Journal of Contemporary Applied Mathematics
V. 4, No 2, 2014, December
ISSN 2222-5498

Multi-sublinear rough maximal operator on product Morrey and product modified Morrey spaces

Sabir Q. Hasanov

Abstract. We will study the boundedness of multi-sublinear maximal operator $\mathcal{M}_{\Omega,m}$ with rough kernels $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s \leq \infty$ on product Morrey and on product modified Morrey spaces. We study the boundedness of the operators $\mathcal{M}_{\Omega,m}$ on product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ and on product modified Morrey spaces $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to modified Morrey spaces $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.

Key Words and Phrases: multi-sublinear maximal operator, product Morrey space, product modified Morrey space.

2000 Mathematics Subject Classifications: 42B20, 42B25, 42B35

1. Introduction

The classical Morrey spaces, introduced by Morrey [10] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations. They appeared to be quite useful in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces. See, for example, [3, 4, 5] for details. The boundedness of fractional integral operators on the classical Morrey spaces was studied by Adams [1], Chiarenza and Frasca [2], see also [11, 12]. In [2], by establishing a pointwise estimate of fractional integrals in terms of the Hardy-Littlewood maximal function, they showed the boundedness of fractional integral operators on the Morrey spaces.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x, r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x, r)$ denote its complement. Let $|B(x, r)|$ be the Lebesgue measure of the ball $B(x, r)$. Also for $\overrightarrow{x} = (x_1, \ldots, x_m) \in \mathbb{R}^{mn}$ and $r > 0$, we denote by $B(\overrightarrow{x}, r)$ the open ball centered at $\overrightarrow{x}$ of radius r, and $B(\overrightarrow{x}, r)$ We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\overrightarrow{y} = (y_1, \ldots, y_m)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

Definition 1. *Let $1 \le p < \infty$, $0 \le \lambda \le \gamma$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, and by $WL_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms*

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

$$\|f\|_{WL_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))}$$

respectively.

Definition 2. *Let $1 \le p < \infty$, $0 \le \lambda \le \gamma$, $[t]_1 = \min\{1, t\}$. We denote by $\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space, and by $W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)$ the weak modified Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms*

$$\|f\|_{\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

$$\|f\|_{W\widetilde{L}_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, r>0} [r]_1^{-\frac{\lambda}{p}} \|f\|_{W\widetilde{L}_p(B(x,r))}$$

respectively.

Note that

$$\widetilde{L}_{p,0}(\mathbb{R}^n) = L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n),$$

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n) \subset\!\succ L_{p,\lambda}(\mathbb{R}^n) \cap L_p(\mathbb{R}^n) \quad \text{and} \quad \max\{\|f\|_{L_{p,\lambda}}, \|f\|_{L_p}\} \le \|f\|_{\widetilde{L}_{p,\lambda}}$$

and if $\lambda < 0$ or $\lambda > n$, then $L_{p,\lambda}(\mathbb{R}^n) = \widetilde{L}_{p,\lambda}(\mathbb{R}^n) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$. The multi-sublinear maximal operator $\mathcal{M}_{\Omega,m}$ with rough kernels Ω is defined by

$$\mathcal{M}_{\Omega,m}(\overrightarrow{f})(x) = \sup_{r>0} \frac{1}{r^{nm}} \int_{B(\overrightarrow{y},r)} |\Omega(\overrightarrow{y})| \prod_{j=1}^{m} |f_i(x-y_i)| d\overrightarrow{y}.$$

If $m = 1$, then $M_\Omega \equiv \mathcal{M}_{\Omega,1}$ is the maximal operator with rough kernel Ω. When $m = 1$ and $\Omega \equiv 1$, then $M \equiv \mathcal{M}_{1,1}$ is the classical Hardy-Littlewood maximal operator.

