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Multi-sublinear rough fractional maximal operator on product Morrey spaces

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Abstract. We will study the boundedness of multi-sublinear fractional maximal operator $\mathcal{M}_{\Omega,\alpha,m}$ with rough kernels $\Omega \in L^s(\mathbb{S}^{n-1})$, $1 < s \le \infty$ on product Morrey spaces. We find for the operator $\mathcal{M}_{\Omega,\alpha,m}$ necessary and sufficient conditions on the parameters of the boundedness on product Morrey spaces $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to Morrey spaces $L^{q,\lambda}(\mathbb{R}^n)$.

Key Words and Phrases: multi-sublinear fractional maximal operator, Morrey space, product Morrey space.
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1. Introduction

The classical Morrey spaces, introduced by Morrey [17] in 1938, have been studied intensively by various authors and together with weighted Lebesgue spaces play an important role in the theory of partial differential equations. They appeared to be quite useful in the study of local behavior of the solutions of elliptic differential equations and describe local regularity more precisely than Lebesgue spaces. See [4, 5, 6] for details. The boundedness of fractional integral operators on the classical Morrey spaces was studied by Adams [1], Chiarenza and Frasca *et al.* [3]. In [3], by establishing a pointwise estimate of fractional integrals in terms of the Hardy-Littlewood maximal function, they showed the boundedness of fractional integral operators on the Morrey spaces.

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, and let $(\mathbb{R}^n)^m = \mathbb{R}^n \times \ldots \times \mathbb{R}^n$ be the m-fold product space ($m \in \mathbb{N}$). For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$. Also for $\overrightarrow{x} = (x_1,\ldots,x_m) \in \mathbb{R}^{mn}$ and $r > 0$, we denote by $B(\overrightarrow{x},r)$ the open ball centered at $\overrightarrow{x} \in \mathbb{R}^{mn}$ of radius r, and $B(\overrightarrow{x},r)$ We denote by $\overrightarrow{f}$ the m-tuple $(f_1, f_2, \ldots, f_m)$, $\overrightarrow{y} = (y_1,\ldots,y_m)$ and $d\overrightarrow{y} = dy_1 \cdots dy_n$.

Definition 1. *Let $1 \le p < \infty$, $0 \le \lambda \le \gamma$, $[t]_1 = \min\{1,t\}$. We denote by $L_{p,\lambda}(\mathbb{R}^n)$ the Morrey space, and by $WL_{p,\lambda}(\mathbb{R}^n)$ the weak Morrey space, the set of locally integrable functions $f(x)$, $x \in \mathbb{R}^n$, with the finite norms*

$$\|f\|_{L_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))},$$

$$\|f\|_{WL_{p,\lambda}} = \sup_{x\in\mathbb{R}^n,\, t>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(B(x,r))}$$

respectively.

The multilinear theory has been well developed in the past twenty years. In 1992, Grafakos [9] first study the following multilinear integrals, defined by

$$I_\alpha^m(\overrightarrow{f})(x) = \int_{\mathbb{R}^n} \frac{1}{|y|^{n-\alpha}} f_1(x-\theta_1 y)\dots f_m(x-\theta_m y)\, dy,$$

where $\theta_i (i=1,\dots,m)$ are fixed distinct and nonzero real numbers and $0<\beta<n$. When $m=1$ and $\Omega\equiv 1$, if let $\theta_1 = 1$, $I_{\Omega,\alpha,m}$ will be the Riesz potential operator I_α [19] given by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(x-y)}{|y|^{n-\alpha}} dy.$$

Grafakos proved that the operator I_α^m is bounded from $L_{p_1}(\mathbb{R}^n)\times\dots\times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $0<1/q = 1/p_1+\dots+1/p_m - \beta/n < 1$, which can be regarded as an extension result for the classical fractional integral on Lebesgue spaces. In [11, 12, 13] was proved a certain O'Neil type inequality for dilated multi-linear convolution operators, including permutations of functions. This inequality was used to extend Grafakoss result [9] to more general multi-linear operators of potential type and the relevant maximal operators.

