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Energy Release Rate at the Front of Penny-shaped Interface Cracks Contained in the PZT/Elastic/PZT Sandwich Circular Plate under Action of the Normal Opening Forces on the Cracks' Edges

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Abstract. This paper studies the Energy Release Rate (ERR) at the front of the penny-shaped interface cracks contained in the PZT/Elastic/PZT sandwich circular plate-disc under action on the cracks edges opening uniformly distributed normal forces. It is assumed that the rotationally symmetric stress-strain state in the plate takes place and the investigations are made by utilizing the exact field equations and relations of electro-elasticity for piezoelectric materials.The solution to the corresponding boundary-value problem is made by utilizing the finite element method (FEM) and the ERR is studied for various piezoelectric (PZT) materials of the face layers and for various metal-elastic materials for the core layer of the plate. The main attention is focused on the influence of the coupling effect of the mechanical and electrical fields on the ERR. At the same time, numerical results on the effect of the geometrical parameters such as face layers thickness, crack's radius and etc. on the ERR are presented and discussed.

Key Words and Phrases: Energy Release Rate,piezoelectric material, penny-shaped interface crack, sandwich circular plate.

1. Introduction

It is known that through Energy Release Rate (ERR) at crack tips or at a crack front the fracture of the material or element of construction contained this crack, is determined. For this purpose it is also used the Stress Intensity Factor (SIF) at the crack tips, however, to use the ERR is more suitable for the cracks located completely in the piezoelectric material or in the interface between the piezoelectric and elastic materials. Therefore, the study of the ERR for the penny-shaped interface cracks located between the piezoelectric face and metal-core layers of the PZT/Elastic/PZT circular sandwich plate, to which the present work relates also, has a great significance in the estimation and prognostication of the fracture mechanics of the smart layered systems.Note that the determination of the ERR requires solving the corresponding boundary value problems for the PZT/Elastic/PZT layered systems contained interface cracks in order to determine the stress-strain state in this system and define the ERR through these stresses and strains. Moreover, note that under formulation and solution to these problems one of the main question is the construction of the electrical conditions across the crack's edges.

Now we consider a brief review of the related investigations and the formulation of the conditions on the penny-shaped crack edges with respect to the electrical quantities. First, we consider the paper by Kudryatsev et al. (1975) in which a special solution of the stress and displacement fields is obtained for the penny shaped crack embedded in a piezoelectric material.In this paper the so-called permeable condition on the crack edges is considered. In other word, in this paper it is assumed that the electrical potential and the normal components of the electrical displacements are continuous across the crack edge surfaces. The same type of conditions on the crack's edge are also used in the papers by Parton (1976), Yang (2004) and other ones listed therein. Note that analyses in the papers by Li, McMeeking and Landis (2008) and Li, Fengand Xu (2009) analyze the various types of conditions formulated on the crack edges in the piezoelectric materials, which is differ from the permeable condition.

In the related investigations besides permeable conditions, the corresponding impermeable conditions are also used on the crack edges, according to which, it is assumed that the electric displacements on the crack's edge surfaces are equal to zero. Such conditions, for instance is used in the paper by Li and Lee (2012) in which an axisymmetric penny-shaped crack problem for the infinite piezoelectric layer in the case where the crack is in the middle plane of the layer is studied. Moreover, the energetically consistent boundary condition, which was proposed by Landis (2004), is also used under consideration the crack problems for the piezoelectric materials (see, for instance, the papers by Zhong (2012), Eskandari et al. (2010) and others).

It should be noted that in all the foregoing works it is assumed that the penny-shaped crack is embedded completely in a piezoelectric material and therefore formulation of the permeable, impermeable, energetically consistent, semi-consistent and other types of conditions for the electrical quantities across the crack's edge surfaces, becomes necessary. However, in the cases where the penny-shaped crack is in the interface between piezoelectric and elastic mediums the necessity for such conditions disappears and on the crack's edge face which relate to the piezoelectric medium, the ordinary "electrically-open" (or "open-circuit") and "electrically-shorted" (or "short-circuit") conditions are satisfied. We recall that the "electrically-open" (or "open-circuit") condition coincides with the aforementioned impermeable condition.

At the same time, we note that the first attempt to study the problem related to the interface penny-shaped crack between the piezoelectric layer and elastic half-space is made in the paper by Ren et al. (2014). This study is carried out for the crack's opening mode in the case where on the crack face, which is in the piezoelectric layer, the "open-circuit" condition is satisfied.With this, we complete the consideration the review of the related works carried out during the last 10 years. Note that the review of the regarding works carried out in earlier years can be found in the papers by Kuna (2006, 2010).

