Journal of Contemporary Applied Mathematics
V. 8, No 1, 2018, July
ISSN 2222-5498

Asymptotic Expansions for the Some Probability Characteristics of the One Stochastic Process with a Heavy Tailed Distributed Component Having Finite Variance

Rovshan Aliyev

Abstract. In present study, renewal reward process $X(t)$ with a heavy tailed distributed rewards having finite variance is considered. The asymptotic expansions as $\beta = S - s \to \infty$ for the ergodic distribution and n-order moment of the process $W_\beta(t) \equiv \frac{1}{\beta}(X(t) - s)$, based on the main result of the study Geluk and Frenk (Renewal theory for random variables with a heavy tailed distribution and finite variance. Statistics and Probability Letters, 2011, 81: 77–82), are obtained. Moreover, special case, when reward sizes has Pareto distribution with parameter $\alpha > 2$ is considered.

Key Words and Phrases: ergodic distribution; moments; renewal function; heavy tailed distributions, subexponential distributions.

1. Introduction and primarily discussions

The studies of renewal reward processes have an important role in the field of stochastic process, because they have been used as a model in many applications, such as; stock control, queuing, reliability and so on. There are many studies on renewal reward processes in the literature. For example, in study Brown and Solomon (1975) was obtained second order approximation for the variance of renewal reward process. Csenki (2000) considered renewal reward process with retrospective reward structure and found asymptotic expansions for the expected value. Levy and Taqqu (2000) also considered the renewal process with stable inter-renewal time intervals and stable rewards. Aliyev and Khaniyev (2014), Aliyev et. al. (2009), Khaniyev et. al. (2013) was studied renewal reward process with discrete interference of chance. Unlike of the above mention studies, in the present paper we will be consider renewal reward process with discrete interference of chance and a heavy tailed distributed rewards having finite variance.

Let $\{\xi_n\}$,$\{\eta_n\}$,$\{\theta_n\}$, and $\{\zeta_n\}$, $n \geq 1$- are independent sequences of random variables defined on probability space$(\Omega, \Im, P)$, such that variables in each sequence independent and identically distributed. Suppose that ξ_n, η_n, θ_n and ζ_n take only positive values and these distribution functions be denoted by

$$\Phi(t) = P\left\{\xi_1 \le t\right\},\ t > 0, F \equiv F(x) = P\left\{\eta_1 \le x\right\}, x > 0,$$

$$H(u) = P\left\{\theta_1 \le u\right\}, u > 0, \pi(z) = P\left\{\zeta_1 \le z\right\},\ z \in [s, S].$$

Let introduce also so called equilibrium distribution
$F_1(x) = \frac{1}{m_1}\int_0^x \bar{F}(u)du$, where $\bar{F}(u) = 1 - F(u)$, $m_1 = E\left(\eta_1\right)$.

Define independent renewal sequence $\{T_n\}$ and $\{Y_n\}$ as follows using the initial sequences of the random variables$\{\xi_n\}$ and $\{\eta_n\}$as:

$$T_n = \sum_{i=1}^{n} \xi_i,\ \ Y_n = \sum_{i=1}^{n} \eta_i,\ n = 1, 2, ...;\ T_0 = Y_0 = 0.$$

Define also sequence of integer valued random variables:

$$N(z - s) = \min\left\{k \ge 1 : z - Y_k < s\right\}; N_0 = 0,$$

$$N_n \equiv N_n(\zeta_n - s) = \min\left\{k \ge N_{n-1} + 1 : \zeta_n - (Y_k - Y_{N_{n-1}}) < s\right\}, n = 1, 2, ...$$

Let the random variables $\tau_{\rm n}$ represents the nth time of the process drops below the control level s and $\gamma_{\rm n}$ represents the nth moment exit from the level s:

$$\tau_0 = 0, \tau_1 = T_{N_1}, \gamma_1 = \tau_1 + \theta_1,$$

$$\tau_2 = \gamma_1 + T_{N_2} - T_{N_1}, \gamma_2 = \tau_2 + \theta_2,$$

...

$$\tau_n = \gamma_{n-1} + T_{N_n} - T_{N_{n-1}},\ \gamma_n = \tau_n + \theta_n, n = 1, 2, ...$$

Define also the counting process $\nu\,(\mathrm{t})$ as:

$$\nu\,(\mathrm{t}) = \max\left\{n \ge 0\ :\ T_n \le t\right\}.$$

Thus the following stochastic process can be constructed using these notations:
$X(t) = \max\{s\,,\ \zeta_{n+1} - Y_{\nu(t)} + Y_{N_n}\}$ as $\gamma_n \le t < \gamma_{n+1}, \quad n = 0, 1, 2, ...$

The considered process $X(t)$ will be called renewal reward process with discrete interference of chance.