In 1999, Kenig and Stein [16] studied the following multilinear fractional integral,

$$I_{\alpha,m}(\overrightarrow{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\dots f_m(y_m)}{|(x-y_1,\dots,x-y_m)|^{nm-\alpha}} dy_1 dy_2\dots dy_m,$$

and showed that $I_{\alpha,m}$ is bounded from product $L_{p_1}(\mathbb{R}^n)\times L_{p_2}(\mathbb{R}^n)\times\dots\times L_{p_m}(\mathbb{R}^n)$ to $L_q(\mathbb{R}^n)$ with $1/q = 1/p_1+\dots+1/p_m-\beta/n>0$ for each $p_i>1 (i=1,\dots,m)$. If some $p_i=1$, then $I_{\alpha,m}$ is bounded $L_{p_1}(\mathbb{R}^n)\times L_{p_2}(\mathbb{R}^n)\times\dots\times L_{p_m}(\mathbb{R}^n)$ to $L_{q,\infty}(\mathbb{R}^n)$. Obviously, the multilinear fractional integral $I_{\alpha,m}$ is a natural generalization of the classical fractional integral $I_\alpha\equiv I_{\alpha,1}$.

Let $1<s\le\infty$, $\Omega\in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$. The multi-sublinear fractional maximal operator $\mathcal{M}_{\alpha,m}$ with rough kernels Ω is defined by

$$\mathcal{M}_{\alpha,m}(\overrightarrow{f})(x) = \sup_{r>0} \frac{1}{r^{nm-\alpha}} \int_{B(\overrightarrow{y},r)} |\Omega(\overrightarrow{y})| \prod_{j=1}^m |f_i(x-y_i)| d\overrightarrow{y}, \qquad 0\le\alpha<nm.$$

If $m=1$, then $M_{\Omega,\alpha}\equiv\mathcal{M}_{\Omega,\alpha,1}$ is the fractional maximal operator with rough kernel Ω. When $m=1$ and $\Omega\equiv 1$, then $M_\alpha\equiv\mathcal{M}_{1,\alpha,1}$ is the classical fractional maximal operator.

In this work, we prove the boundedness of the multi-sublinear fractional maximal operator with rough kernels $\mathcal{M}_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$, if $p>s'$, $1<p_1,\dots,p_m<\infty$, $1/q = 1/p_1+\dots+1/p_m-\alpha/(mn-\lambda)$ and from the space $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\dots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak space $WL^{q,\lambda}(\mathbb{R}^n)$, if $p=s'$,

$1 \le p_1, \ldots, p_m < \infty$ and $1/q = 1/p_1 + \ldots + 1/p_m - \alpha/(n-\lambda)$ and at least one exponent p_i, $1 \le i \le m$ equals one.

Throughout this paper, we assume the letter C always remains to denote a positive constant that may vary at each occurrence but is independent of the essential variables.

2. Boundedness of multi-sublinear fractional maximal operator $\mathcal{M}_{\Omega,\alpha,m}$ on product Morrey spaces

In this part, we investigate the boundedness of multi-sublinear fractional maximal operator $\mathcal{M}_{\Omega,\alpha,m}$ on product Morrey spaces.

Spanne and Adams obtained two remarkable results on Morrey spaces (see Definition 1.1 of the Morrey spaces in Section 1) for I_α. Their results can be summarized as follows.

Theorem 1. *[14, 18] (Spanne, but published by Peetre) Let $0 < \alpha < n$, $0 \le \lambda < n - \alpha p$, $1/q = 1/p - \alpha/n$ and $\mu/q = \lambda/p$. Then for $p > 1$, the operators M_α and I_α are bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\mu}(\mathbb{R}^n)$ and for $p = 1$, I_α is bounded from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\mu}(\mathbb{R}^n)$.*

Theorem 2. *[1, 10] Let $0 < \alpha < n$, $1 \le p < n/\alpha$, $0 \le \lambda < n/\alpha p$.*

(i) If $p > 1$, then condition $1/p - 1/q = \alpha/(n-\lambda)$ is necessary and sufficient for the boundedness of the operator M_α from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$.

(ii) If $p = 1$, then condition $1 - 1/q = \alpha/(n-\lambda)$ s necessary and sufficient for the boundedness of the operator M_α from $L^{1,\lambda}(\mathbb{R}^n)$ to $WL^{q,\lambda}(\mathbb{R}^n)$.

(iii) If $p = \frac{n-\lambda}{\alpha}$, then the operator M_α is bounded from $L^{p,\lambda}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

If $\lambda = 0$, then the statement of Theorems 1 and 2 reduces to the well known Hardy-Littlewood-Sobolev inequality.