Analyzes of reviewed above works show that all the investigations carried out therein for the penny-shaped cracks in piezoelectric materials and in the interface between piezoelectric and elastic materials have been made within the scope of the linear piezoelectric fracture mechanics and within the scope of the assumptions that the layers' dimensions are infinite in the plane on which this crack lies. Namely, these infinities allow using the Hankel integral transformation method for the solution to the corresponding boundary value problems.

However, in the cases where the dimensions of the layers in the planes on which the cracks are located, are finite, such as sandwich PZT/Metal/PZT circular plate-disc the radius of which is commensurable with the radius of the penny-shaped crack, then the methods based on the integral transformations, in general, is not applicable. As in the present paper namely such a case is considered and therefore for a solution to the corresponding boundary value problem the numerical method, i.e. the finite element method (FEM) is employed. It should be noted that the corresponding buckling delamination problems were considered in the papers by Cafarova et al. (2017), Akbarov et al. (2017) and Cafarova and Rzayev (2016). Moreover, note that the corresponding buckling delamination and crack problems for the plane-strain state were considered in the papers by Aklbarov and Yahnioglu (2013, 2016).

2. Formulation of the problem

Consider a circular PZT/Metal/PZT sandwich plate with geometry illustrated in Fig. 1a and assume that the thicknesses and piezoelectric materials of the face layers are the same, and the material of the middle (core) layer is an elastic one. Also, we suppose that between the core and face layers there are penny-shaped cracks whose locations are illustrated in Fig. 1b.At the same time, Fig. 1b indicates the geometric parameters and the external opening forces acting on the cracks edge surfaces.

a **b**

Fig. 1. The sketches of the PZT/Metal/PZT plate-disc (a), the geometry of this disc, interface cracks and external opening forces

We associate with the lower face plane of the plate (Fig. 1a) the cylindrical coordinate system $Or\theta z$, according to which, the plate occupies the region

$\{0 \le r \le \ell/2; 0 \le \theta \le 2\pi; 0 \le z \le h\}\,(h = 2h_F + h_c)$ and the penny-shaped cracks occur in $\{z = h_F \pm 0\,;0 \le r \le \ell_0/2\}$ and in $\{z = h_C + h_F \pm 0\,;0 \le r \le \ell_0/2\}$.

Within these framework, we suppose that on the cracks' edges the uniformly rotational symmetric distributed normal opening forces with intensity $p$act and it is required to determine the ERR at the interface cracks' front in the PZT/Elastic/PZT sandwich plate caused with this mechanical forces. For this purpose, first, we consider formulation of the problems for determination of the electromechanical quantities which appear in the plate as a result of the action of the aforementioned mechanical forces.

As we are considering the rotationally axisymmetric deformation case, therefore under the mathematical formulation of the corresponding problem we will use the corresponding field equations related to this case.Moreover, below we will denote the values related to the upper and lower face layers by upper indices (3) and (1), respectively, whereas the values related to the core layer are denoted by upper index (2).

Assuming that the electro-mechanical state in the sandwich plate under consideration appears within the scope of the linear theory of piezoelectricity for the face layers and the linear theory of elasticity for the core layer, the corresponding field equations, according to the monograph by Yang (2005), can be written as follows.

Equilibrium and electrostatic equations:

$$\frac{\partial \sigma_{rr}^{(j)}}{\partial r} + \frac{\partial \sigma_{zr}^{(j)}}{\partial z} + \frac{1}{r}(\sigma_{rr}^{(j)} - \sigma_{\theta\theta}^{(j)}) = 0, \frac{\partial \sigma_{rz}^{(j)}}{\partial r} + \frac{\partial \sigma_{zz}^{(j)}}{\partial z} + \frac{1}{r}\sigma_{rz}^{(j)} = 0, j = 1,2,3,$$

$$\frac{\partial D_r^{(k)}}{\partial r} + \frac{1}{r}D_r^{(k)} + \frac{\partial D_z^{(k)}}{\partial z} = 0, k = 1,3 \qquad (1)$$

