v_n is the volume of the unit ball in $\mathbb{R}^n$. F. Chiarenza and M. Frasca [8] proved that the maximal operator M is also bounded from $L_{p,\lambda}$ to $L_{p,\lambda}$ for all $1 < p < \infty$ and $0 < \lambda < n$.

In [16] the boundedness of the Riesz potential in modified Morrey spaces is studied and the following statement is proved.

Theorem C. *Let* $0 < \alpha < n$ *and* $0 \le \lambda < n$, $1 \le p \le \frac{n}{\alpha}$.

1) If $1 < p < \frac{n-\lambda}{\alpha}$, *then condition* $\frac{\alpha}{n} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of* I^α *from* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ *to* $\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

2) If $p = 1$, *then condition* $\frac{\alpha}{n} \le 1 - \frac{1}{q} \le \frac{\alpha}{n-\lambda}$ *is necessary and sufficient for the boundedness of* I^α *from* $\widetilde{L}_{1,\lambda}(\mathbb{R}^n)$ *to* $W\widetilde{L}_{q,\lambda}(\mathbb{R}^n)$.

3) If $\frac{n-\lambda}{\alpha} \le p \le \frac{n}{\alpha}$, *then the operator* $\widetilde{I}^\alpha$ *is bounded from* $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ *to* $BMO(\mathbb{R}^n)$.

Moreover, if the integral $I^\alpha f$ *exists almost everywhere for* $f \in \widetilde{L}_{p,\lambda}(\mathbb{R}^n)$, $\frac{n-\lambda}{\alpha} \le p \le \frac{n}{\alpha}$, *then* $I^\alpha f \in BMO(\mathbb{R}^n)$ *and the following inequality is valid*

$$\|I^\alpha f\|_{BMO} \le C \|f\|_{\widetilde{L}_{p,\lambda}},$$

where $C > 0$ *is independent of* f.

If $\lambda = 0$, then $\alpha = \frac{n}{p} - \frac{n}{q}$ and the statement of Theorem C also reduces to the aforementioned result by Hardy-Littlewood-Sobolev.

The maximal operator, singular integral, potential and related topics associated with the Laplace-Bessel differential operator

$$\Delta_B = \sum_{i=1}^{k} \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i} + \sum_{j=1}^{n}\frac{\partial^2}{\partial x_j^2}, \qquad \gamma_i > 0, \quad i = 1, \dots, k.$$

In this paper, we consider the generalized shift operator generated by the Laplace-Bessel differential operator Δ_B in terms of which the B-maximal operator and the B-Riesz potential are investigated. We study the B-Riesz potential in the modified B-Morrey space and B-BMO space. The inequality of Hardy-Littlewood-Sobolev -Morrey type is established for the B-Riesz potentials.

We obtain necessary and sufficient conditions for the operator $I_{\alpha,\gamma}$ to be bounded from the modified B-Morrey space $\widetilde{L}_{p,\lambda,\gamma}$ to $\widetilde{L}_{q,\lambda,\gamma}$.

The structure of the paper is as follows. In Section 1 we present some definitions and auxiliary results. In Section 2 we study some embeddings into the modified B-Morrey spaces. In Section 3 the boundedness of the B-maximal operator on modified B-Morrey space $\widetilde{L}_{p,\lambda,\gamma}$ is proved. The main result of the paper is the inequality of Hardy-Littlewood-Sobolev-Morrey type for the B-Riesz potential, established in Section 4.

1. Definitions, notation and preliminaries

Suppose that $\mathbb{R}^n$ is n-dimensional Euclidean space, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $|x|^2 = \sum_{i=1}^n x_i^2$, $1 \le k \le n$, $n \ge 2$, $x' = (x_1, \dots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \dots, x_n) \in \mathbb{R}^{n-k}$, $x = (x', x'') \in \mathbb{R}^n$, $\mathbb{R}^n_{k,+} = \{x = (x', x'') \in \mathbb{R}^n; x_1 > 0, \dots, x_k > 0\}$, $E(x,t) = \{y \in \mathbb{R}^n_{k,+}\ ;\ |x-y| < t\}$, $E(0,r) = E_r$, $\gamma = (\gamma_1, \dots, \gamma_k)$, $\gamma_1 > 0, \dots, \gamma_k > 0$, $|\gamma| = \gamma_1 + \ldots + \gamma_k$, $(x')^\gamma = x_1^{\gamma_1} \cdot \ldots \cdot x_k^{\gamma_k}$.

For measurable $E \subset \mathbb{R}^n_{k,+}$ suppose $|E|_\gamma = \int_E (x')^\gamma dx$, then $|E_r|_\gamma = \omega(n,k,\gamma) r^Q$, $Q = n + |\gamma|$, where

$$\omega(n,k,\gamma) = \int_{E_1} (x')^\gamma dx = 2^{-k} \pi^{\frac{n-k}{2}}\ \Gamma^{-1}\big((Q+2)/2\big) \prod_{i=1}^{k} \Gamma\big((\gamma_i+1)/2\big).$$

Denote by T^y the generalized shift operator (B–shift operator) acting according to the law

$$T^y f(x) = C_{\gamma,k} \int_0^\pi \dots \int_0^\pi f\left((x', y')_\beta, x'' - y''\right)\ d\nu(\beta),$$

where $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos\beta_i + y_i^2)^{\frac{1}{2}}$, $1 \le i \le k$, $(x', y')_\beta = ((x_1, y_1)_{\beta_1}, \dots, (x_k, y_k)_{\beta_k})$, $d\nu\,(\beta) = \prod\limits_{i=1}^{k} \sin^{\gamma_i - 1}\beta_i\ d\beta_1 \dots d\beta_k$, $1 \le k \le n$ and

$$C_{\gamma,k} = \pi^{-\frac{k}{2}} \quad \prod_{i=1}^{k} \frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{\gamma_i}{2}\right)}.$$

We remark that the generalized shift operator T^y is closely connected with the Bessel differential operator B.

Let $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ be the space of measurable functions on $\mathbb{R}^n_{k,+}$ with finite norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\mathbb{R}^n_{k,+})} = \left(\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma dx \right)^{1/p}, \quad 1 \le p < \infty.$$

For $p = \infty$ the spaces $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$ are defined by means of the usual modification

$$\|f\|_{L_{\infty,\gamma}} = \|f\|_{L_\infty} = \underset{x \in \mathbb{R}^n_{k,+}}{esssup} |f(x)|.$$

The translation operator T^y generates the corresponding B-convolution

$$(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y) T^y g(x) (y')^\gamma dy,$$

for which the Young inequality

$$\|f \otimes g\|_{L_{r,\gamma}} \le \|f\|_{L_{p,\gamma}} \|g\|_{L_{q,\gamma}}, \quad 1 \le p, q, r \le \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1$$

holds.

Lemma 1.

$$\int_{\mathbb{R}^n_{k,+}} g(y)(y')^\gamma dy = C^{-1}_{\gamma,k} \int_{\mathbb{R}^n \times (0,\infty)^k} g\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z'' \right) d\mu(z, \overline{z'}),$$

where $(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k$, $d\mu(z, \overline{z'}) = (\overline{z'})^{\gamma - 1} dz d\overline{z'}$, $d\overline{z'} = d\overline{z}_1 \cdots d\overline{z}_k$, $(\overline{z'})^{\gamma-1} = (\overline{z}_1)^{\gamma_1 - 1} \cdots (\overline{z}_k)^{\gamma_k - 1}$.