When $m \ge 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, in [15] was find out $\mathcal{M}_{\Omega,m}$ also have the same properties by providing the following multi-version result of the Chiarenza and Frasca [3].

Theorem 3. *[15] Let $1 < s \le \infty$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ be a homogeneous function of degree zero on $\mathbb{R}^{mn}$, p be the harmonic mean of $p_1, \ldots, p_m > 1$ and*

$$\frac{\lambda}{p} = \sum_{j=1}^{m} \frac{\lambda_j}{p_j} \quad for \quad 0 \le \lambda_j < n. \tag{1}$$

(i) If $p > s'$, then the operator $\mathcal{M}_{\Omega,m}$ is bounded from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{p,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{L^{p,\lambda}} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = s'$, *then the operator* $\mathcal{M}_{\Omega,m}$ *is bounded from product Morrey space* $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ *to weak Morrey space* $WL^{p,\lambda}(\mathbb{R}^n)$. *Moreover, there exists a positive constant* C *such that for all* $\boldsymbol{f}\in L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,m}\boldsymbol{f}\|_{WL^{p,\lambda}} \le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda_j}}.$$

Lemma 1. *Let* $0 < \alpha < mn$, $1 \le s' < mn/\alpha$, $\Omega \in L^s(\mathbb{S}^{mn-1})$ *be a homogeneous function of degree zero on* $\mathbb{R}^{mn}$, p *be the harmonic mean of* $p_1,\ldots,p_m > 1$ *and* $f \in L^1_{\rm loc}(\mathbb{R}^n)\times\ldots\times L^1_{\rm loc}(\mathbb{R}^n)$. *Then for any* $x\in\mathbb{R}^n$

$$\mathcal{M}_{\Omega,\alpha,m}\boldsymbol{f}(x) \le C_0\prod_{j=1}^{m}\left[M_{\frac{\alpha s'}{m}}(f_j^{s'})(x)\right]^{\frac{1}{s'}} \le C_0\prod_{j=1}^{m}\left[M_{\frac{\alpha s' p_j}{mp}}(f_j^{\frac{s'p_j}{p}})(x)\right]^{\frac{p}{s'p_j}}, \qquad (2)$$

where $C_0 = \frac{\|\Omega\|_{L_s(\mathbb{S}^{mn-1})}}{(mn)^{\frac{1}{s}}}$.

Proof. Since $\Omega \in L^s(\mathbb{S}^{mn-1})$ with $s > 1$, Hölder's inequality yields that

$$\frac{1}{r^{nm-\alpha}}\int_{B(\overrightarrow{y},r)}|\Omega(\overrightarrow{y})|\prod_{j=1}^{m}|f_i(x-y_i)|d\overrightarrow{y}$$
$$\le \frac{1}{r^{nm-\alpha}}\left(\int_{B(\overrightarrow{y},r)}\prod_{j=1}^{m}|f_i(x-y_i)|^{s'}d\overrightarrow{y}\right)^{\frac{1}{s'}}\left(\int_{B(\overrightarrow{y},r)}|\Omega(\overrightarrow{y})|^{s}d\overrightarrow{y}\right)^{\frac{1}{s}}$$
$$= \frac{1}{r^{nm-\alpha}}\left(\int_{B(\overrightarrow{y},r)}\prod_{j=1}^{m}|f_i(x-y_i)|^{s'}d\overrightarrow{y}\right)^{\frac{1}{s'}}\left(\int_0^r\int_{\mathbb{S}^{mn-1}}|\Omega(\xi)|^{s}t^{mn-1}d\xi dt\right)^{\frac{1}{s}}$$

$$= C_0\sup_{r>0}\left(\frac{1}{r^{nm-\alpha s'}}\int_{B(y,r)}\cdots\int_{B(y,r)}\prod_{j=1}^{m}|f_i(x-y_i)|^{s'}dy_1\ldots dy_m\right)^{\frac{1}{s'}}$$
$$\le C_0\prod_{j=1}^{m}\sup_{r>0}\left(\frac{1}{r^{n-\alpha s'/m}}\int_{B(y,r)}|f_i(x-y_i)|^{\frac{s'p_j}{p}}dy_i\right)^{\frac{p}{s'p_j}}$$
$$\le C_0\prod_{j=1}^{m}\left[M_{\frac{\alpha s'}{m}}(f_j^{\frac{s'p_j}{p}})(x)\right]^{\frac{p}{s'p_j}},$$

which implies a pointwise estimate

$$\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(x) \le C_0\prod_{j=1}^{m}\left[M_{\frac{\alpha s'p_j}{mp}}(f_j^{\frac{s'p_j}{p}})(x)\right]^{\frac{p}{s'p_j}}.$$

The lemma is proved.