The electro-mechanical constitutive relations for piezoelectric materials:

$$\sigma_{rr}^{(k)} = c_{1111}^{(k)} s_{rr}^{(k)} + c_{1122}^{(k)} s_{\theta\theta}^{(k)} + c_{1133}^{(k)} s_{zz}^{(k)} - e_{111}^{(k)} E_r^{(k)} - e_{311}^{(k)} E_z^{(k)},$$

$$\sigma_{\theta\theta}^{(k)} = c_{2211}^{(k)} s_{rr}^{(k)} + c_{2222}^{(k)} s_{\theta\theta}^{(k)} + c_{2233}^{(k)} s_{zz}^{(k)} - e_{122}^{(k)} E_r^{(k)} - e_{322}^{(k)} E_z^{(k)},$$

$$\sigma_{zz}^{(k)} = c_{3311}^{(k)} s_{rr}^{(k)} + c_{3322}^{(k)} s_{\theta\theta}^{(k)} + c_{3333}^{(k)} s_{zz}^{(k)} - e_{133}^{(k)} E_r^{(k)} - e_{333}^{(k)} E_z^{(k)},$$

$$\sigma_{rz}^{(k)} = c_{1311}^{(k)} s_{rz}^{(k)} - e_{113}^{(k)} E_r^{(k)} - e_{313}^{(k)} E_z^{(k)},$$

$$D_r^{(k)} = e_{111}^{(k)} s_{rr}^{(k)} + e_{122}^{(k)} s_{\theta\theta}^{(k)} + e_{133}^{(k)} s_{zz}^{(k)} + \varepsilon_{11}^{(k)} E_r^{(k)} + \varepsilon_{13}^{(k)} E_z^{(k)},$$

$$D_z^{(k)} = e_{311}^{(k)} s_{rr}^{(k)} + e_{322}^{(k)} s_{\theta\theta}^{(k)} + e_{333}^{(k)} s_{zz}^{(k)} + \varepsilon_{31}^{(k)} E_r^{(k)} + \varepsilon_{33}^{(k)} E_z^{(k)},$$

$$E_r^{(k)} = -\frac{\partial \varphi^{(k)}}{\partial r}, E_z^{(k)} = -\frac{\partial \varphi^{(k)}}{\partial z}. \qquad (2)$$

Elasticity relations for the core layer material.

$$\sigma_{rr}^{(2)} = \lambda^{(2)} s^{(2)} + 2\mu^{(2)} s_{rr}^{(2)},$$

$$\sigma_{\theta\theta}^{(2)} = \lambda^{(2)} s^{(2)} + 2\mu^{(2)} s_{\theta\theta}^{(2)}, \sigma_{zz}^{(2)} = \lambda^{(2)} s^{(2)} + 2\mu^{(2)} s_{zz}^{(2)},$$

$$\sigma_{rz}^{2} = 2\mu^{2} s_{rz}^{2}, s^{2} = s_{rr}^{2} + s_{\theta\theta}^{2} + s_{zz}^{2}, k = 1, 3. \tag{3}$$

Strain-displacement relations:

$$s_{rr}^{(j)} = \frac{\partial u_r^{(j)}}{\partial r}, s_{\theta\theta}^{(j)} = \frac{u_r^{(j)}}{r}, s_{zz}^{(j)} = \frac{\partial u_z^{(j)}}{\partial z}, s_{rz}^{(j)} = \frac{1}{2}\left(\frac{\partial u_r^{(j)}}{\partial z} + \frac{\partial u_z^{(j)}}{\partial r}\right), j = 1, 2, 3. \tag{4}$$

Note that in (1) – (4) the following notation is used: $\sigma_{rr}^{(j)}, \ldots, \sigma_{rz}^{(j)}$ and $s_{rr}^{(j)}, \ldots, s_{rz}^{(j)}$ are the components of the stress and strain tensors, respectively, $u_r^{(j)}$and $u_z^{(j)}$ are the components of the displacement vector, $D_r^{(k)}$ and $D_z^{(k)}$ are the components of the electrical displacement vector, $E_r^{(k)}$ and $E_z^{(k)}$ are the components of the electrical field vector, $\varphi^{(k)}$ is the electric potential, $\lambda^{(2)}$and $\mu^{(2)}$ are Lame constants of the core layer material, and $c_{ijkl}^{(k)}$, $e_{nij}^{(k)}$ and $\varepsilon_{nj}^{(k)} (k = 1, 2, 3)$ are the elastic, piezoelectric and dielectric constants, respectively.