Lemma 2. *For all* $x \in \mathbb{R}^n_{k,+}$ *the following equality is valid*

$$\int_{E(x,t)} g(y)(y')^\gamma dy = C^{-1}_{\gamma,k} \int_{B((x,0),t)} g\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z'' \right) d\mu(z, \overline{z'}),$$

where $B((x,0),t) = \{(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \; : \; |(x_1 - \sqrt{z_1^2 + \overline{z}_1^2}, \ldots, x_k - \sqrt{z_k^2 + \overline{z}_k^2}, x'' - z'')| < t\}$, $d\mu(z, \overline{z'}) = (\overline{z'})^{\gamma-1} dz d\overline{z'}$, $d\overline{z'} = d\overline{z}_1 \cdots d\overline{z}_k$, $(\overline{z'})^{\gamma-1} = (\overline{z}_1)^{\gamma_1-1} \cdots (\overline{z}_k)^{\gamma_k - 1}$.

Lemma 3. *For all* $x \in \mathbb{R}^n_{k,+}$ *the following equality is valid*

$$\int_{E_t} T^y g(x)(y')^\gamma dy = \int_{E((x,0),t)} g\left(\sqrt{z_1^2 + \overline{z}_1^2}, \ldots, \sqrt{z_k^2 + \overline{z}_k^2}, z'' \right) d\mu(z, \overline{z'}),$$

where $E((x,0),t) = \{(z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k \; : \; \left|(x - z, \overline{z'})\right| < t\}$.

The proof of Lemmas 2, 3 is straightforward via the following substitutions

$$z'' = x'', z_i = x_i \cos\alpha_i, \quad \overline{z_i} = x_i \sin\alpha_i, \quad 0 \le \alpha_i < \pi, \; i = 1, \ldots, k,$$
$$x \in \mathbb{R}^n_{k,+}, \; \overline{z'} = (\overline{z}_1, \ldots, \overline{z}_k), \; (z, \overline{z'}) \in \mathbb{R}^n \times (0,\infty)^k, \; 1 \le k \le n.$$

Lemma 4. *Let* $0 < \alpha < Q$. *Then for* $2|x| \le |y|$ *the following inequality is valid*

$$\left| T^y |x|^{\alpha - Q} - |y|^{\alpha - Q} \right| \le 2^{Q - \alpha + 1} |y|^{\alpha - Q - 1} |x|. \tag{1}$$

2. Some embeddings into the modified B-Morrey spaces

Definition 3. *Let $1 \le p < \infty$. By $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ we denote the weak $L_{p,\gamma}$ space defined as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$ with the finite norms*

$$\|f\|_{WL_{p,\gamma}} = \sup_{r>0} r \left|\left\{x \in \mathbb{R}^n_{k,+} \ : \ |f(x)| > r\right\}\right|_{\gamma}^{1/p}.$$

Definition 4. *[11] Let $1 \le p < \infty$, $0 \le \lambda \le Q$, $[t]_1 = \min\{1, t\}$. We denote by $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the modified Morrey space, ($\equiv$ modified B-Morrey space) associated with the Laplace-Bessel differential operator as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with the finite norms*

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left([t]_1^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p}$$

respectively.

Note that

$$\widetilde{L}_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+}),$$

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\gamma}(\mathbb{R}^n_{k,+}) \quad \text{and} \quad \|f\|_{L_{p,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \tag{2}$$

and if $\lambda < 0$ or $\lambda > Q$, then $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$.

Note that

$$\widetilde{L}_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+}),$$

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\gamma}(\mathbb{R}^n_{k,+}) \quad \text{and} \quad \|f\|_{L_{p,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}, \tag{3}$$

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \quad \text{and} \quad \|f\|_{L_{p,\lambda,\gamma}} \le \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \tag{4}$$

and if $\lambda < 0$ or $\lambda > Q$, then $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n_{k,+}$.

Lemma 5. *Let $1 \le p < \infty$, $0 \le \lambda \le Q$. Then*

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+})$$

and for $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ $\|f\|_{L_{p,\lambda,\gamma}} = \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$.

Proof. Let $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. Then by (3) and (4) we have

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+}).$$

Therefore, $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+})$ and the embedding $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+})$ is valid.

Let $f \in L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+})$. Then

$$\|f\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},t>0} \left([t]_1^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p}$$

$$= \max\left\{ \sup_{x\in\mathbb{R}^n_{k,+},0<t\le 1} \left(t^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p}, \sup_{x\in\mathbb{R}^n_{k,+},t>1} \left(\int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p} \right\}$$

$$\le \max\left\{ \|f\|_{L_{p,\lambda,\gamma}}, \|f\|_{L_{p,\gamma}} \right\}.$$

Therefore, $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and the embedding $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ is valid.

Thus $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Let now $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. Then

$$\|f\|_{L_{p,\lambda,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},t>0} \left(t^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p}$$

$$= \sup_{x\in\mathbb{R}^n_{k,+},t>0} \left(t^{-1}[t]_1\right)^{\frac{\lambda}{p}} \left([t]_1^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p}$$

$$= \sup_{x\in\mathbb{R}^n_{k,+},t>0} \left([t]_1^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p} = \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

It is known that for $1 \le p < \infty$

$$L_{p,Q,\gamma}(\mathbb{R}^n_{k,+}) = L_\infty(\mathbb{R}^n_{k,+}) \quad \text{and} \quad \|f\|_{L_{p,Q,\gamma}} = \omega(n,k,\gamma)^{1/p} \|f\|_{L_\infty}. \tag{5}$$

From (5) and Lemma 5 for $1 \le p < \infty$ we have

$$\widetilde{L}_{p,Q,\gamma}(\mathbb{R}^n_{k,+}) = L_\infty(\mathbb{R}^n_{k,+}) \cap L_{p,\gamma}(\mathbb{R}^n_{k,+}). \tag{6}$$

Lemma 6. *Let $1 \le p < \infty$, $0 < \alpha < Q$, $0 \le \lambda < Q$. Then for $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$*

$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset\!\!\succ \widetilde{L}_{1,Q-\alpha,\gamma}(\mathbb{R}^n_{k,+})$$

and for $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the following inequality

$$\|f\|_{\widetilde{L}_{1,Q-\alpha,\gamma}} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

is valid.