When $m \geq 2$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$, we find out $\mathcal{M}_{\Omega,\alpha,m}$ also have the same properties by providing the following multi-version of the Theorem 2.

Theorem 4. *Let $0 < \alpha < mn$, $1 \leq s' < \frac{mn}{\alpha}$ and $\Omega \in L^s(\mathbb{S}^{mn-1})$. Suppose $\sum_{j=1}^m \frac{\lambda_j}{p_j} = \frac{\lambda}{p}$, $\frac{1}{p_j} - \frac{1}{q_j} = \frac{\alpha}{n-\lambda_j}$ and $0 \leq \lambda_j < n - \frac{\alpha p'}{m}$.*

(i) If $p > s'$ and $\sum_{j=1}^m \frac{n-\lambda_j}{p_j} = \frac{mn-\lambda}{p}$, then the condition $\frac{mn-\lambda}{p} - \frac{n-\lambda}{q} = \alpha$ is necessary and sufficient for the boundedness of the operator $\mathcal{M}_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,\alpha,m}\boldsymbol{f}\|_{L^{q,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

(ii) If $p = s'$ and $\sum_{j=1}^m \frac{n-\lambda_j}{p_j} = \frac{mn-\lambda}{s'}$, then the condition $\frac{mn-\lambda}{s'} - \frac{n-\lambda}{q} = \alpha$ is necessary and sufficient for the boundedness of the operator $\mathcal{M}_{\Omega,\alpha,m}$ from product Morrey space $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to the weak Morrey space $WL^{q,\lambda}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $\boldsymbol{f} \in L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$

$$\|\mathcal{M}_{\Omega,\alpha,m}\boldsymbol{f}\|_{WL^{q,\lambda}} \leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

(iii) If $p = \frac{mn-\lambda}{\alpha} \geq s'$, then the operator $\mathcal{M}_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$.

Proof.

(i) *Sufficiency.* If $p > s'$, by (2) and the Hölder inequality, we get

$$\frac{1}{t^\lambda}\int_{B(x,t)} |\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy \leq C_0 \frac{1}{t^\lambda}\int_{B(x,t)} \prod_{j=1}^m \left[M_{\frac{\alpha s'}{m}}(f_j^{s'})(y)\right]^{\frac{q}{s'}} dy$$
$$\leq C_0 \prod_{j=1}^m \left(\frac{1}{t^{\lambda_j}}\int_{B(x,t)} \left[M_{\frac{\alpha s'}{m}}(f_j^{s'})(y)\right]^{\frac{q_j}{s'}} dy\right)^{\frac{q}{q_j}}$$

for all $x \in \mathbb{R}^n$ and $t > 0$. Taking the q-th root of both sides, applying Theorem 2 with $p/s' > 1$, the fact $\frac{1}{p_j} - \frac{1}{q_j} = \frac{\alpha}{n-\lambda_j}$ and $f_j^{s'} \in L^{p/s',\lambda_j}$, we get

$$\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = \sup_{x\in\mathbb{R}^n, t>0}\left(\frac{1}{t^\lambda}\int_{B(x,t)} |\mathcal{M}_{\Omega,\alpha,m}f(y)|^q dy\right)^{1/q}$$
$$\leq C \prod_{j=1}^m \sup_{x\in\mathbb{R}^n, t>0}\left(\frac{1}{t^{\lambda_j}}\int_{B(x,t)} \left[M_{\frac{\alpha s'}{m}}(f_j^{s'})(y)\right]^{\frac{q_j}{s'}} dy\right)^{\frac{1}{q_j}}$$

$$= C \prod_{j=1}^{m} \left\| M_{\frac{\alpha s'}{m}} (f_j^{s'}) \right\|_{L^{q_j/s',\lambda_j}}^{1/s'}$$

$$\leq C \prod_{j=1}^{m} \left\| f_j^{s'} \right\|_{L^{p_j/s',\lambda_j}}^{1/s'} = C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}},$$

which is the desired inequality.