Notethat the piezoelectric material exhibits the characteristics of orthotropic materials with the corresponding elastic symmetry axes and becomes electrically polarized under mechanical loads or mechanical deformation placed in an electrical field. According to the monograph by Yang (2005) and other related references, the polled direction of the piezoelectric material will change according to the position of the material constants in the constitutive relations in (2). In the present paper, under numerical calculations, it is assumed that the Oz axis direction is the polarized direction. Moreover, in general, in the theory of the piezoelectricity for simplicity the following notation is used.

$$c_{1111}^{(k)} = c_{11}^{(k)}, c_{2211}^{(k)} = c_{1122}^{(k)} = c_{12}^{(k)}, c_{3311}^{(k)} = c_{1133}^{(k)} = c_{13}^{(k)}, c_{2222}^{(k)} = c_{22}^{(k)},$$

$$c_{3322}^{(k)} = c_{2233}^{(k)} = c_{23}^{(k)}, c_{3333}^{(k)} = c_{33}^{(k)}, c_{1313}^{(k)} = c_{55}^{(k)}, e_{111}^{(k)} = e_{11}^{(k)}, e_{311}^{(k)} = e_{31}^{(k)},$$

$$e_{122}^{(k)} = e_{12}^{(k)}, e_{322}^{(k)} = e_{32}^{(k)}, e_{133}^{(k)} = e_{13}^{(k)}, e_{333}^{(k)} = e_{33}^{(k)}, e_{313}^{(k)} = e_{35}^{(k)}, e_{113}^{(k)} = e_{15}^{(k)}. \tag{5}$$

Thus, the equations and relations in (1) – (6) completes the writing of the field equations. Now we consider mathematical formulation of the boundary conditions.

Boundary conditions on the cracks' edges:

$$\sigma_{zr}^{(3)}\Big|_{z=h_F+h_C+0} = 0,\ \sigma_{zz}^{(3)}\Big|_{z=h_F+h_C+0} = -p,\ \sigma_{zr}^{(2)}\Big|_{z=h_F+h_C-0} = 0,\ \sigma_{zz}^{(2)}\Big|_{z=h_F+h_C-0} = -p,$$

$$\sigma_{zr}^{(2)}\Big|_{z=h_F+0}=0,\ \sigma_{zz}^{(2)}\Big|_{z=h_F+0}=-p,\ \sigma_{zr}^{(1)}\Big|_{z=h_F-0}=0,\ \sigma_{zz}^{(1)}\Big|_{z=h_F-0}=-p,$$

$$D_z^3\big|_{z=h_F+h_C+0}=0,\ D_z^1\big|_{z=h_F-0}=0,\quad \text{for}\quad 0\le r\le \ell_0/2. \tag{6}$$

Contact conditions between the layers in the areas which are out of the cracks:

$$\sigma_{zz}^{(3)}\Big|_{z=h_F+h_C}=\sigma_{zz}^{(2)}\Big|_{z=h_F+h_C},\ \sigma_{zr}^{(3)}\Big|_{z=h_F+h_C}=\sigma_{zr}^{(2)}\Big|_{z=h_F+h_C},\ u_z^{(3)}\Big|_{z=h_F+h_C}=u_z^{(2)}\Big|_{z=h_F+h_C},$$

$$u_r^{(3)}\Big|_{z=h_F+h_C}=u_r^{(2)}\Big|_{z=h_F+h_C},\ \sigma_{zz}^{(2)}\Big|_{z=h_F}=\sigma_{zz}^{(1)}\Big|_{z=h_F},\ \sigma_{zr}^{(2)}\Big|_{z=h_F}=\sigma_{zr}^{(1)}\Big|_{z=h_F},$$

$$u_r^2\big|_{z=h_F}=u_r^1\big|_{z=h_F},\ D_z^3\big|_{z=h_F+h_C}=0,\ D_z^1\big|_{z=h_F}=0,\ \text{for}\ \ \ell_0/2\le r\le \ell/2. \tag{7}$$

Boundary conditions on the face planes of the plate:

$$\sigma_{zz}^{(3)}\Big|_{z=2h_F+h_C}=0,\ \sigma_{zr}^{(3)}\Big|_{z=2h_F+h_C}=0,\ \sigma_{zz}^{(1)}\Big|_{z=0}=0,$$

$$\sigma_{zr}^{(1)}\Big|_{z=0}=0,\ D_z^3\big|_{z=2h_F+h_C}=0,\ D_z^1\big|_{z=0}=0,\quad \text{for}\ \ 0\le r\le \ell/2. \tag{8}$$

Conditions on the lateral boundary of the plate:
$\sigma_{rr}^{(j)}\Big|_{r=\ell/2}=0,\ u_z^{(j)}\Big|_{r=\ell/2}=0$, for $j=1,2,3;\ \varphi^{(k)}\big|_{r=\ell/2}=0$

$$\text{for} k=1,3,\quad \text{under} 0\le z\le 2h_F+h_C. \tag{9}$$

This completes the formulation of all the boundary and contact conditions for the problem under consideration.