Our purpose in this paper is to investigate asymptotic behaviour as sufficiently large values of parameter $\beta = S - s$ of the ergodic distribution and n-order $(n = 1, 2, ...)$ moment of the process $X(t)$, when η_n has a heavy tailed distribution and finite variance. Note that, heavy tailed distributions play an important role in applied probability, especially when modeling the samples where large values appear with non remissible probability. There are

some important classes of heavy tailed distributions. One of the subclass of subexpanential distributions was introduce by Chistyakov (1964).
Definition 1.1 (Chistyakov (1964)). A distribution F on $[0,\infty)$ is said to be subexponential, written as $F\in$, if $\bar{F}(x)=1-F(x)>0$ for all $x>0$ and
$\bar{F}^{*(2)}\left(x\right)\sim 2\bar{F}\left(x\right)$,as $x\to\infty$,
where $F^{*(2)}\left(x\right)$denote convolution product and $\bar{F}^{*(2)}\left(x\right)=1-F^{*(2)}\left(x\right)$.

Another subclass of heavy tailed distributions introduce in study Klppelberg (1988).
Definition 1.2 (Klppelberg (1988)). A distribution F on $[0,\infty)$ is said to belong to the class L^*, if $F(x)>0$ for all $x>0$, $m_1=\int_0^\infty \bar{F}(x)dx<\infty$, and
$\int_0^x \bar{F}(x-t)\bar{F}\left(t\right)dt\sim 2m_1\bar{F}\left(x\right)$, as $x\to\infty$.

The class L^* forms an important subclass of the subexponential class . Klppelberg (1988) first introduced the class L^* and pointed out that the class L^*contains almost all cited subexponential distributions with finite means. In recent studies in applied probability, researchers have discovered that the class L^*enjoys a lot of nicer properties than the class L. The standard examples of subexponential distributions with a finite variance such as the lognormal, Weibull with $\bar{F}(x)=\exp\left(-x^\alpha\right)$, $0<\alpha<1$ and Pareto distribution (about class L^*see, Asmussen (2000), also, Foss, Korshunov, Zachary (2011)).

2. Ergodic distribution and n-order moment of ergodic distribution of the process

In this section we will give exact formulas for the ergodic distribution and n-order moment of ergodic distribution of the investigated process$X(t)$.
Proposition 2.1. *Let initial sequences* $\{\xi_n\}$,$\{\eta_n\}$,$\{\theta_n\}$, and $\{\zeta_n\}$, $n\geq 1-$ *satisfies the following supplementary conditions:*

$$1)E\xi_1<\infty;$$

$$2)E\theta_1<\infty;$$

3) distribution function$F(x)$*of the random variable* η_1*is non-singular and*$F_1\in\mathrm{L}^*$*;*
4) random variables$\{\zeta_n\}$, $n\geq 1$*has the uniform distribution on* $[s,S]$.
Then the process $X(t)$ *is ergodic and ergodic distribution function has the following explicit form:*

$$Q_X(x)=1-\frac{1*U(\beta-x+s)}{1*U(\beta)+K\beta},\beta=S-s,s\leq x\leq S, \tag{1}$$

where$K=E\theta_1/E\xi_1-$ *delay coefficient,* $1*U(t)=\int_0^t U(x)dx$ *and* $U(x)=\sum_{n=0}^\infty F^{*(n)}(x)$ *– is a renewal function generated by the sequence* $\{\eta_n\}$, $n\geq 1$, $F^{*(n)}(x)$ *is denoted n-*

fold convolution of the distribution function $F(x)$*with itself and define as* $F^{*(n)}(x) = \int_0^x F^{*(n-1)}(x-y)dF(y)$, $n \geq 1$.

By using Proposition 2.1 can be obtain n-order $(n = 1, 2, ...)$ moment of ergodic distribution of the process $X(t)$.

