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On determination of coefficient at lowest term in the Cauchy problem for string vibrations equation

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Abstract. In the paper we consider an inverse problem for determining the coefficient at lowest term in the Cauchy problem for a string vibrations equation. This problem is reduced to an optimal control problem for which a theorem on the existence and uniqueness of optimal control, Frechet differentiability of functional is proved are necessary condition of optimality is derived.

Cauchy problem, coefficient of equation, inverse problem, optimal control.

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1. Introduction

Starting from the mid XX century the inverse problems became the object of systematic study and application in physics, medicine, astronomy, ecology and generally in all spheres of knowledge, wherein mathematical methods are applicable. In direct problems of mathematical physics, the researchers try to find a function describing different physical phenomena. Herewith, the properties of the studied medium (coefficients of equations) are assumed to be known. But just the properties of the medium are often unknown. And there arise inverse problems wherein according to some information about the solution of the direct problem it is required to determine the coefficient of equation. As in known, the inverse problems are called coefficient inverse problems (see e.i [6],[7]). There are a lot of methods and approaches for solving inverse problems. One of such methods is the optimization method for solving inverse problems. This method was most developed for parabolic equations (see e.i [1], [5], [7], [11]).

In this paper, we suggest optimizational statement of a problem for finding the coefficient at lowest term of a string vibrations equation in the Cauchy problem. The problem is reduced to an optimal control problem and is studied by the methods of optimal control theory.

2. Problem statement

Consider a problem of determination of the pair of functions $(u(x,t), \upsilon(x)) \in W_2^1(\Pi) \times V$ from the following relations

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \upsilon(x) u = f(x,t), (x,t) \in \Pi \equiv (-\infty, \infty) \times (0,T), \tag{1}$$

$$u(x,0) = u_0(x), \frac{\partial u(x,0)}{\partial t} = u_1(x), \, x \in (-\infty, \infty), \tag{2}$$

$$u(x,T) = g(x), x \in (-\infty, \infty) \tag{3}$$

where $f \in L_2(\Pi)$, $u_0 \in W_2^1(-\infty,\infty)$, $u_1 \in L_2(-\infty,\infty)$, $g \in L_2(-\infty,\infty)$ are the given functions, $V = \{\upsilon(x) \in L_2(-\infty,\infty) : \, a \leq \upsilon(x) \leq b \; a.e.on \; (-\infty,\infty)\}$ is the given set, $T > 0$, a, b are the given finite numbers.

Under the given function $\upsilon(x)$ the Cauchy problem (1), (2) is called a direct problem in the strip Π, the problem (1)-(3) of finding $v(x)$ is called inverse to problem. We reduce problem (1)-(3) to the following optimal control problem: to find a function $\upsilon(x)$ from the set V that minimizes the functional

$$J_0(\upsilon) = \frac{1}{2} \int_{-\infty}^{\infty} [u(x,T;\upsilon) - g(x)]^2 \, dx \tag{4}$$

under constraints (1), (2) where $u(x,t;\upsilon)$ is the solution of problem (1),(2) corresponding to the function $\upsilon = \upsilon(x)$. We call the function $\upsilon(x)$ a control, V a class of admissible controls. Note that if $\min_{\upsilon \in V} J_0(\upsilon) = 0$, then additional condition (3) is fulfilled. We regularize problem (1), (2), (4) in the following way: to find a function $\upsilon(x)$ from V that minimizes the functional

$$J_\alpha(\upsilon) = J_0(\upsilon) + \frac{\alpha}{2} \int_{-\infty}^{\infty} (\upsilon(x) - \omega(x))^2 \, dx \tag{5}$$

where $\alpha > 0$ is the given number, $\omega \in L_2(-\infty,\infty)$ is the given function (1), (2), (5). We call this problem as problem (1), (2), (5).