3. Method of solution. FEM modeling of the problem

As the analytical or approximate analytical solution to the problem under consideration is impossible therefore the formulated problem is solved numerically by employing FEM. For FEM modeling of the problem, according to Yang (2005) and others, the following functional is introduced.

$$\Pi(\,u_r^{(1)},u_r^{(2)},u_r^{(3)},u_z^{(1)},u_z^{(2)},u_z^{(3)},\varphi^{(1)},\varphi^{(3)})=$$

$$\frac{1}{2}2\pi\sum_{k=1}^{3}\iint_{\Omega^{(k)}}\left[\sigma_{rr}^{(k)}\frac{\partial u_r^{(k)}}{\partial r}+\sigma_{\theta\theta}^{(k)}\frac{u_r^{(k)}}{r}+\sigma_{rz}^{(k)}\frac{\partial u_z^{(k)}}{\partial r}+\sigma_{zr}^{(k)}\frac{\partial u_r^{(k)}}{\partial z}+\sigma_{zz}^{(k)}\frac{\partial u_z^{(k)}}{\partial z}\right]rdrdz$$

$$+\frac{1}{2}2\pi\iint_{\Omega^{(1)}}\left[E_r^{(1)}D_r^{(1)}+E_z^{(1)}D_z^{(1)}\right]rdrdz+\frac{1}{2}2\pi\iint_{\Omega^{(3)}}\left[E_r^{(3)}D_r^{(3)}+E_z^{(3)}D_z^{(3)}\right]rdrdz-$$

$$2\pi\int_0^{\ell_0/2}p\,u_z^{(1)}\Big|_{z=h_F}rdr-2\pi\int_0^{\ell_0/2}p\,u_r^{(2)}\Big|_{z=h_F}rdr-$$

$$2\pi\int_0^{\ell_0/2}p\,u_z^{(2)}\Big|_{z=h_F+h_C}rdr-2\pi\int_0^{\ell_0/2}p\,u_z^{(3)}\Big|_{z=h_F+h_C}rdr, \tag{10}$$

where

$$\Omega^{(1)}=\{0\le r\le\ell/2;0\le z\le h_F\}-\{z=h_F-0;0\le r\le\ell_0/2\};$$

$$\Omega^{(2)}=\{0\le r\le\ell/2;h_F\le z\le h_F+h_C\}-\{z=h_F+0;0\le r\le\ell_0/2\}$$

$$-\{z=h_F+h_C-0;0\le r\le\ell_0/2\};$$

$$\Omega^{(3)}=\{0\le r\le\ell/2;h_F+h_C\le z\le 2h_F+h_C\}-\{z=h_F+h_C+0;0\le r\le\ell_0/2\}. \tag{11}$$

Equating to zero the first variation of the functional (10), i.e. from the relation

$$\delta\Pi=\sum_{k=1}^{3}\frac{\partial\Pi}{\partial u_r^{(k)}}\delta u_r^{(k)}+\sum_{k=1}^{3}\frac{\partial\Pi}{\partial u_z^{(k)}}\delta u_z^{(k)}+\frac{\partial\Pi}{\partial\varphi^{(1)}}\delta\varphi^{(1)}+\frac{\partial\Pi}{\partial\varphi^{(3)}}\delta\varphi^{(3)}=0 \tag{12}$$

anddoing well-known mathematical manipulations we obtain the equations in (1) and all the corresponding boundary and contact conditions in (7) – (9) with respect to the forces and electrical displacements. In this way it is proven that the equations in (1) are the Euler equations for the functional (10), and the boundary and contact conditions in (7) – (9) which are given with respect to the forces and electrical displacements, are the related natural boundary and contact conditions.

As an usual procedureof FEM modelling, the solution domains indicated in (11) are divided into a finite number of finite elements. For the considered problem, each of the finite elements is selected as a standard rectangular Lagrange family quadratic finite element with nine nodes and each node has three degrees of freedom, i.e. radial displacement $u_r^{(j)}$, transverse displacement $u_z^{(j)}(j=1,2,3)$ and electric potential $\varphi^{(k)}(k=1,2)$. We recall that under FEM modelling of the region containing the crack's tip, as did our predecessors, we use ordinary (not singular) finite elements. This is because up to now finite elements with oscillating singularity which appear at the interface crack tips have not been found. Furthermore, as shown in the references Akbarov (2013), Akbarov and Yahnioglu (2016), Akbarov and Turan (2009),Henshell and Shaw (1975) and other ones listed therein, under calculation of the fracture characteristics of the element of construction (such as the critical forces, ERR and etc.) the results obtained by the use of the “ordinary” singular finite elements coincide, with very high accuracy, with the results obtained by the use of the ordinary finite elements.