Proof. Let $0 < \alpha < Q$, $0 \le \lambda < Q$, $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ and $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$. By the Hölder's inequality we have

$$\|f\|_{\widetilde{L}_{1,Q-\alpha,\gamma}} = \sup_{x\in\mathbb{R}^n_{k,+},t>0} [t]_1^{\alpha-n-\gamma} \int_{E_t} T^y |f(x)| (y')^\gamma dy$$

$$\leq \omega(n,k,\gamma)^{1/p'} \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left([t]_1\, t^{-1}\right)^{-n/p'} [t]_1^{\alpha-\frac{Q-\lambda}{p}} \left([t]_1^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p}$$

$$= \omega(n,k,\gamma)^{1/p'} \sup_{x\in\mathbb{R}^n_{k,+},\, t>0} \left([t]_1\, t^{-1}\right)^{Q-\alpha} \left([t]_1\, t^{-1}\right)^{-\frac{Q}{p'}} [t]_1^{\alpha-\frac{Q-\lambda}{p}} \left([t]_1^{-\lambda} \int_{E_t} T^y |f(x)|^p (y')^\gamma dy\right)^{1/p}$$

$$\leq \omega(n,k,\gamma)^{1/p'} \, \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \sup_{t>0} \left([t]_1\, t^{-1}\right)^{\frac{Q}{p}-\alpha} [t]_1^{\alpha-\frac{Q-\lambda}{p}}.$$

Note that

$$\sup_{t>0} \left([t]_1\, t^{-1}\right)^{\frac{Q}{p}-\alpha} [t]_1^{\alpha-\frac{Q-\lambda}{p}} = \max\{\sup_{0<t\leq 1} t^{\alpha-\frac{Q-\lambda}{p}}, \sup_{t>1} t^{\alpha-\frac{Q}{p}}\} < \infty$$

$$\Longleftrightarrow \frac{Q-\lambda}{\alpha} \leq p \leq \frac{Q}{\alpha}.$$

Therefore $f \in \widetilde{L}_{1,Q-\alpha,\gamma}(\mathbb{R}^n_{k,+})$ and

$$\|f\|_{\widetilde{L}_{1,Q-\alpha,\gamma}} \leq \omega(n,k,\gamma)^{1/p'} \, \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Definition 5. *[13] Let $1 \leq p < \infty$, $0 \leq \lambda \leq Q$. We denote by $W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ the modified weak B-Morrey space as the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$ with finite norms*

$$\|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} = \sup_{r>0} r \sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \left([t]_1^{-\lambda} \int_{\{y\in E_t:\, T^y|f(x)|>r\}} (y')^\gamma dy\right)^{1/p}$$

respectively.

Note that

$$W L_{p,\gamma}(\mathbb{R}^n_{k,+}) = W\widetilde{L}_{p,0,\gamma}(\mathbb{R}^n_{k,+}),$$
$$\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \subset W\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \text{ and } \|f\|_{W\widetilde{L}_{p,\lambda,\gamma}} \leq \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Definition 6. *[10] We denote by $BMO_\gamma(\mathbb{R}^n_{k,+})$ B–BMO space the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n_{k,+}$, with finite norms*

$$\|f\|_{*,\gamma} = \sup_{t>0,\ x\in\mathbb{R}^n_{k,+}} |E_t|_\gamma^{-1} \int_{E_t} |T^x f(y) - f_{E_t}(x)| (y')^\gamma dy < \infty,$$

where $f_{E_t}(x) = |E_t|_\gamma^{-1} \int_{E_t} [T^x f(y)\,](y')^\gamma dy$.

3. $\widetilde{L}_{p,\lambda,\gamma}$-boundedness of the B-maximal operator

In this section we study the $L_{p,\lambda,\gamma}$-boundedness of the B-maximal operator (see [10])

$$M_\gamma f(x) = \sup_{t>0} |E_t|_\gamma^{-1} \int_{E_t} T^y |f(x)| (y')^\gamma dy.$$

Theorem 1. *1. If $f \in \widetilde{L}_{1,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $0 \le \lambda < Q$, then $M_\gamma f \in W\widetilde{L}_{1,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and*

$$\|M_\gamma f\|_{W\widetilde{L}_{1,\lambda,\gamma}} \le C_{1,\lambda,\gamma} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}},$$

where $C_{1,\lambda,\gamma}$ is independent of f.

2.If $f \in \widetilde{L}_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $1 < p < \infty$,$0 \le \lambda < Q$, then $M_\gamma f \in \widetilde{L}_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and

$$\|M_\gamma f\|_{\widetilde{L}_{p,\lambda,\gamma}} \le C_{p,\lambda,\gamma} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

where $C_{p,\lambda,\gamma}$ depends only on p,λ,γ,k and n.

Proof. We need to introduce the maximal operator defined on a space of homogeneous type (Y, d, ν). By this we mean a topological space $Y = \mathbb{R}^n \times (0,\infty)^k$ equipped with a continuous pseudometric d and a positive measure ν satisfying

$$\nu(E((x,\overline{x'}), 2t)) \le C_1 \nu(E((x,\overline{x'}), t)) \tag{7}$$

with a constant C_1 independent of $(x,\overline{x'})$ and $t > 0$. Here $E((x,\overline{x'}),t) = \{(y,\overline{y'}) \in Y : d(((x,\overline{x'}),(y,\overline{y'})) < t\}$, $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1} dy\, d\overline{y'}$, $(\overline{y'})^{\gamma-1} = (\overline{y_1})^{\gamma_1-1} \cdots (\overline{y_k})^{\gamma_k-1}$, $d((x,\overline{x'}),(y,\overline{y'})) = |(x,\overline{x'}) - (y,\overline{y'})| \equiv (|x-y|^2 + (\overline{x'}-\overline{y'})^2)^{\frac{1}{2}}$.

Let (Y, d, ν) be a space of homogeneous type. Define

$$M_\nu \overline{f}(x,\overline{x'}) = \sup_{t>0} \nu(E((x,\overline{x'}),t))^{-1} \int_{E((x,\overline{x'}),t)} \left|\overline{f}(y,\overline{y'})\right| d\nu(y),$$

where $\overline{f}(x,\overline{x'}) = f\left(\sqrt{x_1^2+\overline{x}_1^2}, \dots, \sqrt{x_k^2+\overline{x}_k^2}, x''\right)$.

It is well known that the maximal operator M_ν is of weak type (1, 1) and is bounded on $L_p(Y, d\nu)$ for $1 < p < \infty$ (see [7]). Here we are concerned with the maximal operator defined by $d\nu(y,\overline{y'}) = (\overline{y'})^{\gamma-1} dy\, d\overline{y'}$. It is clear that this measure satisfies the doubling condition (7).

It can be proved that

$$M_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2}, \dots, \sqrt{z_k^2+\overline{z}_k^2}, z''\right) = M_\nu \overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2}, \dots, \sqrt{z_k^2+\overline{z}_k^2}, z'', 0\right), \tag{8}$$

$$M_\gamma f(x) = M_\nu \overline{f}(x, 0). \tag{9}$$

Indeed, Lemma 3

$$\int_{E_t} T^y \left| f\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\right| (y')^\gamma \, dy$$
$$= \int\limits_{E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),t\right)} \left|\overline{f}(y,\overline{y'})\right| \, d\nu(y,\overline{y'})$$

and

$$|E_t|_\gamma = \nu E\left(\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right),t\right)$$

imply (8). Furthermore, taking $\overline{z}_k = 0$ in (8) we get (9).