Under the solution of problem for (1), (2) the given function $\upsilon \in V$ we mean a function $u(x,t;\upsilon) \in W_2^1(\Pi)$ that at $t = 0$ equals $u_0(x)$and the following integral identity

$$\int_\Pi \left[-\frac{\partial u}{\partial t}\frac{\partial \eta}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial \eta}{\partial x} + \upsilon u \eta \right] dxdt - \int_{-\infty}^{\infty} u_1(x) \eta(x,0) \, dx = \int_\Pi f \eta dxdt \tag{6}$$

is fulfilled for the arbitrary function $\eta = \eta(x,t)$ from $W_2^1(\Pi)$, $\eta(x,T) = 0$.

Such a solution of problem (1), (2) is called a generalized solution. From the results of the papers [9], [10] it follows that for each control $\upsilon \in V$ under the above condition imposed on the data of problem (1), (2), it has a unique generalized solution from $W_2^1(\Pi)$ and the estimation

$$\|u\|_{W_2^1(-\infty,\infty)} \leq c \left[\|u_0\|_{W_2^1(-\infty,\infty)} + \|u_1\|_{L_2(-\infty,\infty)} + \|f\|_{L_2(\Pi)} \right], \forall t \in [0,T] \tag{7}$$

is valid.

Here and in what follows, c denotes various constants independent on the estimated variables and admissible controls.

3. Existence and uniqueness of optimal control in problem (1), (2), (5).

Theorem 1. *Let the condition in the statement of problem* (1), (2), (5) *be fulfilled. Then in* $L_2(-\infty,\infty)$ *there exists a dense set* G *such that for all* $\omega\in G$ *at* $\alpha>0$ *problem* (1), (2), (5) *has a unique solution.*

Proof. Prove the continuity of functional (4) in the norm of the space $L_2(-\infty,\infty)$ on the set V. Let $\delta\upsilon\in L_2(-\infty,\infty)$ be the increment of the control on the element $\upsilon\in V$ such that $\upsilon+\delta\upsilon\in V$. Denote $\delta u(x,t)=u(x,t;\upsilon+\delta\upsilon)-u(x,t;\upsilon)$. Then the function $\delta u(x,t)$ is the generalized solution from $W_2^1(\Pi)$ of the Cauchy problem

$$\frac{\partial^2\delta u}{\partial t^2}-\frac{\partial^2\delta u}{\partial x^2}+(\upsilon+\delta\upsilon)\,\delta u=-u\delta\upsilon,\,(x,t)\in\Pi, \tag{8}$$

$$\delta u\,|_{t=0}=0,\frac{\partial\delta u}{\partial t}\,|_{t=0}=0,x\in(-\infty,\infty)\,. \tag{9}$$

The generalized solution from $W_2^1(\Pi)$ of problem (8), (9) equals zero for $t=0$ and satisfies the identity

$$\int_\Pi\left[\frac{\partial\delta u}{\partial t}\frac{\partial\eta}{\partial t}-\frac{\partial\delta u}{\partial x}\frac{\partial\eta}{\partial x}\right]dxdt=\int_\Pi\left[(\upsilon+\delta\upsilon)\,\delta u\eta+u\delta\upsilon\eta\right]dxdt \tag{10}$$

for all $\eta=\eta(x,t)\in W_2^1(\Pi)$ equal $t=T$.

Prove that for the solution of problem (8), (9) the following estimation is valid:

$$\|\delta u\|_{W_2^1(\Pi)}\le c\,\|\delta\upsilon\|_{L_2(-\infty,\infty)}\,. \tag{11}$$

To this end, we use the Faedo- Galerkin method. Let $\{\varphi_k(x)\}$ be a fundamental system in $W_2^1(-\infty,\infty)$. The functions $\varphi_k(x)$, $k=1,2,...$, are called Hermite-type functions ([8] p. 401). We look for the approximate solution of problem (8), (9) in the form $\delta u^N(x,t)=\sum_{k=1}^N C_k^N(t)\,\varphi_k(x)$ from the relations

$$\int_{-\infty}^{\infty}\frac{\partial^2\delta u^N}{\partial t^2}\varphi_m(x)\,dx+\int_{-\infty}^{\infty}\frac{\partial\delta u^N}{\partial x}\frac{d\varphi_m(x)}{dx}dx+\int_{-\infty}^{\infty}(\upsilon+\delta\upsilon)\,\delta u^N\varphi_m(x)\,dx$$