Table 1The values of the mechanical, piezoelectrical and dielectrical constants of the selected piezoelectric materials

Mater. (Source Ref.)	$c_{11}^{(r_1)}$	$c_{12}^{(r_1)}$	$c_{13}^{(r_1)}$	$c_{33}^{(r_1)}$	$c_{44}^{(r_1)}$	$c_{66}^{(r_1)}$	$e_{31}^{(r_1)}$	$e_{33}^{(r_1)}$	$e_{15}^{(r_1)}$	$\varepsilon_{11}^{(r_1)}$	$\varepsilon_{33}^{(r_1)}$
PZT-4 (Yang, 2005)	13.9	7.78	7.40	11.5	2.56	3.06	-5.2	15.1	12.7	0.646	0.562
PZT-5H (Yang, 2005)	12.6	7.91	8.39	11.7	2.30	2.35	-6.5	23.3	17.0	1.505	1.302
					$\times 10^{10} N/m$		C/m^2			$\times 10^{-8} C/Vm$	

The algorithm and the programs to obtain the numerical results are coded within the foregoing assumptions by the author in the FORTRAN programming language (FTN77). Employing the standard Ritz technique detailed in many references, for instance, in the book by Zienkiewicz and Taylor (1989), we determine the displacements and electrical potential at the selected nodes. After this determination, according to the relation

$$\gamma = \frac{\partial U}{\pi \ell_0 \partial \ell_0}, \tag{13}$$

the energy release rate γ (ERR) is determined, where

$$U = \frac{1}{2} 2\pi \sum_{k=1}^{3} \iint_{\Omega^{(k)}} \left[\sigma_{rr}^{(k)} \frac{\partial u_r^{(k)}}{\partial r} + \sigma_{\theta\theta}^{(k)} \frac{u_r^{(k)}}{r} + \sigma_{rz}^{(k)} \frac{\partial u_z^{(k)}}{\partial r} + \sigma_{zr}^{(k)} \frac{\partial u_r^{(k)}}{\partial z} + \sigma_{zz}^{(k)} \frac{\partial u_z^{(k)}}{\partial z} \right] rdrdz +$$

$$\frac{1}{2} 2\pi \iint_{\Omega^{(1)}} \left[E_r^{(1)} D_r^{(1)} + E_z^{(1)} D_z^{(1)} \right] rdrdz + \frac{1}{2} 2\pi \iint_{\Omega^{(3)}} \left[E_r^{(3)} D_r^{(3)} + E_z^{(3)} D_z^{(3)} \right] rdrdz, \tag{14}$$

is the electro-mechanical strain enegy.

This completes the consideration the solution method of the problem through the FEM modeling.

4. Numerical results and discussions

In the present paper we consider only the numerical results related to the ERR and all the numerical results are obtained for the piezoelectric materials PZT - 4 and PZT -5H which are selected for the face layers, however the metal materials - aluminum (Al) and steel (St) are taken as the core layer materials. The values of the elastic, piezoelectric and dielectric constants of the selected piezoelectric materials and the references used for this purpose are given in Table 1. According to the monograph by Guz (2004), the values of Lame's constants of the core layer material are selected as follows: for the Al: $\lambda = 48.1\,GPa$ and $\mu = 27.1\,GPa$; and for the St: $\lambda = 92.6\,GPa$ and $\mu = 77.5\,GPa$.

In order to analyze the coupling effects of the electro-mechanical fields on the ERR, the numerical results are obtained for the following two cases:

Case 1.

$$e_{ij}^{(r_n)} = 0,\ \varepsilon_{ii}^{(r_n)} = 0, \tag{15}$$

Case 2.

$$e_{ij}^{(r_n)} \neq 0, \varepsilon_{ii}^{(r_n)} \neq 0. \tag{16}$$

Numerical results obtained in Case 1 (15) relate to the pure mechanical ERR, however the numerical results obtained in Case 2 (16) relate to the total electro-mechanical ERR and comparison of the results obtained in Case 2 with the corresponding ones obtained in Case 1 will give the information for estimation of the influence of the coupling electro-mechanical effect on the studied quantities.