Using Lemma 3 and equality (8) we have

$$\int_{E_t} T^y (M_\gamma f(x))^p (y')^\gamma \, dy$$
$$= \int_{E((x,0),t)} \left(M_\gamma f\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z''\right)\right)^p \, d\nu(z,\overline{z'})$$
$$= \int_{E((x,0),t)} \left(M_\nu \overline{f}\left(\sqrt{z_1^2+\overline{z}_1^2},\ldots,\sqrt{z_k^2+\overline{z}_k^2},z'',0\right)\right)^p \, d\nu(z,\overline{z'}).$$

In [19] it was proved that the analogue of the Fefferman-Stein theorem for the maximal operator defined on a space of homogeneous type is valid, if condition (7) is satisfied. Therefore

$$\int_{E((x,\overline{x'}),t)} \left(M_\nu \varphi(y,\overline{y'})\right)^p \psi(y,\overline{y'}) d\nu(y,\overline{y'})$$
$$\le C_2 \int_{E((x,\overline{x'}),t)} |\varphi(y,\overline{y'})|^p M_\nu \psi(y,\overline{y'}) \, d\nu(y,\overline{y'}). \tag{10}$$

Then taking $\varphi(y,\overline{y'}) = \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)$ and $\psi(y,\overline{y'}) \equiv 1$ we obtain from inequality (10) and Lemma 3 that

$$\int_{E_t} T^y (M_\gamma f(x))^p (y')^\gamma dy$$
$$= \int_{E((x,0),t)} \left(M_\nu \overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right)^p \, d\nu(y,\overline{y'})$$
$$\le C_2 \int_{E((x,0),t)} \left|\overline{f}\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y'',0\right)\right|^p \, d\nu(y,\overline{y'})$$
$$= C_2 \int_{E((x,0),t)} \left|f\left(\sqrt{y_1^2+\overline{y}_1^2},\ldots,\sqrt{y_k^2+\overline{y}_k^2},y''\right)\right|^p \, d\nu(y,\overline{y'})$$
$$= C_2 \int_{E_t} T^y |f(x)|^p (y')^\gamma dy \le C_2 \; [t]_1^\lambda \; \|f\|^p_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Corollary 1. *Let* $f \in L_{1,\gamma}^{loc}\left(\mathbb{R}_{k,+}^n\right)$, *then*

$$\lim_{t\to 0} |E_t|_\gamma^{-1} \int_{E_t} T^y f(x)\ (y')^\gamma dy = f(x)$$

for almost $x \in \mathbb{R}_{k,+}^n$.

Corollary 2. *[12]*

1.If $f \in L_{1,\gamma}\left(\mathbb{R}_{k,+}^n\right)$, *then* $M_\gamma f \in WL_{1,\gamma}(\mathbb{R}_{k,+}^n)$ *and*

$$\|M_\gamma f\|_{WL_{1,\gamma}} \le C_{1,\gamma} \|f\|_{L_{1,\gamma}},$$

where $C_{1,\gamma}$ *is independent of* f.

2. If $f \in L_{p,\gamma}(\mathbb{R}_{k,+}^n)$, $1 < p \le \infty$, *then* $M_\gamma f \in L_{p,\gamma}(\mathbb{R}_{k,+}^n)$ *and*

$$\|M_\gamma f\|_{L_{p,\gamma}} \le C_{p,\gamma} \|f\|_{L_{p,\gamma}},$$

where $C_{p,\gamma}$ *depends only on* p, γ, k *and* n.

In Theorem 1 if we take $\lambda = 0$, we obtained Corollary 2.

4. Hardy-Littlewood-Sobolev-Morrey type inequality for B-Riesz potential

We consider the fractional B-maximal operator

$$M_{\alpha,\gamma} f(x) = \sup_{r>0} |E_r|_\gamma^{\frac{\alpha}{Q}-1} \int_{E_r} T^y |f(x)| (y')^\gamma dy, \quad 0 \le \alpha < Q,$$

the B-Riesz potential

$$I_{\alpha,\gamma} f(x) = \int_{\mathbb{R}_{k,+}^n} T^y |x|^{\alpha-Q} f(y) (y')^\gamma dy, \quad 0 < \alpha < Q,$$

and the modified B-Riesz potential

$$\widetilde{I}_{\alpha,\gamma} f(x) = \int_{\mathbb{R}_{k,+}^n} \left(T^y |x|^{\alpha-Q} - |y|^{\alpha-Q} \chi_{E_1^*}(y)\right) f(y) (y')^\gamma dy,$$

where $E_1^* = \mathbb{R}_{k,+}^n \backslash E_1$.

The examples show that the B-Riesz potential $I_{\alpha,\gamma}$ is not defined for all functions $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}_{k,+}^n)$, $0 \le \lambda < Q$, if $p \ge \frac{Q}{\alpha}$.

For the B-Riesz potential the following Hardy-Littlewood-Sobolev-Morrey type inequality is valid.

Theorem 2. *Let $0<\alpha<Q$, $0\le\lambda<Q$ and $1\le p\le\frac{Q-\lambda}{\alpha}$.*

1) If $1<p<\frac{Q-\lambda}{\alpha}$, then condition $\frac{\alpha}{Q}\le\frac{1}{p}-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of $I_{\alpha,\gamma}$ from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p=1$, then condition $\frac{\alpha}{Q}\le 1-\frac{1}{q}\le\frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of $I_{\alpha,\gamma}$ from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

3) If $\frac{Q-\lambda}{\alpha}\le p\le\frac{Q}{\alpha}$, then the operator $\widetilde{I}_{\alpha,\gamma}$ is bounded from $\widetilde{L}_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ to $BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$.

Moreover, for $f\in\widetilde{L}_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $\frac{Q-\lambda}{\alpha}\le p\le\frac{Q}{\alpha}$ the integral $I_{\alpha,\gamma}f$ exists almost everywhere, then $I_{\alpha,\gamma}f\in BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$ and the following inequality is valid

$$\|I_{\alpha,\gamma}f\|_{BMO_\gamma}\le C\|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

where $C>0$ is independent of f.

Proof. 1) *Sufficiency.* Let $f\in\widetilde{L}_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$. Then

$$I_{\alpha,\gamma}f(x)=\left(\int_{E_t}+\int_{\mathbb{R}^n_{k,+}\setminus E_t}\right)T^yf(x)|y|^{\alpha-Q}(y')^\gamma dy$$
$$\equiv A(x,t)+C(x,t). \quad (11)$$

For $A(x,t)$ we have

$$|A(x,t)|\le\int_{E_t}T^y\left|f(x)\right|\left|y\right|^{\alpha-Q}(y')^\gamma dy$$
$$\le\sum_{j=-\infty}^{-1}\left(2^jt\right)^{\alpha-Q}\int_{E_{2^{j+1}t}\setminus E_{2^jt}}T^y|f(x)|(y')^\gamma dy.$$

Hence

$$|A(x,t)|\le C_3t^\alpha M_\gamma f(x)\qquad\text{with}\qquad C_3=\frac{\omega(n,k,\gamma)2^Q}{2^\alpha-1}. \quad (12)$$

For $C(x,t)$ by the Hölder's inequality we have

$$|C(x,t)|\le\left(\int_{\mathbb{R}^n_{k,+}\setminus E_t}|y|^{-\beta}T^y\left|f\left(x\right)\right|^p(y')^\gamma dy\right)^{1/p}$$
$$\times\left(\int_{\mathbb{R}^n_{k,+}\setminus E_t}|y|^{\left(\frac{\beta}{p}+\alpha-Q\right)p'}(y')^\gamma dy\right)^{1/p'}=J_1\cdot J_2. \quad (13)$$