Necessity. Suppose that $\mathcal{M}_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n) \times \ldots \times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $L_{q,\lambda}(\mathbb{R}^n)$. Define $\mathbf{f}_\varepsilon(x) = \big(f_1(\varepsilon x), \ldots, f_m(\varepsilon x)\big)$ for $\varepsilon > 0$. Then it is easy to show that

$$\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y) = \varepsilon^{-\alpha}\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y). \tag{3}$$

Thus

$$\begin{aligned}
\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{L^{q,\lambda}} &= \varepsilon^{-\alpha} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^\lambda} \int_{B(x,t)} |\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)|^q dy\Big)^{1/q} \\
&= \varepsilon^{-\alpha-n/q} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^\lambda} \int_{B(\varepsilon x,\varepsilon t)} |\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{1/q} \\
&= \varepsilon^{-\alpha-n/q+\lambda/q} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{(\varepsilon t)^\lambda} \int_{B(\varepsilon x,\varepsilon t)} |\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(y)|^q dy\Big)^{1/q} \\
&= \varepsilon^{-\alpha-(n-\lambda)/q} \|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}}.
\end{aligned}$$

Since $\mathcal{M}_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$ to $L^{q,\lambda}$, we have

$$\begin{aligned}
&\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = \varepsilon^{\alpha+(n-\lambda)/q} \|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{L^{q,\lambda}} \\
&\leq C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^{m} \|f_j(\varepsilon\cdot)\|_{L^{p_j,\lambda_j}} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^{m} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^{\lambda_j}} \int_{B(x,t)} |f_j(\varepsilon y)|^{p_j} dy\Big)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^{m} \varepsilon^{-n/p_j} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{t^{\lambda_j}} \int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy\Big)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q} \prod_{j=1}^{m} \varepsilon^{(\lambda_j-n)/p_j} \sup_{x\in\mathbb{R}^n, t>0} \Big(\frac{1}{(\varepsilon t)^{\lambda_j}} \int_{B(\varepsilon x,\varepsilon t)} |f_j(y)|^{p_j} dy\Big)^{1/p_j} \\
&= C\varepsilon^{\alpha+(n-\lambda)/q-(mn-\lambda)/p} \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}},
\end{aligned}$$

where C is independent of ε.

If $(mn-\lambda)/p < (n-\lambda)/q+\alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = 0$ as $\varepsilon \to 0$.

If $(mn-\lambda)/p > (n-\lambda)/q+\alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{L^{q,\lambda}} = 0$ as $\varepsilon \to \infty$.

Therefore we get $(mn-\lambda)/p = (n-\lambda)/q+\alpha$.

(ii) *Sufficiency.* If $p = s'$, for any $\beta > 0$, let $\varepsilon_0 = \beta$, $\varepsilon_m = 1$ and $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m-1} > 0$ be arbitrary which will be chosen later. From the pointwise estimate (2), we get

$$\begin{aligned}
&\{y \in B(x,t) : |\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(y)| > \beta\} \\
&\subset \bigcup_{j=1}^{m} \Big\{ y \in B(x,t) : \Big[M_{\frac{\alpha s' p_j}{mp}} (f_j^{\frac{s' p_j}{p}})(x) \Big]^{\frac{p}{s' p_j}} > \frac{\varepsilon_{j-1}}{t^{(\lambda-\lambda_j)/p_j} \varepsilon_j} \Big\} \\
&= \bigcup_{j=1}^{m} \Big\{ y \in B(x,t) : \Big[M_{\frac{\alpha p_j}{m}} (f_j^{p_j})(x) \Big]^{\frac{1}{p_j}} > \frac{\varepsilon_{j-1}}{t^{(\lambda-\lambda_j)/p_j} \varepsilon_j} \Big\}.
\end{aligned}$$

Let us now take $\varepsilon_1, \varepsilon_2, ..., \varepsilon_{m-1} > 0$ such that

$$\Big(\frac{\varepsilon_j}{\varepsilon_{j=1}}\Big)^{p_j q_j} = \frac{\Big[\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}\Big]^q}{\beta^q \|f_j\|_{L^{p_j,\lambda_j}}}, \qquad j = 1, 2, ..., m.$$