Proposition 2.2. *Let the conditions of Proposition 2.1 be satisied. If n-order moment of the ergodic distribution of the process* $\bar{X}(t) = X(t) - s$ *exists and finite, then it can be represented as follows:*

$$E(\bar{X}^n) = \frac{n\ (1 * U_n(\beta))}{1 * U(\beta) + K\beta}, \tag{2}$$

where $\mathrm{U_n}(\beta) \equiv \beta^{\mathrm{n}-1} * \mathrm{U}(\beta) = \int_0^\beta (z-t)^{n-1} U(t)dt$, $n = 1,\ 2,\ \ldots$.

Proof. By using (1) it is not difficult to see that :

$$Q_{\bar{X}}(x) = P\left\{\bar{X} \leq x\right\} = P\left\{X \leq s + x\right\} = Q_X(s+x) = 1 - \frac{1 * U(\beta - x)}{1 * U(\beta) + K\beta}. \tag{3}$$

Using (3) not difficult to see that,

$$\mathrm{E}\left(\bar{X}^n\right) = \int_0^{S-s} \mathrm{x^n P}\left\{\bar{\mathrm{X}} \leq \mathrm{x}\right\} = \mathrm{n} \int_0^\beta \mathrm{x^{n-1}}\left(1 - Q_{\bar{X}}(x)\right)\mathrm{dx} =$$

$$= \frac{n}{1 * U(\beta) + K\beta} \int_{\mathrm{o}}^\beta \mathrm{x^{n-1}}\left(1 * U(\beta - x)\right)\ \mathrm{dx}. \tag{4}$$

On the other hand

$$\int_0^\beta \mathrm{x^{n-1}} \int_0^\beta \mathrm{U\,(z-x)\,dzdx} = \int_0^\beta \mathrm{x^{n-1}} \int_{\mathrm{x}}^\beta \mathrm{U}\,(z - \mathrm{x})\,\mathrm{dzdx}$$

$$= \int_0^\beta \mathrm{x^{n-1}} \int_{\mathrm{x}}^\beta \mathrm{U}\,(z - \mathrm{x})\,\mathrm{dzdx} = \int_0^\beta \mathrm{dz} \int_0^z \mathrm{x^{n-1}U}\,(z - \mathrm{x})\,\mathrm{dx}$$

$$= \int_0^\beta \int_0^z (z-t)^{n-1} U\,(t)\,\mathrm{dtdz} = \int_0^\beta U_n(\mathrm{z})\mathrm{dz} = 1 * U_n\,(\beta)\,. \tag{5}$$

Taking (5) into account in (4) can be obtained statement of Proposition 2.2.

3. Asymptotic behovior of ergodic distribution and n-order moment of the process $X(t)$

In previous section the exact expressions for the ergodic distribution and its n-order moment are obtained. It is very difficult to solve real world problems with exact formulas because of the complexity in mathematics. Therefore, in this section, our aim is to obtain an asymptotic expansions as sufficiently large values of parameter $\beta = S - s$ for the ergodic distribution $Q_{W_\beta}(x)$ of the process $W_\beta(t) \equiv \frac{1}{\beta}(X(t) - s)$, and its n-order moment.

From exact formulas (1) and (2) seen that both characteristics expressed by renewal function $U(x)$. For this we can use asymptotic expansions for the renewal function.

From renewal theory well-known the so-called sharper form of the renewal theorem in case $m_2 = E\eta_1^2 < \infty$ for the light tailed distributions (see, for example, Feller, (1971), p.366) as $x \to \infty$:

$$U(x) - \frac{x}{m_1} \to \frac{m_2}{2m_1^2}. \tag{6}$$

This expansion is sufficient to obtain the asymptotic expansion for the characteristics of the process with any light tailed rewards, which investigated in studies Aliyev et. al. (2009), Khaniyev et.al. (2013), Aliyev and Khaniyev (2014) , Aliyev and Bayramov (2017) and etc.

Quite a lot of studies have been done about heavy tailed distributions by now. For a comprehensive survey see the books by Embrechts et. al. (1997), Asmussen (2000).

In study Geluk and Frenk (2011) asymptotic expansion (6) is extended for the some subclass of the subexpanentional distributions.