Let $\lambda < \beta < Q - \alpha p$. For J_1 we get

$$J_1 = \Big(\sum_{j=0}^{\infty} \int_{E_{2^{j+1}t} \setminus E_{2^j t}} T^y |f(x)|^p |y|^{-\beta} (y')^\gamma dy \Big)^{1/p}$$

$$\leq t^{-\frac{\beta}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \Big(\sum_{j=0}^{\infty} 2^{-\beta j} [2^{j+1} t]_1^{\lambda} \Big)^{1/p}$$

$$= t^{-\frac{\beta}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \left(\begin{array}{ll} \Big(2^{\lambda} t^{\lambda} \sum\limits_{j=0}^{\log_2[\frac{1}{2t}]} 2^{(\lambda-\beta)j} + \sum\limits_{j=\log_2[\frac{1}{2t}]+1}^{\infty} 2^{-\beta j} \Big)^{1/p}, & 0 < t < 1 \\ \Big(\sum\limits_{j=0}^{\infty} 2^{-\beta j} \Big)^{1/p}, & t \geq 1 \end{array} \right).$$

$$= t^{-\frac{\beta}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \left\{ \begin{array}{ll} (C_4 t^{\lambda} + C_5 t^{\beta})^{\frac{1}{p}}, & 0 < t < 1, \\ C_6, & t \geq 1, \end{array} \right.$$

$$= C_7 [t]_1^{\frac{\lambda}{p}} t^{-\frac{\beta}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} . \tag{14}$$

For J_2 we obtain

$$J_2 = \left(\int_{\mathbb{S}^{n-1}_{k,+}} (\xi')^\gamma d\xi \int_t^{\infty} r^{Q-1+\left(\frac{\beta}{p}+\alpha-Q\right)p'} dr \right)^{\frac{1}{p'}} = C_8 t^{\frac{\beta}{p}+\alpha-\frac{Q}{p}}. \tag{15}$$

From (15) and inequality (14) we have

$$|C(x,t)| \leq C_9 [t]_1^{\frac{\lambda}{p}} t^{\alpha-\frac{Q}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} . \tag{16}$$

Thus, (12) and (16) implies

$$|I_{\alpha,\gamma} f(x)| \leq C_{10} (t^{\alpha} M_\gamma f(x) + [t]_1^{\frac{\lambda}{p}} t^{\alpha-\frac{Q}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}})$$

$$\leq C_{10} \min\{ t^{\alpha} M_\gamma f(x) + t^{\alpha-\frac{Q-\lambda}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} , t^{\alpha} M_\gamma f(x) + t^{\alpha-\frac{Q}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \}.$$

Minimizing with respect to t, at $t = \Big[(M_\gamma f(x))^{-1} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \Big]^{p/(Q-\lambda)}$ or $t = \Big[(M_\gamma f(x))^{-1} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \Big]^{p/Q}$ we have

$$|I_{\alpha,\gamma} f(x)| \leq C_{11} \min \left\{ \left(\frac{M_\gamma f(x)}{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}} \right)^{1-\frac{p\alpha}{Q-\lambda}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} , \left(\frac{M_\gamma f(x)}{\|f\|_{\widetilde{L}_{p,\lambda,\gamma}}} \right)^{1-\frac{p\alpha}{Q}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \right\}.$$

Then

$$|I_{\alpha,\gamma}f(x)| \leq C_{12}\left(M_\gamma f(x)\right)^{p/q}\|f\|^{1-p/q}_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Hence, by Theorem 1, we have

$$\int_{E_t} T^y\left|I_{\alpha,\gamma}f(x)\right|^q (y')^\gamma dy$$
$$\leq C_{12}\|f\|^{q-p}_{\widetilde{L}_{p,\lambda,\gamma}}\int_{E_t} T^y\left(M_\gamma f(y)\right)^p (y')^\gamma dy \leq C_{13}[t]_1^\lambda \|f\|^q_{\widetilde{L}_{p,\lambda,\gamma}},$$

which implies that $I_{\alpha,\gamma}$ is bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Necessity. Let $1 < p < \frac{Q-\lambda}{\alpha}$, $f \in \widetilde{L}_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and $I_{\alpha,\gamma}$ be bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Define $f_t(x) =: f(tx)$. Then

$$\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}}\left([r]_1^{-\lambda}\int_{E_r} T^y|f_t(x)|^p (y')^\gamma dy\right)^{1/p}$$

$$= t^{-\frac{Q}{p}}\sup_{r>0,\, x\in\mathbb{R}^n_{k,+}}\left([r]_1^{-\lambda}\int_{E_{tr}} T^y|f(x)|^p (y')^\gamma dy\right)^{1/p}$$

$$= t^{-\frac{Q}{p}}\sup_{r>0}\left(\frac{[tr]_1}{[r]_1}\right)^{\lambda/p}\sup_{r>0,\, x\in\mathbb{R}^n_{k,+}}\left([tr]_1^{-\lambda}\int_{E_{tr}} T^y\left|f(x)\right|^p (y')^\gamma dy\right)^{1/p}$$

$$= t^{-\frac{Q}{p}}[t]_{1,+}^{\frac{\lambda}{p}}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

where $[t]_{1,+} = \max\{1, t\}$ and

$$\|I_{\alpha,\gamma}f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} = t^{-\alpha}\sup_{r>0,\, x\in\mathbb{R}^n_{k,+}}\left([r]_1^{-\lambda}\int_{E_r} T^{ty}\left|I_{\alpha,\gamma}f(tx)\right|^q (y')^\gamma dy\right)^{1/q}$$

$$= t^{-\alpha-\frac{Q}{q}}\sup_{r>0}\left(\frac{[tr]_1}{[r]_1}\right)^{\lambda/q}\sup_{r>0,\, x\in\mathbb{R}^n_{k,+}}\left([tr]_1^{-\lambda}\int_{E_{tr}} T^y\left|I_{\alpha,\gamma}f(x)\right|^q (y')^\gamma dy\right)^{1/q}$$

$$= t^{-\alpha-\frac{Q}{q}}[t]_{1,+}^{\frac{\lambda}{q}}\|I_{\alpha,\gamma}f\|_{\widetilde{L}_{q,\lambda,\gamma}}.$$

By the boundedness of $I_{\alpha,\gamma}$ from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$

$$\|I_{\alpha,\gamma}f\|_{\widetilde{L}_{q,\lambda,\gamma}} \leq C_{p,q,\lambda,\gamma}t^{\alpha+\frac{Q}{q}-\frac{Q}{p}}[t]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}}\|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

where $C_{p,q,\lambda,\gamma}$ depends only on p,q,λ,γ,k and n.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{Q}$, then in the case $t \to 0$ we have $\|I_{\alpha,\gamma}f\|_{\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then as $t \to \infty$ we obtain $\|I_{\alpha,\gamma} f\|_{\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Therefore $\frac{\alpha}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$.