$$=-\int_{-\infty}^{\infty}u\delta\upsilon\varphi_m(x)\,dx,\;m=1,...,N, \tag{12}$$

$$C_k^N(0)=0,\frac{dC_k^N(t)}{dt}\,|_{t=0}=0. \tag{13}$$

The system of equation (12) is a system of linear ordinary differential equations with respect to unknown functions $C_k^N(t)$,$k=1,...,N$ solved with respect to $\frac{d^2C_k^N}{dt^2}$. This system is uniquely solvable with unitial conditions (13) and $\frac{d^2C_k^N}{dt^2} \in L_2(0,T)$. Multiplying the both hand sides of equation (12) by its $\frac{d}{dt}C_m^N(t)$ and summing over m from 1 to N, we get

$$\int_{-\infty}^{\infty} \frac{\partial^2 \delta u^N}{\partial t^2} \frac{\partial \delta u^N}{\partial t} dx + \int_{-\infty}^{\infty} \frac{\partial \delta u^N}{\partial x} \frac{\partial^2 \delta u^N}{\partial t \partial x} dx$$

$$= -\int_{-\infty}^{\infty} (\upsilon + \delta \upsilon)\, \delta u^N \frac{\partial \delta u^N}{\partial t} dx - \int_{-\infty}^{\infty} u \delta \upsilon \frac{\partial \delta u^N}{\partial t} dx$$

or

$$\frac{1}{2}\frac{d}{dt}\left[\int_{-\infty}^{\infty} \left(\frac{\partial \delta u^N}{\partial t}\right)^2 + \left(\frac{\partial \delta u^N}{\partial x}\right)^2\right] dx$$

$$= -\int_{-\infty}^{\infty} (\upsilon + \delta \upsilon)\, \delta u^N \frac{\partial \delta u^N}{\partial t} dx - \int_{-\infty}^{\infty} u \delta \upsilon \frac{\partial \delta u^N}{\partial t} dx.$$

Hence, integrating with respect to t from 0 to t, we have

$$\int_{-\infty}^{\infty} \left[\left(\frac{\partial \delta u^N}{\partial t}\right)^2 + \left(\frac{\partial \delta u^N}{\partial x}\right)^2\right] dx$$

$$\leq c \int_0^t \int_{-\infty}^{\infty} \left[\left(\delta u^N\right)^2 + \left(\frac{\partial \delta u^N}{\partial t}\right)^2\right] dx ds + c \int_0^t \int_{-\infty}^{\infty} (u \delta \upsilon)^2 \, dx ds. \tag{14}$$

By estimation (7) and the theorem on bounded imbedding $W_2^1(-\infty,\infty) \subset C(-\infty,\infty)$ [2], the solutions of the problem for all $\upsilon \in V$ possess the property $u \in C([0,T]; C(-\infty,\infty)) \equiv C(\Pi)$,, therefore these functions are uniformly bounded with respect to υ in the Π..

Then, from inequality (14) it follows

$$\int_{-\infty}^{\infty} \left[\left(\frac{\partial \delta u^N}{\partial t}\right)^2 + \left(\frac{\partial \delta u^N}{\partial x}\right)^2\right] dx$$

$$\leq c \int_0^t \int_{-\infty}^{\infty} \left[\left(\delta u^N\right)^2 + \left(\frac{\partial \delta u^N}{\partial t}\right)^2\right] dx ds + c \int_{-\infty}^{\infty} |\delta \upsilon|^2 \, dx.$$