Then, applying Lemma 1 with $p/s' = 1$ and the fact $f_j^{p_j} \in L^{1,\lambda_j}$, we get

$$\begin{aligned}
&\Big|\Big\{ y \in B(x,t) : \big|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(y)\big| > \beta \Big\}\Big| \\
&\le C \sum_{j=1}^m \Big| \Big\{ y \in B(x,t) : M_{\frac{\alpha p_j}{m}} (f_j^{p_j})(x) > \Big(\frac{\varepsilon_{j-1}}{t^{(\lambda-\lambda_j)/p_j} \varepsilon_j} \Big)^{p_j} \Big\} \Big| \\
&\le C \sum_{j=1}^m t^{\lambda_j} \Big(\frac{t^{(\lambda-\lambda_j)/p_j} \varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j q_j} \|f_j^{p_j}\|_{L^{1,\lambda_j}}^{q_j} \\
&= C \sum_{j=1}^m t^{\lambda} \Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big)^{p_j q_j} \|f_j\|_{L^{p_j,\lambda_j}}^{p_j q_j} \\
&= C \sum_{j=1}^m t^{\lambda} \Big[\Big(\frac{\varepsilon_j}{\varepsilon_{j-1}} \Big) \|f_j\|_{L^{p_j,\lambda_j}} \Big]^{p_j q_j} \\
&= C \sum_{j=1}^m t^{\lambda} \Big(\frac{1}{\beta} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}} \Big)^q \\
&= C t^{\lambda} \Big(\frac{1}{\beta} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}} \Big)^q.
\end{aligned}$$

Hence, we obtain the following inequality

$$\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{p,\lambda}} = \sup_{\beta>0} \beta \sup_{x\in\mathbb{R}^n,t>0} \Big(\frac{1}{t^\lambda}\Big|\Big\{y \in B(x,t) : |\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(y)| > \beta\Big\}\Big|\Big)^{\frac{1}{p}}$$
$$\leq C\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}}.$$

This is the conclusion (ii) of Theorem 4.

Necessity. Suppose that $\mathcal{M}_{\Omega,\alpha,m}$ is bounded from $L^{p_1,\lambda_1}(\mathbb{R}^n)\times\ldots\times L^{p_m,\lambda_m}(\mathbb{R}^n)$ to $WL_{q,\lambda}(\mathbb{R}^n)$. From equality (3) we get

$$\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{WL^{q,\lambda}} = \sup_{\tau>0}\tau \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{\big\{y\in B(x,t):\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}_\varepsilon(y)>\tau\big\}} dy\Big)^{1/q}$$
$$= \sup_{\tau>0}\tau \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{\big\{y\in B(x,t):\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\big\}} dy\Big)^{1/q}$$
$$= \varepsilon^{-\frac{n}{q}}\sup_{\tau>0}\tau \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^\lambda}\int_{\big\{y\in B(x,\varepsilon t):\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\big\}} dy\Big)^{1/q}$$
$$= \varepsilon^{-\alpha-\frac{n}{q}+\frac{\lambda}{q}}\sup_{\tau>0}\tau\varepsilon^\alpha \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{(\varepsilon t)^\lambda}\int_{\big\{y\in B(x,\varepsilon t):\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(\varepsilon y)>\tau\varepsilon^\alpha\big\}} dy\Big)^{1/q}$$
$$= \varepsilon^{-\alpha-(n-\lambda)/q}\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}}.$$

By the boundedness of the operator $\mathcal{M}_{\Omega,\alpha,m}$ from $L^{p_1,\lambda_1}\times\ldots\times L^{p_m,\lambda_m}$ to $WL^{q,\lambda}$, we have

$$\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} = \varepsilon^{\alpha+(n-\lambda)/q}\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}_\varepsilon\|_{WL^{q,\lambda}}$$
$$\leq C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m \|f_j(\varepsilon\cdot)\|_{L^{p_j,\lambda_j}}$$
$$= C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m \sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^{\lambda_j}}\int_{B(x,t)}|f_j(\varepsilon y)|^{p_j}dy\Big)^{1/p_j}$$
$$= C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m \varepsilon^{-n/p_j}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{t^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)}|f_j(y)|^{p_j}dy\Big)^{1/p_j}$$
$$= C\varepsilon^{\alpha+(n-\lambda)/q}\prod_{j=1}^m \varepsilon^{(\lambda_j-n)/p_j}\sup_{x\in\mathbb{R}^n,t>0}\Big(\frac{1}{(\varepsilon t)^{\lambda_j}}\int_{B(\varepsilon x,\varepsilon t)}|f_j(y)|^{p_j}dy\Big)^{1/p_j}$$
$$= C\varepsilon^{\alpha+(n-\lambda)/q-(mn-\lambda)/p}\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda_j}},$$

where C is independent of ε.