2) *Sufficiency.* Let $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. We have

$$\begin{aligned}
&|\{y \in E_t \ : \ T^y \, |I_{\alpha,\gamma} f(x)| > 2\beta\}|_\gamma \\
&\le |\{y \in E_t \ : \ T^y |A(x,t)| > \beta\}|_\gamma \\
&+ |\{y \in E_t \ : \ T^y |C(x,t)| > \beta\}|_\gamma .
\end{aligned}$$

Taking into account inequality (12) and Theorem 1, we have

$$\begin{aligned}
&|\{y \in E_t \ : \ T^y |A(x,t)| > \beta\}|_\gamma \\
&\le \left| \left\{ y \in E_t \ : \ T^y \left(M_\gamma f(x) \right) > \frac{\beta}{C_3 t^\alpha} \right\} \right|_\gamma \\
&\le \frac{C_{14} t^\alpha}{\beta} \cdot [t]_1^\lambda \, \|f\|_{\widetilde{L}_{1,\lambda,\gamma}}
\end{aligned}$$

and thus if $C_9 [t]_1^\lambda t^{\alpha - Q} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}} = \beta$, then $|C(x,t)| \le \beta$ and consequently, $|\{y \in E_t \ : \ T^y \, |C(x,t)| > \beta\}|_\gamma = 0$.

Then

$$|\{y \in E_t \ : \ T^y \, |I_{\alpha,\gamma} f(x)| > 2\beta\}|_\gamma \le C_{14} [t]_1^\lambda t^\alpha \frac{\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}}{\beta}$$

$$\le C_{15} [t]_1^\lambda \min \left\{ \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}}{\beta} \right)^{\frac{Q-\lambda}{Q-\lambda-\alpha}}, \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}}{\beta} \right)^{\frac{Q}{Q-\alpha}} \right\}.$$

Finally

$$|\{y \in E_t \ : \ T^y \, |I_{\alpha,\gamma} f(x)| > 2\beta\}|_\gamma \le C_{15} [t]_1^\lambda \left(\frac{\|f\|_{\widetilde{L}_{1,\lambda,\gamma}}}{\beta} \right)^q .$$

Necessity. Let $I_{\alpha,\gamma}$ be bounded from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. We have

$$\begin{aligned}
\|I_{\alpha,\gamma} f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}} &= \sup_{r>0} r \sup_{\tau>0,\, x \in \mathbb{R}^n_{k,+}} \left([\tau]_1^{-\lambda} \int_{\{y \in E_\tau \ : \ T^y |I_{\alpha,\gamma} f_t(x)| > r\}} (y')^\gamma dy \right)^{1/q} \\
&= \sup_{r>0} r \sup_{\tau>0,\, x \in \mathbb{R}^n_{k,+}} \left([\tau]_1^{-\lambda} \int_{\{y \in E_\tau \ : \ T^{ty} |I_{\alpha,\gamma} f(tx)| > r t^\alpha\}} (y')^\gamma dy \right)^{1/q}
\end{aligned}$$

$$= t^{-\alpha-\frac{Q}{q}} \sup_{\tau>0} \left(\frac{[t\tau]_1}{[\tau]_1} \right)^{\lambda/q} \sup_{r>0} rt^{\alpha} \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left([t\tau]_1^{-\lambda} \int_{\{y\in E_{t\tau}\,:\, T^y|I_{\alpha,\gamma}f(x)|>rt^{\alpha}\}} (y')^{\gamma} dy \right)^{1/q}$$

$$= t^{-\alpha-\frac{Q}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \|I_{\alpha,\gamma} f\|_{W\widetilde{L}_{q,\lambda,\gamma}}.$$

By the boundedness of $I_{\alpha,\gamma}$ from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$

$$\|I_{\alpha,\gamma} f\|_{W\widetilde{L}_{q,\lambda,\gamma}} \le C_{1,q,\lambda,\gamma} t^{\alpha+\frac{Q}{q}\ -Q} [t]_{1,+}^{\lambda-\frac{\lambda}{q}} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}},$$

where $C_{1,q,\lambda,\gamma}$ depends only on q,λ,γ,k and n.

If $1 < \frac{1}{q}+\frac{\alpha}{Q}$, then in the case $t \to 0$ we have $\|I_{\alpha,\gamma} f\|_{W\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then as $t \to \infty$ we obtain $\|I_{\alpha,\gamma} f\|_{W\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Therefore $\frac{\alpha}{Q} \le 1 - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$.

3) Let $f \in \widetilde{L}_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$. For given $t > 0$ we denote

$$f_1(z) = f(z)\chi_{E_{2t}}(z), \quad f_2(z) = f(z) - f_1(z), \tag{17}$$

where $\chi_{E_{2t}}$ is the characteristic function of the set E_{2t}. Then

$$\widetilde{I}_{\alpha,\gamma} f(z) = \widetilde{I}_{\alpha,\gamma} f_1(z) + \widetilde{I}_{\alpha,\gamma} f_2(z) = F_1(z) + F_2(z), \tag{18}$$

where

$$F_1(z) = \int_{E_{2t}} \left(T^y|z|^{\alpha-Q} - |y|^{\alpha-Q}\chi_{E_1^*}(y) \right) f(y)(y')^{\gamma} dy,$$

$$F_2(z) = \int_{\mathbb{R}^n_{k,+}\setminus E_{2t}} \left(T^y|z|^{\alpha-Q} - |y|^{\alpha-Q}\chi_{E_1^*}(y) \right) f(y)(y')^{\gamma} dy.$$

Note that the function f_1 has compact (bounded) support and thus

$$a_1 = -\int_{E_{2t}\setminus E_{\min\{1,2t\}}} |y|^{\alpha-Q} f(y)(y')^{\gamma} dy$$

is finite.

Note also that

$$F_1(z) - a_1 = \int_{E_{2t}} T^y|z|^{\alpha-Q}\, f(y)(y')^{\gamma} dy - \int_{E_{2t}\setminus E_{\min\{1,2t\}}} |y|^{\alpha-Q} f(y)(y')^{\gamma} dy + \int_{E_{2t}\setminus E_{\min\{1,2t\}}} |y|^{\alpha-Q} f(y)(y')^{\gamma} dy$$

$$= \int_{\mathbb{R}^n_{k,+}} T^y |z|^{\alpha-Q} f_1(y)(y')^\gamma dy.$$

Therefore

$$|F_1(z) - a_1| \le \int_{\mathbb{R}^n_{k,+}} |y|^{\alpha-Q}\, T^y |f_1(z)|\, (y')^\gamma dy$$

$$= \int_{\{y\in\mathbb{R}^n_{k,+} : T^y|z|<2t\}} |y|^{\alpha-Q}\, T^y |f(z)|\, (y')^\gamma dy.$$

Further, for $z \in E_t$, $T^y|z| < 2t$ we have

$$|y| \le |z| + |z-y| \le |z| + T^y|z| < 3t.$$

Consequently

$$|F_1(z) - a_1| \le \int_{E_{3t}} |y|^{\alpha-Q}\, T^y |f(z)|\, (y')^\gamma dy, \tag{19}$$

if $z \in E_t$.