Summing up the inequality $\int_{-\infty}^{\infty} \left(\delta u^N(x,t)\right)^2 dx \leq c \int_0^t \int_{-\infty}^{\infty} \left(\frac{\partial \delta u^N(x,s)}{\partial t}\right)^2 ds$ with the previous inequality and exsessing the right hand side, we get

$$\int_{-\infty}^{\infty} \left[\left|\delta u^N\right|^2 + \left|\frac{\partial \delta u^N}{\partial t}\right|^2 + \left|\frac{\partial \delta u^N}{\partial x}\right|^2\right] dx$$

$$\leq c \int_0^t \int_{-\infty}^{\infty} \left[\left|\delta u^N(x,s)\right|^2 + \left|\frac{\partial \delta u^N(x,s)}{\partial t}\right|^2 + \left|\frac{\partial \delta u^N(x,s)}{\partial x}\right|\right] dx ds + c \int_{-\infty}^{\infty} |\delta \upsilon|^2 \, dx.$$

Applying the Gronwall lemma, hence we have

$$\int_{-\infty}^{\infty}\left[\left|\delta u^{N}(x,t)\right|^{2}+\left|\frac{\partial \delta u^{N}(x,t)}{\partial t}\right|^{2}+\left|\frac{\partial \delta u^{N}(x,t)}{\partial x}\right|^{2}\right]dx$$

$$\leq c\int_{-\infty}^{\infty}\left|\delta v(x)\right|^{2}dx, \forall t\in[0,T].$$

Hence, integrating with respect to t from 0 to T, we get

$$\left\|\delta u^{N}\right\|_{W_2^1(\Pi)}\leq c\left\|\delta v\right\|_{L_2(-\infty,\infty)}.$$

This estimation shows that from the sequence $\left\{\delta u^{N}(x,t)\right\}$ we can choose a subsequence (we also denote it by $\left\{\delta u^{N}(x,t)\right\}$, it converges weakly in $W_2^1(\Pi)$ to some element $\delta u\in W_2^1(\Pi)$. As the norm in Hilbert space is weakly lower semi-continuous, for the limit function$\delta u(x,t)$ the estimation

$$\left\|\delta u\right\|_{W_2^1(\Pi)}\leq c\left\|\delta v\right\|_{L_2(-\infty,\infty)}$$

is valid

Thus, estimation (11) is proved. By the imbedding theorem, $W_2^1(\Pi)$ is boundedly imbedded in $L_2(-\infty,\infty)$ [2]. Then from (11) it follows

$$\left\|\delta u(x,T)\right\|_{L_2(-\infty,\infty)}\leq c\left\|\delta u\right\|_{W_2^1(\Pi)}\leq c\left\|\delta v\right\|_{L_2(-\infty,\infty)}.$$

Therefore

$$\left\|\delta u(x,T)\right\|_{L_2(-\infty,\infty)}\to 0 \text{ for } \left\|\delta v\right\|_{L_2(-\infty,\infty)}\to 0. \tag{15}$$

We can write the increment of the functional $J_0(v)$ in the form

$$\Delta J_0(v)=\int_{-\infty}^{\infty}\left[u(x,T;v)-g(x)\right]\delta u(x,T)\,dx+\frac{1}{2}\int_{-\infty}^{\infty}\left|\delta u(x,T)\right|^{2}dx.$$

Hence and from (15) we get that the functional $J_0(v)$ is continuous in the norm $L_2(-\infty,\infty)$ on the set V. Thus, the functional $J_0(v)$ is continuous and lower bounded on V the set V is closed and bounded in uniformly convex Banach space $L_2(-\infty,\infty)$ ([3], p.510).

Then the statement of theorem 3.1 follows from the known theorem below [4].

Theorem 2. *[4] Let B be uniformly convex Banach space; V is closed and bounded set from B; $J: V\to R$ is bounded below and semicontinous below functional. Then there exists a dense subset B_0 of B, such that for any $w\in B_0$ at $r>1$, $\alpha>0$, the functional $I(v)=J(v)+\alpha\left\|v-w\right\|_B^r$ attains its lower boundary on V in unique element $v_*\in V$.*

Theorem 3.1 is proved. □

4. On solvability of adjoint problem.

Introduce the following boundary value problem to the extremal problem (1), (2), (5).

$$\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}+\upsilon\psi=0,\ (x,t)\in\Pi, \tag{16}$$

$$\psi\,|_{t=T}\,=0,\ \frac{\partial\psi}{\partial t}\,|_{t=T}\,=u\left(x,T;\upsilon\right)-g\left(x\right),x\in\left(-\infty,\infty\right). \tag{17}$$

This problem is called adjoint to problem (1), (2), (5).