In this work, we prove the boundedness of the multi-sublinear maximal operator with rough kernels $\mathcal{M}_{\Omega,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$, if $p > s'$, $1 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \ldots + 1/p_m$ and from the space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $WL^{p,\lambda}(\mathbb{R}^n)$, if $p = s'$, $1 \le p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \ldots + 1/p_m$ and at least one exponent p_i, $1 \le i \le m$ equals one. Also we prove the boundedness of $\mathcal{M}_{\Omega,m}$ from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times\widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$, if $p > s'$, $1 < p_1, \ldots, p_m < \infty$, $1/p = 1/p_1 + \ldots + 1/p_m$ and from the space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times\widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$, if $p = s'$, $1 \le p_1, \ldots, p_m < \infty$ and $1/p = 1/p_1 + \ldots + 1/p_m$ and at least one exponent p_i, $1 \le i \le m$ equals one.

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Boundedness of multi-sublinear maximal operator $\mathcal{M}_{\Omega,m}$ on product Morrey spaces

In this part, we investigate the boundedness of multi-sublinear maximal operator $\mathcal{M}_{\Omega,m}$ on product Morrey spaces.

The boundedness of Hardy-Littlewood maximal operator on the classical Morrey spaces was studied by Chiarenza and Frasca [2]. Their results can be summarized as follows.

Theorem 1. *[2] Let $1 \le p < \infty$ and $0 \le \lambda < n$. Then for $p > 1$, the operator M is bounded on $L^{p,\lambda}(\mathbb{R}^n)$ and for $p = 1$, the operator M is bounded from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{1,\lambda}(\mathbb{R}^n)$.*

If $\lambda = 0$, then the statement of Theorem 1 reduces to the well known Hardy-Littlewood theorem.

Lemma 1. *Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \dots, p_m > 1$ and $f \in L^1_{\rm loc}(\mathbb{R}^n) \times \dots \times L^1_{\rm loc}(\mathbb{R}^n)$. Then there exists a constant $C > 0$ such that for any $x \in \mathbb{R}^n$*

$$\mathcal{M}_{\Omega,m}\boldsymbol{f}(x) \le C_0 \prod_{j=1}^{m} \Big[M\big(f_j^{\frac{s'p_j}{p}}\big)(x)\Big]^{\frac{p}{s'p_j}}, \tag{1}$$

where $C_0 = \frac{\|\Omega\|_{L^s(\mathbb{S}^{mn-1})}}{(mn)^{\frac{1}{s}}}$.

Proof. Since $\Omega \in L^s(\mathbb{S}^{mn-1})$ with $s > 1$, Hölder's inequality yields that

$$\begin{aligned}
&\frac{1}{r^{nm}} \int_{B(\overrightarrow{y},r)} |\Omega(\overrightarrow{y})| \prod_{j=1}^{m} |f_i(x - y_i)| d\overrightarrow{y} \\
&\le \frac{1}{r^{nm}} \bigg(\int_{B(\overrightarrow{y},r)} \prod_{j=1}^{m} |f_i(x - y_i)|^{s'} d\overrightarrow{y} \bigg)^{\frac{1}{s'}} \Big(\int_{B(\overrightarrow{y},r)} |\Omega(\overrightarrow{y})|^s d\overrightarrow{y} \Big)^{\frac{1}{s}} \\
&= \frac{1}{r^{nm}} \bigg(\int_{B(\overrightarrow{y},r)} \prod_{j=1}^{m} |f_i(x - y_i)|^{s'} d\overrightarrow{y} \bigg)^{\frac{1}{s}} \Big(\int_0^r \int_{\mathbb{S}^{mn-1}} |\Omega(\xi)|^s t^{mn-1} d\xi dt \Big)^{\frac{1}{s}} \\
&\le C_0 \sup_{r>0} \frac{1}{r^{nm(1-\frac{1}{s})}} \bigg(\int_{B(y,r)} \cdots \int_{B(y,r)} \prod_{j=1}^{m} |f_i(x - y_i)|^{s'} dy_1 \dots dy_m \bigg)^{\frac{1}{s'}} \\
&\le C_0 \prod_{j=1}^{m} \sup_{r>0} \Big(\frac{1}{r^n} \int_{B(y,r)} |f_i(x - y_i)|^{\frac{s'p_j}{p}} dy_i \Big)^{\frac{p}{s'p_j}} \\
&\le C_0 \prod_{j=1}^{m} \Big[M\big(f_j^{\frac{s'p_j}{p}}\big)(x)\Big]^{\frac{p}{s'p_j}},
\end{aligned}$$

which implies a pointwise estimate

$$\mathcal{M}_{\Omega,m}\mathbf{f}(x) \le C_0 \prod_{j=1}^{m} \Big[M\big(f_j^{\frac{s'p_j}{p}}\big)(x)\Big]^{\frac{p}{s'p_j}}.$$