Thus, from (19) and Lemma 6, we have

$$\int_{E_t} |T^x F_1(z) - a_1|\, (z')^\gamma dz \le \int_{E_t} T^x |F_1(z) - a_1| (z')^\gamma dz$$

$$= \int_{E_t} \left(\int_{E_{3t}} |y|^{\alpha-Q} T^x T^y |f(z)| (y')^\gamma dy \right) (z')^\gamma dz$$

$$= \int_{E_{3t}} |y|^{\alpha-Q}\; T^x \left(\int_{E_t} T^y |f(z)| (z')^\gamma dz \right) (y')^\gamma dy$$

$$\le [t]_1^{Q-\alpha}\, \|f\|_{\widetilde{L}_{1,Q-\alpha,\gamma}} \int_{E_{3t}} |y|^{\alpha-Q} (y')^\gamma dy$$

$$\le [t]_1^{Q-\alpha} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \int_{E_{3t}} |y|^{\alpha-Q} (y')^\gamma dy. \tag{20}$$

In inequality (20) we take into account that $f \in \widetilde{L}_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and

$$\int_{E_{3t}} |y|^{\alpha-Q} (y')^\gamma dy \le C_{19} t^\alpha, \tag{21}$$

respectively, where $C_{19} = \frac{3^\alpha}{\alpha} \omega(n,k,\gamma)$.

Therefore,

$$\sup_{t>0,\, x\in\mathbb{R}^n_{k,+}} \frac{1}{t^Q} \int_{E_t} |T^x F_1(z) - a_1|\, (z')^\gamma dz$$

$$\le C_{19} \sup_{t>0} \left(\frac{[t]_1}{t} \right)^{Q-\alpha} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \le C_{19}\, \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{22}$$

(22) and condition $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$ implies the following inequality

$$|E_t|^{-1} \int_{E_t} |T^x F_1(z) - a_1| \, (z')^\gamma dz \le C_{16} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{23}$$

Denote

$$a_2 = \int_{E_{\max\{1,2t\}} \setminus E_{2t}} |y|^{\alpha-Q} f(y)(y')^\gamma dy.$$

Let's estimate $|F_2\,(z) - a_2|$ for $z \in E_t$.

$$|F_2(z) - a_2| \le \int_{\mathbb{R}^n_{k,+} \setminus E_{2t}} |f(y)| \left| T^y |z|^{\alpha-Q} - |y|^{\alpha-Q} \right| (y')^\gamma dy.$$

Applying Lemma 4 and Hölder's inequality, we have

$$|F_2(z) - a_2| \le 2^{Q-\alpha+1} |z| \int_{\mathbb{R}^n_{k,+} \setminus E_{2t}} |f(y)||y|^{\alpha-Q-1} (y')^\gamma dy$$

$$\le 2^{Q-\alpha+1} |z| \sum_{j=0}^{\infty} \int_{E_{2^{j+2}t} \setminus E_{2^{j+1}t}} |f(y)||y|^{\alpha-Q-1} (y')^\gamma dy$$

$$\le 2^{Q-\alpha+1} |z| \sum_{j=0}^{\infty} (2^{j+1} t)^{\alpha-Q-1} \int_{E_{2^{j+2}t}} |f(y)|(y')^\gamma dy$$

$$\le 2^{Q-\alpha+1} |z| \sum_{j=0}^{\infty} (2^{j+1} t)^{\alpha-Q-1} (2^{j+2} t)^{\frac{Q}{p'}} \Big(\int_{E_{2^{j+2}t}} |f(y)|^p (y')^\gamma dy \Big)^{1/p}$$

$$\le 4^{Q-\alpha+1} |z| \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \sum_{j=0}^{\infty} (2^{j+2} t)^{\alpha - \frac{Q}{p} - 1} [2^{j+2} t]_1^{\frac{\lambda}{p}}$$

$$\le 4^{Q-\alpha+1} t^{\alpha-\frac{Q}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}} \times$$

$$\times \begin{cases} t^{\frac{\lambda}{p}} \displaystyle\sum_{j=0}^{\log_2[\frac{1}{2t}]} 2^{(j+2)(\alpha-\frac{Q}{p}-1+\frac{\lambda}{p})} + \sum_{j=\log_2[\frac{1}{2t}]+1}^{\infty} 2^{(j+2)(\alpha-\frac{Q}{p}-1)}, & 0 < t < 1/2, \\ \displaystyle\sum_{j=0}^{\infty} 2^{(j+2)(\alpha-\frac{Q}{p}-1)}, & t \ge 1/2 \end{cases}$$

$$= 4^{Q-\alpha+1} C_{17} [t]_1^{\frac{\lambda}{p}} t^{\alpha-\frac{Q}{p}} \, \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}$$

$$= C_{18} [t]_1^{\frac{\lambda}{p}} t^{\alpha-\frac{Q}{p}} \, \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}. \tag{24}$$

(24) and condition $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$ implies the following inequality

$$|T^x F_2(z) - a_2| \le T^x |F_2(z) - a_2| \le C_{18} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}, \; for \; all \; x \in \mathbb{R}^n_{k,+}, \; z \in E_t. \qquad (25)$$

Finally, from (23) and (25) we have

$$\sup_{x,t} \frac{1}{|E_t|_\gamma} \int_{E_t} \left| T^x \widetilde{I}_{\alpha,\gamma} f(y) - a_f \right| (y')^\gamma dy \le (C_{16} + C_{18}) \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Thus

$$\left\| \widetilde{I}_{\alpha,\gamma} f \right\|_{BMO_\gamma} \le 2 \sup_{x,t} \frac{1}{|E_t|_\gamma} \int_{E_t} \left| T^x \widetilde{I}_{\alpha,\gamma} f(y) - a_f \right| (y')^\gamma dy \le C \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Theorem 2 is proved.

Theorem 3. *Let $0 < \alpha < Q$, $0 \le \lambda < Q$ and $1 \le p \le \frac{Q-\lambda}{\alpha}$.*

1) If $1 < p < \frac{Q-\lambda}{\alpha}$, then condition $\frac{\alpha}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of $M_{\alpha,\gamma}$ from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p = 1$, then condition $\frac{\alpha}{Q} \le 1 - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$ is necessary and sufficient for the boundedness of $M_{\alpha,\gamma}$ from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Proof. In view of the well known pointwise estimate $M_{\alpha,\gamma} f(x) \le C(I_{\alpha,\gamma}|f|)(x)$, it suffices to treat only the case of the operator $I_{\alpha,\gamma}$.

1) *Necessity.* Let $1 < p < \frac{Q-\lambda}{\alpha}$, $f \in \widetilde{L}_{p,\lambda,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ and $M_{\alpha,\gamma}$ be bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Define $f_t(x) =: f(tx)$. Then

$$\|f_t\|_{\widetilde{L}_{p,\lambda,\gamma}} = \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda} \int_{E_r} T^y |f_t(x)|^p (y')^\gamma dy \right)^{1/p}$$

$$= t^{-\frac{Q}{p}} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda} \int_{E_{tr}} T^y |f(x)|^p (y')^\gamma dy \right)^{1/p}$$

$$= t^{-\frac{Q}{p}} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1} \right)^{\lambda/p} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([tr]_1^{-\lambda} \int_{E_{tr}} T^y \left| f(x) \right|^p (y')^\gamma dy \right)^{1/p}$$

$$= t^{-\frac{Q}{p}} [t]_{1,+}^{\frac{\lambda}{p}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

where $[t]_{1,+} = \max\{1, t\}$ and

$$\|M_{\alpha,\gamma} f_t\|_{\widetilde{L}_{q,\lambda,\gamma}} = t^{-\alpha} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([r]_1^{-\lambda} \int_{E_r} T^{ty} \left| M_{\alpha,\gamma} f(tx) \right|^q (y')^\gamma dy \right)^{1/q}$$

$$= t^{-\alpha-\frac{Q}{q}} \sup_{r>0} \left(\frac{[tr]_1}{[r]_1}\right)^{\lambda/q} \sup_{r>0,\, x\in\mathbb{R}^n_{k,+}} \left([tr]_1^{-\lambda} \int_{E_{tr}} T^y \left|M_{\alpha,\gamma} f(x)\right|^q (y')^\gamma dy\right)^{1/q}$$

$$= t^{-\alpha-\frac{Q}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \left\|M_{\alpha,\gamma} f\right\|_{\widetilde{L}_{q,\lambda,\gamma}}.$$