If $(mn-\lambda)/p < (n-\lambda)/q+\alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} = 0$ as $\varepsilon \to 0$.

If $(mn-\lambda)/p > (n-\lambda)/q+\alpha$, then for all $\mathbf{f} \in L^{p_1,\lambda_1} \times \ldots \times L^{p_m,\lambda_m}$, we have $\|\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}\|_{WL^{q,\lambda}} = 0$ as $\varepsilon \to \infty$.

Therefore we get $(mn-\lambda)/p = (n-\lambda)/q+\alpha$.

(iii) Since $\Omega \in L^s(\mathbb{S}^{mn-1})$ with $s' \le p = \frac{mn-\lambda}{\alpha}$, Hölder's inequality yields that

$$\mathcal{M}_{\Omega,\alpha,m}\mathbf{f}(x) = \sup_{r>0} \frac{1}{r^{nm-\alpha}} \int_{B(\overrightarrow{y},r)} |\Omega(\overrightarrow{y})| \prod_{j=1}^{m} |f_i(x-y_i)| d\overrightarrow{y}$$

$$\le \sup_{r>0} \frac{1}{r^{nm-\alpha}} \left(\int_{B(\overrightarrow{y},r)} \prod_{j=1}^{m} |f_i(x-y_i)|^p d\overrightarrow{y} \right)^{\frac{1}{p}} \Big(\int_{B(\overrightarrow{y},r)} |\Omega(\overrightarrow{y})|^{p'} d\overrightarrow{y} \Big)^{\frac{1}{p'}}$$

$$\le \sup_{r>0} \frac{1}{r^{nm-\alpha}} \left(\int_{B(\overrightarrow{y},r)} \prod_{j=1}^{m} |f_i(x-y_i)|^p d\overrightarrow{y} \right)^{\frac{1}{p}} \Big(\int_{B(\overrightarrow{y},r)} |\Omega(\overrightarrow{y})|^{s} d\overrightarrow{y} \Big)^{\frac{1}{s}} \big|B(\overrightarrow{y},r)\big|^{\frac{1}{p'}-\frac{1}{s}}$$

$$\le C \sup_{r>0} \frac{r^{nm(\frac{1}{p'}-\frac{1}{s})}}{r^{nm-\alpha}} \left(\int_{B(\overrightarrow{y},r)} \prod_{j=1}^{m} |f_i(x-y_i)|^p d\overrightarrow{y} \right)^{\frac{1}{p}} \Big(\int_{B(\overrightarrow{y},r)} |\Omega(\overrightarrow{y})|^{s} d\overrightarrow{y} \Big)^{\frac{1}{s}}$$

$$= C \sup_{r>0} r^{\alpha - \frac{nm}{p} - \frac{nm}{s}} \prod_{j=1}^{m} \left(\int_{B(y,r)} |f_i(x-y_i)|^{p_j} d\overrightarrow{y} \right)^{\frac{1}{p_j}} \Big(\int_0^r \int_{\mathbb{S}^{mn-1}} |\Omega(\xi)|^s t^{mn-1} d\xi dt \Big)^{\frac{1}{s}}$$

$$= C \sup_{r>0} r^{\alpha - \frac{nm}{p}} \prod_{j=1}^{m} \left(\int_{B(y,r)} |f_i(x-y_j)|^{p_j} dy_j \right)^{\frac{1}{p_j}}$$

$$\le C \sup_{r>0} r^{\alpha - \frac{nm-\lambda}{p}} \prod_{j=1}^{m} \Big(\frac{1}{t^{\lambda_j}} \int_{B(y,r)} |f_i(x-y_i)|^{p_j} dy_i \Big)^{\frac{1}{p_j}}$$

$$\le C \prod_{j=1}^{m} \sup_{r>0} \Big(\frac{1}{t^{\lambda_j}} \int_{B(y,r)} |f_i(x-y_i)|^{p_j} dy_i \Big)^{\frac{1}{p_j}}$$

$$\le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda_j}}.$$

The theorem is proved.

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