When $m \ge 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, we find out $\mathcal{M}_{\Omega,m}$ also have the same properties by providing the following multi-version of the Theorem 1.

Theorem 2. *Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \dots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \quad for \quad 0 \le \lambda_j < n. \tag{2}$$

(i) If $p > s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{L^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \dots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{WL^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

Proof.

(i) If $p > s'$, by (1) and the Hölder inequality, we get

$$\begin{aligned}\Big(\frac{1}{t^\lambda}\int_{B(x,t)} |\mathcal{M}_{\Omega,m}\mathbf{f}(y)|^p dy\Big)^{\frac{1}{p}} &\le C_0\Big(\frac{1}{t^\lambda}\int_{B(x,t)} \Big|\prod_{j=1}^{m}\Big[M(f_j^{\frac{s'p_j}{p}})(y)\Big]^{\frac{p}{s'p_j}}\Big|^p dy\Big)^{\frac{1}{p}} \\ &\le C_0 \prod_{j=1}^{m}\Big(\frac{1}{t^{\lambda_j}}\int_{B(x,t)} \Big[M(f_j^{\frac{s'p_j}{p}})(y)\Big]^{\frac{p}{s'}} dy\Big)^{\frac{1}{p_j}}.\end{aligned}$$

Taking the p-th root of both sides and applying Theorem 1 with $p/s' > 1$ and $f_j^{\frac{s'p_j}{p}} \in L^{\frac{p}{s'},\lambda_j}(\mathbb{R}^n)$, we get

$$\|\mathcal{M}_{\Omega,m}\mathbf{f}\|_{L^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} \Big(\frac{1}{t^\lambda}\int_{B(x,t)} |\mathcal{M}_{\Omega,m}\mathbf{f}(y)|^p dy\Big)^{\frac{1}{p}}$$

$$= C_0 \prod_{j=1}^{m} \left\| M\big(f_j^{\frac{s'p_j}{p}}\big) \right\|_{L^{\frac{p_j}{s'},\lambda_j}}^{\frac{1}{s'}}$$

$$\leq C \prod_{j=1}^{m} \left\| f_j^{\frac{s'p_j}{p}} \right\|_{L^{\frac{p_j}{s'},\lambda_j}}^{\frac{1}{s'}} = C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}},$$

which is the desired inequality.

(ii) If $p = s'$, for any $\tau > 0$, let $\varepsilon_0 = \tau$, $\varepsilon_m = 1$ and $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m-1} > 0$ be arbitrary which will be chosen later. From the pointwise estimate (1), we get

$$\{y \in B(x,t) : |\mathcal{M}_{\Omega,m}\mathbf{f}(y)| > \tau\}$$
$$\subset \bigcup_{j=1}^{m} \Big\{ y \in B(x,t) : \Big[M\big(f_j^{\frac{s'p_j}{p}}\big)(y) \Big]^{\frac{p}{s'p_j}} > \frac{\varepsilon_{j-1}}{t^{\frac{\lambda-\lambda_j}{p_j}}\varepsilon_j} \Big\}.$$

Let us now take $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m-1} > 0$ such that

$$\frac{\varepsilon_j}{\varepsilon_{j=1}} = \frac{\Big[\prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}\Big]^{s'/p_j}}{\tau^{s'/p_j} \|f_j\|_{L^{p_j,\lambda_j}}}, \qquad j = 1, 2, ..., m.$$