By the boundedness of $I_{\alpha,\gamma}$ from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$

$$\|I_{\alpha,\gamma} f\|_{\widetilde{L}_{q,\lambda,\gamma}} \le C_{p,q,\lambda,\gamma} t^{\alpha+\frac{Q}{q}-\frac{Q}{p}} [t]_{1,+}^{\frac{\lambda}{p}-\frac{\lambda}{q}} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}},$$

where $C_{p,q,\lambda,\gamma}$ depends only on p,q,λ,γ,k and n.

If $\frac{1}{p} < \frac{1}{q} + \frac{\alpha}{Q}$, then in the case $t \to 0$ we have $\|M_{\alpha,\gamma} f\|_{\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

As well as if $\frac{1}{p} > \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then as $t \to \infty$ we obtain $\|M_{\alpha,\gamma} f\|_{\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Therefore $\frac{\alpha}{Q} \le \frac{1}{p} - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$.

2) *Necessity.* Let $M_{\alpha,\gamma}$ be bounded from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. We have

$$\|M_{\alpha,\gamma} f_t\|_{W\widetilde{L}_{q,\lambda,\gamma}} = \sup_{r>0} r \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left([\tau]_1^{-\lambda} \int_{\{y\in E_\tau \,:\, T^y|M_{\alpha,\gamma} f_t(x)|>r\}} (y')^\gamma dy\right)^{1/q}$$

$$= \sup_{r>0} r \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left([\tau]_1^{-\lambda} \int_{\{y\in E_\tau \,:\, T^{ty}|M_{\alpha,\gamma} f(tx)|>rt^\alpha\}} (y')^\gamma dy\right)^{1/q}$$

$$= t^{-\alpha-\frac{Q}{q}} \sup_{\tau>0} \left(\frac{[t\tau]_1}{[\tau]_1}\right)^{\lambda/q} \sup_{r>0} rt^\alpha \sup_{\tau>0,\, x\in\mathbb{R}^n_{k,+}} \left([t\tau]_1^{-\lambda} \int_{\{y\in E_{t\tau} \,:\, T^y|M_{\alpha,\gamma} f(x)|>rt^\alpha\}} (y')^\gamma dy\right)^{1/q}$$

$$= t^{-\alpha-\frac{Q}{q}} [t]_{1,+}^{\frac{\lambda}{q}} \|M_{\alpha,\gamma} f\|_{W\widetilde{L}_{q,\lambda,\gamma}}.$$

By the boundedness of $M_{\alpha,\gamma}$ from $\widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $W\widetilde{L}_{q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$

$$\|M_{\alpha,\gamma} f\|_{W\widetilde{L}_{q,\lambda,\gamma}} \le C_{1,q,\lambda,\gamma} t^{\alpha+\frac{Q}{q}\ -Q} [t]_{1,+}^{\lambda-\frac{\lambda}{q}} \|f\|_{\widetilde{L}_{1,\lambda,\gamma}},$$

where $C_{1,q,\lambda,\gamma}$ depends only on q,λ,γ,k and n.

If $1 < \frac{1}{q} + \frac{\alpha}{Q}$, then in the case $t \to 0$ we have $\|M_{\alpha,\gamma} f\|_{W\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Similarly, if $1 > \frac{1}{q} + \frac{\alpha}{Q-\lambda}$, then as $t \to \infty$ we obtain $\|M_{\alpha,\gamma} f\|_{W\widetilde{L}_{q,\lambda,\gamma}} = 0$ for all $f \in \widetilde{L}_{1,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Therefore $\frac{\alpha}{Q} \le 1 - \frac{1}{q} \le \frac{\alpha}{Q-\lambda}$.

Theorem 4. *Let $0 < \alpha < Q$, $0 \le \lambda < Q$ and $\frac{Q-\lambda}{\alpha} \le p \le \frac{Q}{\alpha}$. Then the operator $M_{\alpha,\gamma}$ is bounded from $\widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to $L_\infty(\mathbb{R}^n_{k,+})$.*

Proof. Let $f \in \widetilde{L}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. Then by Lemma 6, we have

$$\sup_{t>0} t^{\alpha-Q} \int_{E_t} T^y |f(x)| (y')^\gamma dy \le \sup_{t>0} \left(\frac{[t]_1}{t} \right)^{Q-\alpha} \|f\|_{\widetilde{L}_{1,Q-\alpha,\gamma}}$$

$$\le \|f\|_{\widetilde{L}_{1,Q-\alpha,\gamma}} \le \omega(n,k,\gamma)^{1/p'} \|f\|_{\widetilde{L}_{p,\lambda,\gamma}}.$$

Theorem 4 is proved.

Corollary 3. *[12] Let $0 < \alpha < Q$ and $1 \le p \le \frac{Q}{\alpha}$.*

1) If $1 < p < \frac{Q}{\alpha}$, then condition $\frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q}$ is necessary and sufficient for the boundedness of $I_{\alpha,\gamma}$ from $L_{p,\gamma}(\mathbb{R}^n_{k,+})$ to $L_{q,\gamma}(\mathbb{R}^n_{k,+})$.

2) If $p = 1$, then condition $1 - \frac{1}{q} = \frac{\alpha}{Q}$ is necessary and sufficient for the boundedness of $I_{\alpha,\gamma}$ from $L_{1,\gamma}(\mathbb{R}^n_{k,+})$ to $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.

3) If $1 < p = \frac{Q}{\alpha}$, then the operator $\widetilde{I}_{\alpha,\gamma}$ is bounded from $L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$ to $BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$.

Moreover, if for $f \in L_{p,\gamma}\left(\mathbb{R}^n_{k,+}\right)$, $1 < p = \frac{Q}{\alpha}$ the integral $I_{\alpha,\gamma} f$ exists almost everywhere, then $I_{\alpha,\gamma} f \in BMO_\gamma\left(\mathbb{R}^n_{k,+}\right)$ and the following inequality is valid

$$\|I_{\alpha,\gamma} f\|_{BMO_\gamma} \le C \|f\|_{L_{p,\gamma}},$$

where $C > 0$ is independent of f.

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Javanshir J. Hasanov
Institute of Mathematics and Mechanics, Baku, AZ1141, Azerbaijan
Azerbaijan State Pedagogical University, Baku, AZ1000, Azerbaijan
E-mail: hasanovjavanshir@yahoo.com.tr

Xayyam A. Badalov
Agjabady branch of Azerbaijan Teachers Institute, AZ0400, Azerbaijan

Ayna E. Fleydanli
Institute of Mathematics and Mechanics, Baku, AZ1141, Azerbaijan

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