For the given $\upsilon\in V$ under the generalized solution from $W_2^1\left(\Pi\right)$ of problem (16), (17), we understand a function $\psi=\psi\left(x,t;\upsilon\right)$ that equals zero for $t=T$ and satisfies the integral identity

$$\int_{\Pi}\left[-\frac{\partial\psi}{\partial t}\frac{\partial\mu}{\partial t}+\frac{\partial\psi}{\partial x}\frac{\partial\mu}{\partial x}+\upsilon\psi\mu\right]dxdt=-\int_{-\infty}^{\infty}\left[u\left(x,T;\upsilon\right)-g\left(x\right)\right]\times\mu\left(x,T\right)dx. \tag{18}$$

for any function $\mu=\mu\left(x,t\right)\in W_2^1\left(\Pi\right)$, $\mu\left(x,0\right)=0$.

Note that problems (1), (2) and (16), (17) are of the same type.

Therefore, using the results of the papers [9], [10], it is easy to get that for each fixed $\upsilon\in V$ problem (16), (17) has a unique generalized solution $\psi=\psi\left(x,t;\upsilon\right)$ from $W_2^1\left(\Pi\right)$ and for this solution, the estimation

$$\left\|\psi\right\|_{W_2^1(\Pi)}\leq c\left\|u\left(x,T;\upsilon\right)-g\left(x\right)\right\|_{L_2(-\infty,\infty)}. \tag{19}$$

is valid.

Taking into account estimation (7) and boundedness of the imbedding $W_2^1\left(\Pi\right)\to L_2\left(-\infty,\infty\right)$ from (19) we get

$$\left\|\psi\right\|_{W_2^1(\Pi)}\leq c\left[\left\|u_0\right\|_{W_2^1(-\infty,\infty)}+\left\|u_1\right\|_{L_2(-\infty,\infty)}+\left\|f\right\|_{L_2(\Pi)}+\left\|g\right\|_{L_2(-\infty,\infty)}\right]. \tag{20}$$

5. Calculation of differential of functional (5) and optimality condition

Theorem 3. *Let the conditions of theorem 3.1 be fulfilled. Then functional* (5) *is continuously Frechet differentiable on the set* V *over the norm of the space* $\left[L_\infty\left(-\infty,\infty\right)\right]$*, and its differential at the point* $\upsilon\in V$ *under the increment* $\delta\upsilon\in L_\infty\left(-\infty,\infty\right)$*,* $\upsilon+\delta\upsilon\in V$ *is determined by the expression.*

$$\left\langle J_\alpha'\left(\upsilon\right),\delta\upsilon\right\rangle=\int_{\Pi}u\psi\delta\upsilon dxdt+\alpha\int_{-\infty}^{\infty}\left(\upsilon-\omega\right)\delta\upsilon dx. \tag{21}$$

Proof. Let us consider the increment of the functional (5):

$$\Delta J_{\alpha}(v)=J_{\alpha}(v+\delta v)-J_{\alpha}(v)=\frac{1}{2}\int_{-\infty}^{\infty}\left[u(x,T;v+\delta v)-g(x)\right]^{2}dx$$

$$+\frac{\alpha}{2}\int_{-\infty}^{\infty}(v+\delta v-\omega)^{2}dx-\frac{1}{2}\int_{-\infty}^{\infty}\left[u(x,T;v)-g(x)\right]^{2}dx-\frac{\alpha}{2}\int_{-\infty}^{\infty}(v-\omega)^{2}dx$$

$$=\int_{-\infty}^{\infty}\left[u(x,T;v)-g(x)\right]\delta u(x,T)\,dx$$

$$+\alpha\int_{-\infty}^{\infty}(v-\omega)\,\delta v dx+\frac{1}{2}\int_{-\infty}^{\infty}\left|\delta u(x,T)\right|^{2}dx+\frac{\alpha}{2}\int_{-\infty}^{\infty}\left|\delta v(x)\right|^{2}dx. \tag{22}$$