Then, applying Theorem 1 with $p/s' = 1$ and the fact $f_j^{p_j} \in L^{1,\lambda_j}(\mathbb{R}^n)$, we get

$$\Big| \Big\{ y \in B(x,t) : \big|\mathcal{M}_{\Omega,m}\mathbf{f}(y)\big| > \tau \Big\} \Big|$$
$$\leq C \sum_{j=1}^{m} \Big| \Big\{ y \in B(x,t) : M(f_j^{p_j})(y) > \Big(\frac{\varepsilon_{j-1}}{t^{(\lambda-\lambda_j)/p_j}\varepsilon_j} \Big)^{p_j} \Big\} \Big|$$
$$\leq C \sum_{j=1}^{m} t^{\lambda_j} \Big(\frac{t^{(\lambda-\lambda_j)/p_j}\varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j} \|f_j^{p_j}\|_{L^{1,\lambda_j}}$$
$$= C \sum_{j=1}^{m} t^{\lambda} \Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j} \|f_j\|_{L^{p_j,\lambda_j}}^{p_j}$$

$$= C \sum_{j=1}^{m} t^{\lambda} \Big[\Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big) \|f_j\|_{L^{p_j,\lambda_j}} \Big]^{p_j}$$
$$= C \sum_{j=1}^{m} t^{\lambda} \left(\frac{1}{\tau} \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}} \right)^{s'}$$
$$= C t^{\lambda} \left(\frac{1}{\tau} \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}} \right)^{p}.$$

Hence, we obtain the following inequality

$$\|\mathcal{M}_{\Omega,m}\mathbf{f}\|_{WL^{p,\lambda}} = \sup_{\tau>0}\tau \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\Big|\Big\{y\in B(x,t):|M_{\Omega,\alpha,m}\mathbf{f}(y)|>\tau\Big\}\Big|\Big)^{\frac{1}{p}}$$
$$\le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.$$

This is the conclusion (ii) of Theorem 2.

3. Boundedness of multi-sublinear maximal operator $\mathcal{M}_{\Omega,m}$ on product modified Morrey spaces

In this part, we investigate the boundedness of multi-sublinear maximal operator $\mathcal{M}_{\Omega,m}$ on product modified Morrey spaces.

The boundedness of Hardy-Littlewood maximal operator on the modified Morrey spaces was studied by Guliyev, Hasanov and Zeren [6], see also [7, 8, 9]. Their results can be summarized as follows.

Theorem 3. *[6] Let $1\le p<\infty$ and $0\le\lambda<n$. Then for $p>1$, the operator M is bounded on $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ and for $p=1$, the operator M is bounded from $\widetilde{L}^{1,\lambda}(\mathbb{R}^n)$ to $W\widetilde{L}^{1,\lambda}(\mathbb{R}^n)$.*

If $\lambda=0$, then the statement of Theorem 3 reduces to the well known Hardy-Littlewood theorem.

The following lemmas was proved in [6], see also [7, 8, 9].

Lemma 2. *Let $1\le p<\infty$, $0\le\lambda\le n$. Then*

$$\widetilde{L}_{p,\lambda}(\mathbb{R}^n)=L_{p,\lambda}(\mathbb{R}^n)\cap L_p(\mathbb{R}^n)$$

and

$$\|f\|_{\widetilde{L}_{p,\lambda}}=\max\Big\{\|f\|_{L_{p,\lambda}},\|f\|_{L_p}\Big\}.$$

Lemma 3. *Let $1\le p<\infty$, $0\le\lambda\le n$. Then*

$$W\widetilde{L}_{p,\lambda}(\mathbb{R}^n)=WL_{p,\lambda}(\mathbb{R}^n)\cap WL_p(\mathbb{R}^n)$$

and

$$\|f\|_{W\widetilde{L}_{p,\lambda}}=\max\Big\{\|f\|_{WM_{p,\lambda}},\|f\|_{WL_p}\Big\}.$$

When $m\ge2$ and $\Omega\in L^s(\mathbb{S}^{mn-1})$, we find out $\mathcal{M}_{\Omega,m}$ also have the same properties by providing the following multi-version of the Theorem 3.