As noted above, in the present paper we consider the numerical results related to the dimensionless ERR determined through the expression $\gamma/(c_{44}^{PZT}\ell)$ and the influence of the problem parameters on this ERR. Under obtaining these results, the values of γ are calculated through the expression (13) and under this calculation the following approximate relation is used.

$$\gamma \approx \frac{\Delta U}{\pi \ell_0 \Delta \ell_0}; \Delta U = U(\ell_0 + \Delta \ell_0) - U(\ell_0), \Delta \ell_0/\ell = 10^{-8}. \tag{17}$$

Note that the number 10^{-8} shown in (17) for the ratio $\Delta\ell_0/\ell$ is determined from the corresponding convergence requirement which appears for the numerical calculation of the derivative $\partial U/\partial \ell_0$.

Under obtaining all the numerical results illustrated in the present paper, we assume that the piezoelectric materials are polarized along the plate thickness, i.e. the polarized direction of the PZT materials coincides with the Oz axis. Moreover, all the numerical results are obtained in the case where $h/\ell = 0.2$.

Under FEM modelling we use the symmetry of the problem with respect to the plane $z = h_F + h_C/2$ and the axial symmetry with respect to the Oz (Fig. 1a) axis and according to these symmetries, we consider only the region $\{0 \leq r \leq \ell/2;\ 0 \leq z \leq h_F + h_C/2\}$under FEM modelling and divide this regioninto 500 finite elements along the radial direction and 40 finite elements along the plate's thickness direction. Under fixed numbers of the finite elements, the NDOF depends on the length (or radius) of the penny-shaped crack and the NDOF increases with increasing of this length. For instance, in the case where $\ell_0/\ell = 0.5$ we have 243499 NDOF, however, in the case where $\ell_0/\ell = 0.3$ we have 242899 NDOF. All the corresponding PC programs are composed by the author of the paper.

Table 2.Convergence of the numerical results with respect to the number of FE selected in the radial direction in the case where $\ell_0/\ell = 0.5$, $h_F/\ell = 0.05$, and $h_C/\ell = 0.1$and the number of FE in the Oz axis direction is 12 for PZT-5H/Al/PZT-5H

Number of FE in the radial direct.	NDOF	$\gamma/(c_{44}^{PZT-5H}\ell)$	
		Case 1	Case 2
40	5039	5.17384	3,74138
60	7559	5.25945	3.80980
80	10079	5.31453	3.85354
100	12599	5.35750	3.88634
120	15119	5.39413	3.91292
140	17639	5.42579	3.93496
160	20159	5.45335	3.95347
200	25199	5.49826	3.98258
300	37799	5.56864	4.02635
400	50399	5.60464	4.04842
500	62999	5.62642	4.06177

Table 3.Convergence of the numerical results with respect to the number of FE selected in the Oz axis direction in the case where $\ell_0/\ell = 0.5$, $h_F/\ell = 0.05$, and $h_C/\ell = 0.1$and the number of FE in the radial direction is 100 for PZT-5H/Al/PZT-5H

Number of the FE in the Oz axis direc.	NDOF	$\gamma/(c_{44}^{PZT-5H}\ell)$	
		Case 1	Case 2
12	12599	5.35750	3.88634
18	18599	5.33552	3.86970
20	20599	5.33017	3.86545
24	24599	5.32377	3.85928
28	28599	5.32045	3.85535
30	39599	5.31515	3.85174
40	40599	5.30807	3.84444

Now we consider the convergence of the numerical results with respect to the number of finite elements (FE) selected in the radial and Oz axis directions. For this purpose, consider the numerical results related to the dimensionless ERR, i.e. to the $\gamma/(c_{44}^{PZT-5H}\ell)$ for the PZT-5H/Al/PZT-5H plate. The results obtained for various values of the FE selected in the radial direction (in the Oz axis direction) are given in Table 2 (in Table 3). It follows from these tables that the convergence of the numerical results is more sensitive with respect to the FE numbers selected in the radial direction. These and other similar results which are not given here allow us to conclude that in the convergence sense of the numerical results, it is enough to select 500 FE in the radial direction and 40 FE in the Oz axis direction in order to obtain results, the relative errors of which are less than 0.4%. Note that 40 FE in the Oz axis direction are divided in half between the face layer and half thickness of the core layer.