In formula (10) for η we put $\psi(x,t;v)$, in formula (18) for μ we put $\delta u(x,t)$, sum them and have

$$\int_{-\infty}^{\infty}\left[u(x,T;v)-g(x)\right]\delta u(x,T)\,dx=\int_{\Pi}u\psi\delta v dxdt+\int_{\Pi}\psi\delta v\delta u dxdt.$$

Talking this equality into account in (22), we get

$$\Delta J_{\alpha}(v)=\int_{\Pi}u\psi\delta v dxdt+\alpha\int_{-\infty}^{\infty}(v-\omega)\,\delta v dx+R, \tag{23}$$

where

$$R=\int_{\Pi}\psi\delta v\delta u dxdt+\frac{1}{2}\int_{-\infty}^{\infty}\left|\delta u(x,T)\right|^{2}dx+\frac{\alpha}{2}\int_{-\infty}^{\infty}\left|\delta v\right|^{2}dx$$

is a remainder.

It is clear that for the given $v\in V$ the right hand side in (21) is a linear functional with respect to δv. Moreover,

$$\left|\int_{\Pi}u\psi\delta v dxdt+\alpha\int_{-\infty}^{\infty}(v-\omega)\,\delta v dx\right|\le\|u\|_{L_2(\Pi)}\|\psi\|_{L_2(\Pi)}\cdot\|\delta v\|_{L_\infty(-\infty,\infty)}$$

$$+\alpha\|v-\omega\|_{L_2(-\infty,\infty)}\|\delta v\|_{L_2(-\infty,\infty)}.$$

Since v, $v+\delta v\in V$, then

$$\left|\int_{\Pi}u\psi\delta v dxdt+\alpha\int_{-\infty}^{\infty}(v-\omega)\,\delta v dx\right|$$

$$\le c\left[\|u\|_{W_2^1(\Pi)}\|\psi\|_{W_2^1(\Pi)}+\alpha\|v-\omega\|_{L_2(-\infty,\infty)}\right]\|\delta v\|_{L_\infty(-\infty,\infty)}.$$

Taking here into account estimates (7) and (20), we get the boundedness of functional (21) with respect to δv on $L_\infty(-\infty,\infty)$.

Now estimate the residual term R in formula (23).

Using the Cauchy-Bunyakvosky inequality, we have

$$|R| \le \|\psi\|_{L_2(\Pi)} \cdot \|\delta u\|_{L_2(\Pi)} \|\delta v\|_{L_\infty(-\infty,\infty)} + \frac{1}{2} \|\delta u(x,T)\|^2_{L_2(-\infty,\infty)}$$

$$+\frac{\alpha}{2} \|\delta v\|^2_{L_\infty(-\infty,\infty)} \le c \|\psi\|_{W_2^1(\Pi)} \|\delta u\|_{W_2^1(\Pi)} \|\delta v\|_{L_\infty(-\infty,\infty)}$$

$$+\frac{1}{2} \|\delta u(x,T)\|^2_{L_2(-\infty,\infty)} + \frac{\alpha}{2} \|\delta v\|^2_{L_\infty(-\infty,\infty)} .$$

As the imbedding $W_2^1(\Pi) \to L_2(-\infty,\infty)$ is bounded [2] and estimates (11), (20) hold, hence we get

$$|R| \le c \|\delta v\|^2_{L_\infty(-\infty,\infty)} .$$

Then from (23) it follows that functional (5) is Frechet differentiable on V and formula (21) is valid.

Show that the mapping $v \to J'_\alpha(v)$ defined by equality (21) continuously acts from V to the space $(L_\infty(-\infty,\infty))^*$ that is adjoint to $L_\infty(-\infty,\infty)$.