Theorem 4. *Let $1<s\le\infty$, $\Omega\in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1,\dots,p_m>1$ and satisfy (2).*

(i) If $p > s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to modified Morrey space $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\boldsymbol{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{\widetilde{L}^{p,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

(ii) If $p = s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product modified Morrey space $\widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$ to weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that the following inequality is valid for all $\boldsymbol{f} \in \widetilde{L}^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times \widetilde{L}^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{W\widetilde{L}^{p,\lambda}} \leq C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.$$

Proof.

(i) If $p > s'$, by (1) and the Hölder inequality, we get

$$\Big(\frac{1}{[t]_1^{\lambda}} \int_{B(x,t)} |\mathcal{M}_{\Omega,m}\mathbf{f}(y)|^p dy\Big)^{\frac{1}{p}} \leq C_0 \Big(\frac{1}{[t]_1^{\lambda}} \int_{B(x,t)} \Big| \prod_{j=1}^{m} \Big[M(f_j^{\frac{s'p_j}{p}})(y)\Big]^{\frac{p}{s'p_j}} \Big|^p dy\Big)^{\frac{1}{p}}$$
$$\leq C_0 \prod_{j=1}^{m} \Big(\frac{1}{[t]_1^{\lambda_j}} \int_{B(x,t)} \Big[M(f_j^{\frac{s'p_j}{p}})(y)\Big]^{\frac{p}{s'}} dy\Big)^{\frac{1}{p_j}}.$$

Taking the p-th root of both sides and applying Theorem 3 with $p/s' > 1$ and $f_j^{\frac{s'p_j}{p}} \in \widetilde{L}^{\frac{p}{s'},\lambda_j}(\mathbb{R}^n)$, we get

$$\|\mathcal{M}_{\Omega,m}\mathbf{f}\|_{\widetilde{L}^{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} \Big(\frac{1}{[t]_1^{\lambda}} \int_{B(x,t)} |\mathcal{M}_{\Omega,m}\mathbf{f}(y)|^p dy\Big)^{\frac{1}{p}}$$
$$= C_0 \prod_{j=1}^{m} \Big\|M(f_j^{\frac{s'p_j}{p}})\Big\|_{\widetilde{L}^{\frac{p_j}{s'},\lambda_j}}^{\frac{1}{s'}}$$
$$\leq C \prod_{j=1}^{m} \Big\|f_j^{\frac{s'p_j}{p}}\Big\|_{\widetilde{L}^{\frac{p_j}{s'},\lambda_j}}^{\frac{1}{s'}} = C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}},$$

which is the desired inequality.

(ii) If $p = s'$, for any $\tau > 0$, let $\varepsilon_0 = \tau$, $\varepsilon_m = 1$ and $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m-1} > 0$ be arbitrary which will be chosen later. From the pointwise estimate (1), we get

$$\{y \in B(x,t) : |\mathcal{M}_{\Omega,m}\mathbf{f}(y)| > \tau\}$$

$$\subset \bigcup_{j=1}^{m} \Big\{ y \in B(x,t) : \Big[M(f_j^{\frac{s' p_j}{p}})(y) \Big]^{\frac{p}{s' p_j}} > \frac{\varepsilon_{j-1}}{t^{\frac{\lambda-\lambda_j}{p_j}} \varepsilon_j} \Big\}.$$

Let us now take $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m-1} > 0$ such that

$$\frac{\varepsilon_j}{\varepsilon_{j=1}} = \frac{\Big[\prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}} \Big]^{s'/p_j}}{\tau^{s'/p_j} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}}, \qquad j = 1, 2, ..., m.$$