The convergence of the numerical results illustrated above gives some confidence on the reliability of the calculation algorithm and PC programs. However, for more detailed verification of the PC programs and FEM modelling used we consider a comparison of the numerical results obtained within the scope of the present algorithm and PC programs with the corresponding ones obtained within the scope of the analytical solution method developed in the paper by Li and Lee (2012). We recall that the paper by Li and Lee (2012) studies an axisymmetric penny-shaped crack problem for the infinite piezoelectric layer in the case where the crack is in the middle plane of the layer and the new analytical method is developed for determination of the corresponding fundamental solutions and, by employing this method numerical results related to the ERR are presented and discussed. Let us employ, in some particular cases, i.e. in the cases where on the crack edges the electric displacements are equal to zero and these edges are loaded with uniformly distributed mechanical opening forces with intensityσ_0, our FEM modelling and the PC programs for obtaining the numerical results considered in the by Li and Lee (2012). Note that under FEM modelling of the problem considered in the paper by Li and Lee (2012) we assume that $\ell = 1\,m$, $\ell_0 = 0.003\,m$ and $h = 0.02\,m$. The values selected for ℓ_0 and h coincide with the corresponding ones selected in the paper by Li and Lee (2012), however, the parameter ℓ does not exist in the paper by Li and Lee (2012) because in that paper it is assumed that the length of the piezoelectric layer in the radial direction is infinite.

Thus, within the scope of the foregoing assumptions, we compare the numerical results obtained with employing of the present FEM modelling with the corresponding ones obtained in the paper by Li and Lee (2012) for the PZT-5H material. These results are given in Table 4 and it follows from the corresponding comparison that the FEM modelling and PC programs developed in the present paper are reliable enough.

Table 4.Numerical results related to $\gamma\,(N/m)$ (i.e. ERR) for the penny-shaped crack in the middle plane of the infinite PZT-5H piezoelectric layer in the case where $\ell = 1\,m$, $h = 0.02\,m$ and $\ell_0 = 0.003\,m$

Sources of the results	σ_0		
	10Mpa	20Mpa	30Mpa
Present results	3.3164	13.2655	29.8473
Results obtained in Li and Lee (2012)	3.2000	12.8000	29.4000

Fig. 2. The graphs of the dependence between dimensionless ERR and crack radius for the PZT-5H/Al/PZT-5H plate

Now we consider the results given in Figs. 2, 3 and 4 which illustrate how an increase in the crack radius acts on the ERR. Note that these results relate to the PZT-5H/Al/PZT-5H (Fig.2), PZT-4/Al/PZT-4 (Fig. 3) and PZT-5H/St/PZT-5H (Fig. 4) plates and show the graphs between the dimensionless ERR (denoted as $\gamma/(c_{44}^{PZT}\ell)$) and the dimensionless crack radius (denoted as ℓ_0/ℓ). Note that in these figures, the dashed lines relate to Case1, however the solid lines relate to Case 2.

Fig. 3 The graphs of the dependence between dimensionless ERR and crack radius for the PZT-4/Al/PZT-4 plate

Fig. 4. The graphs of the dependence between dimensionless ERR and crack radius for the PZT-5H/St/PZT-5H plate

Thus, it follows from Figs. 2, 3 and 4 that for all the cases under consideration the piezoelectricity of the face layers causes to decrease the ERR in the front of the interface penny-shaped crack and the magnitude of this decrease increase with the crack's radius. Moreover, the analyzes of the graphs given in these figures show that the values of the ERR increase with decreasing of the face layers thickness.

5. Conclusions

Thus, in the present paper, the ERR at the penny-shape interface crack contained in the PZT/Elastic/PZT sandwich plate-disc is studied within the scope of the exact equations and relations of the electro-elasticity for the piezoelectric bodies. The axisymmetric stress-strain state is considered and the corresponding boundary value problem is solved numerically by employing FEM. Numerical results are presented and discussed for the PZT-5H/Al/PZT-5H, PZT-5H/St/PZT-5H and PZT-4/Al/PZT-4 plates. The convergence of the algorithm and PC programs is tested with respect to the concrete cases. Moreover, the validation of the PC programs and algorithm used in the present investigation is examined with respect to the known results obtained in the paper by Li and Lee (2012). According to analyzes of the aforementioned numerical results obtained for the ERR it can be drawn the following concrete conclusions:

1. The piezoelectricity of the face layers' materials causes to decrease the values of the ERR;

2. The values of the ERR increase (decrease) with the ratio ℓ_0/ℓ (with the ratio h_F/ℓ);

3. The magnitude of the ERR depends not only on the electro-mechanical properties of the face layers' materials, but also on the mechanical properties of the elastic core layer. For instance, the values of the ERR obtained for the plate with the St core layer are significantly less than the corresponding ones obtained for the same plate with the Al core layer.

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