Let $\delta\psi(x,t) = \psi(x,t;v+\delta v) - \psi(x,t;v)$. From (16), (17) it follows that $\delta\psi(x,t)$ is the generalized solution from $W_2^1(\Pi)$ of the problem.

$$\frac{\partial^2 \delta\psi}{\partial t^2} - \frac{\partial^2 \delta\psi}{\partial x^2} + (v+\delta v)\,\delta\psi = -\psi\delta v,\ (x,t) \in \Pi, \tag{24}$$

$$\delta\psi\,|_{t=T} = 0,\ \frac{\partial \delta\psi}{\partial t}\,|_{t=T} = \delta u(x,T;v),\ x \in (-\infty,\infty). \tag{25}$$

Arguing as in obtaining estimation (11), we can show that for solving problem (24), (25) the following estimation

$$\|\delta\psi\|_{W_2^1(\Pi)} \le c \|\delta v\|_{L_\infty(-\infty,\infty)} + c \|\delta u(x,T)\|_{L_2(-\infty,\infty)}$$

is valid.

Using the boundedness of the imbedding $W_2^1(\Pi) \to L_2(-\infty,\infty)$, hence we get

$$\|\delta\psi\|_{W_2^1(\Pi)} \le c \|\delta v\|_{L_\infty(-\infty,\infty)} + c \|\delta u\|_{W_2^1(\Pi)} . \tag{26}$$

Then (11) and (26) yield the estimation

$$\|\delta\psi\|_{W_2^1(\Pi)} \le c \|\delta v\|_{L_\infty(-\infty,\infty)} . \tag{27}$$

Using the Cauchy-Bunyakovsky inequality and (27) it is easy to be convinced in validity of the inequality

$$\left\|J'_\alpha(v+\delta v) - J'_\alpha(v)\right\|_{(L_\infty(-\infty,\infty))^*}$$

$$\le c\left[\|u\|_{L_2(\Pi)} \|\delta\psi\|_{L_2(\Pi)} + \|\psi\|_{L_2(\Pi)} \|\delta u\|_{L_2(\Pi)} + \|\delta u\|_{L_2(\Pi)} \|\delta\psi\|_{L_2(\Pi)}\right] + \alpha \|\delta v\|_{L_\infty(-\infty,\infty)} .$$

From (11) and (27) it follows that the right hand side of this inequality converges to zero as $\|\delta v\|_{L_\infty(-\infty,\infty)} \to 0$. Hence it follows that $v \to J'_\alpha(v)$ is a continuous mapping from V to $(L_\infty(-\infty,\infty))^*$. Theorem 5.1 is proved. □

Theorem 4. *Let the conditious of theorem 5.1 be fulfilled. Then for the control* $\upsilon_* = \upsilon_*(x) \in V$ *to be optimal in problem* (1), (2), (5) *it is necessary that the inequality*

$$\int_{\Pi} u_*(x,t)\,\psi_*(x,t)\,(\upsilon(x) - \upsilon_*(x))\,dxdt$$
$$+\alpha \int_{-\infty}^{\infty} (\upsilon_*(x) - \omega(x))\,(\upsilon(x) - \upsilon_*(x))\,dx \ge 0 \tag{28}$$

be fulfilled for any $\upsilon = \upsilon(x) \in V$, *where* $u_*(x,t) = u(x,t;\upsilon_*)$ *and* $\psi_*(x,t) = \psi(x,t;\upsilon_*)$ *are the solutions of problems* (1), (2) *and* (16), (17), *respectivelly, for* $\upsilon = \upsilon_*(x)$.

Proof. The set is convex in $L_\infty(-\infty,\infty)$. Furthermore, by theorem 5.1, the functional $J_\alpha(\upsilon)$ is Frechet continuosly differentiable on V and its differential at the point $\upsilon \in V$ is determined by equality (21). Then by theorem 5.2 from (see e.i [12], p. 28) on the element $\upsilon_* \in V$ the inequality $\langle J'_\alpha(\upsilon_*), \upsilon - \upsilon_* \rangle \ge 0$ should be fulfilled for all $\upsilon \in V$. Hence and from (21) it follows validity of inequality (28) for all $\upsilon \in V$. Theorem 5.2 is proved. □

Remark 1. *Based on the obtained necessary condition of optimality, we can suggest an iterative algorithm for approximate finding of the solution of problem* (1), (2), (5).

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