Then, applying Theorem 3 with $p/s' = 1$ and the fact $f_j^{p_j} \in \widetilde{L}^{1,\lambda_j}(\mathbb{R}^n)$, we get

$$\begin{aligned}
&\Big| \Big\{ y \in B(x,t) : \big| \mathcal{M}_{\Omega,m} \mathbf{f}(y) \big| > \tau \Big\} \Big| \\
&\leq C \sum_{j=1}^{m} \Big| \Big\{ y \in B(x,t) : M(f_j^{p_j})(y) > \Big(\frac{\varepsilon_{j-1}}{[t]_1^{(\lambda-\lambda_j)/p_j} \varepsilon_j} \Big)^{p_j} \Big\} \Big| \\
&\leq C \sum_{j=1}^{m} [t]_1^{\lambda_j} \Big(\frac{[t]_1^{(\lambda-\lambda_j)/p_j} \varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j} \|f_j^{p_j}\|_{\widetilde{L}^{1,\lambda_j}} \\
&= C \sum_{j=1}^{m} [t]_1^{\lambda} \Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j} \|f_j\|^{p_j}_{\widetilde{L}^{p_j,\lambda_j}}
\end{aligned}$$

$$\begin{aligned}
&= C \sum_{j=1}^{m} [t]_1^{\lambda} \Big[\Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big) \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}} \Big]^{p_j} \\
&= C \sum_{j=1}^{m} [t]_1^{\lambda} \left(\frac{1}{\tau} \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}} \right)^{s'} \\
&= C t^{\lambda} \left(\frac{1}{\tau} \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}} \right)^{p}.
\end{aligned}$$

Hence, we obtain the following inequality

$$\begin{aligned}
\|\mathcal{M}_{\Omega,m} \mathbf{f}\|_{W\widetilde{L}^{p,\lambda}} &= \sup_{\tau>0} \tau \sup_{x \in \mathbb{R}^n, t>0} \Big(\frac{1}{t^{\lambda}} \Big| \Big\{ y \in B(x,t) : |M_{\Omega,\alpha,m} \mathbf{f}(y)| > \tau \Big\} \Big| \Big)^{\frac{1}{p}} \\
&\leq C \prod_{j=1}^{m} \|f_j\|_{\widetilde{L}^{p_j,\lambda_j}}.
\end{aligned}$$

This is the conclusion (ii) of Theorem 4.

References

[1] D.R. Adams, *A note on Riesz potentials*, Duke Math. 42 (1975), 765-778.

[2] F. Chiarenza, M. Frasca, *Morrey spaces and Hardy-Littlewood maximal function*, Rend Mat. 7 (1987), 273-279.

[3] F. Chiarenza, M. Frasca, P. Longo, *Interior* $W^{2,p}$*-estimates for nondivergence ellipic equations with discontinuous coefficients*, Ricerche Mat. 40 (1991), 149-168.

[4] F. Chiarenza, M. Frasca, P. Longo, $W^{2,p}$*-solvability of Dirichlet problem for nondivergence ellipic equations with VMO coefficients*, Trans. Amer. Math. Soc. 336 (1993), 841-853.

[5] G. Di Fazio, D.K. Palagachev, M.A. Ragusa, *Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients*, J. Funct. Anal. 166 (2) (1999), 179-196.

[6] V. Guliyev, J. Hasanov and Y. Zeren, *Necesary and sucient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal. 5 (2011), 491-506.

[7] V. Guliyev, K. Rahimova, *Parabolic fractional maximal operator and modified parabolic Morrey spaces*, Journal of Function Spaces and Applications, 2012, Article ID 543475, 20 pages, 2012.

[8] V. Guliyev, K. Rahimova *Parabolic fractional integral operator in modified parabolic Morrey spaces*, Proc. Razmadze Mathematical Institute, 163 (2013), 85-106.

[9] V. Guliyev, Y.Y.Mammadov, *Riesz potential on the Heisenberg group and modified Morrey spaces*, Analele Stiintifice ale Universitatii Ovidius Constanta, Seria Matematica, 20 (1) 2012, 189-212.

[10] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. 43 (1938), 126-166.

[11] E.M. Stein, Singular integrals and differentiability properties of functions. *Princeton Univ. Press, Princeton, NJ*, 1970.

[12] E. M. Stein, *Harmonic Analysis: Real-Variable methods, Orthogonality, and Oscillatory Integrals*, Princeton N. J. Princeton Univ Press, 1993.

Sabir Q. Hasanov
Gandja State University, Gandja, Azerbaijan
E-mail: sabirhasanov@gmail.com

Received 09 November 2014
Accepted 29 November 2014