Theorem 3.1. (Geluk and Frenk (2011)). *Suppose $F(x)$ is non-singular and $F_1 \in \mathrm{L}^*$, then as $x \to \infty$:*

$$U(x) = \frac{x}{m_1} + \frac{m_2}{2m_1^2} - \frac{1}{m_1}\bar{G}_1(x) + O\left(\bar{F}_1(x)\right), \tag{7}$$

where $\bar{G}_1(x) = \int_x^\infty \bar{F}_1(u)du$, $F_1(x) = \frac{1}{m_1}\int_0^x \bar{F}(u)du$, $\bar{F}(u) = 1 - F(u)$.
Note that, $F_1 \in \mathrm{L}^*$implies that $m_2 = E\eta_1^2 < \infty$.

Based on Theorem 3.1, we can state the first main result of this study as follows:

Theorem 3.2. *Let the conditions of Proposition 2.1 be satisfied. Suppose $F(x)$ is non-singular and$F_1 \in \mathrm{L}^*$. Then for the each $x \in (0,1)$ as $\beta \to \infty$*

$$Q_{W_\beta}(x) = x(2-x) + B(x)\frac{1}{\beta} + 2\left(1 * \bar{G}_1(\beta(1-x))\right)\frac{1}{\beta^2} + O\left(\frac{1 * \bar{F}_1(\beta(1-x)}{\beta^2}\right), \tag{8}$$

where $B(x) = 2(1-x)\left(Km_1(1-x) - m_{21}x\right)$.

Proof. Using (1) ergodic distribution function of process $W_\beta(t) \equiv \frac{1}{\beta}(X(t) - s)$, can be written as follows:

$$Q_{W_\beta}(x) = 1 - \frac{1 * U(\beta(1-x))}{1 * U(\beta) + K\beta}, \tag{9}$$

where$1 * U(t) = \int_0^t U(x)dx$, $\beta \equiv S - s$.
From (7) as $\beta \to \infty$ can be obtained:

$$1 * U(\beta) = \int_0^\beta U(x)dx = \frac{1}{2m_1}\beta^2 + \frac{m_2}{2m_1^2}\beta - \frac{1}{m_1}\left(1 * \bar{G}_1(\beta)\right) + J_1(\beta), \tag{10}$$

where $J_1(\beta) = \int_0^\beta g(x)\,dx$ and $g(x) = O\left(\bar{F}_1(x)\right)$.
According to defination, $g(x) = O\left(\bar{F}_1(x)\right)$ means that there are exist $C > 0$ such that

$$|g(x)| \leq C\left|\bar{F}_1(x)\right| = C\bar{F}_1(x). \tag{11}$$

Taking (11) into account, can be written:

$$|J_1(\beta)| = \left|\int_0^\beta g(x)\,dx\right| \leq \int_0^\beta |g(x)|\,dx \leq C\int_0^\beta \bar{F}_1(x)dx = C\left(1 * \bar{F}_1(\beta)\right)$$

Conseqvently,

$$J_1(\beta) = O(1 * \bar{F}_1(\beta)). \tag{12}$$

Taking (12) into account, (10) can be rewritten as $\beta \to \infty$:

$$1 * U(\beta) = \int_0^\beta U(x)dx = \frac{1}{2m_1}\beta^2 + \frac{m_2}{2m_1^2}\beta - \frac{1}{m_1}\left(1 * \bar{G}_1(\beta)\right) + O(1 * \bar{F}_1(\beta)). \tag{13}$$

Using (13) as $\beta \to \infty$ can be obtained:

$$1 * U(\beta) + K\beta = \frac{\beta^2}{2m_1}\left(1 + 2\left(m_{21} + Km_1\right)\frac{1}{\beta^2} - 2\left(1 * \bar{G}_1(\beta)\right)\frac{1}{\beta^2} + O\left(\frac{1 * \bar{F}_1(\beta)}{\beta^2}\right)\right). \tag{14}$$

From (14) as $\beta \to \infty$ can be get:

$$\left(1 * U(\beta) + K\beta\right)^{-1} = \frac{2m_1}{\beta^2}\left(1 - 2\left(m_{21} + Km_1\right)\frac{1}{\beta^2} + 2\left(1 * \bar{G}_1(\beta)\right)\frac{1}{\beta^2} + O\left(\frac{1 * \bar{F}_1(\beta)}{\beta^2}\right)\right). \tag{15}$$

On the other hand, from (13) for the each $x \in (0, 1)$ as $\beta \to \infty$ can be obtained:

$$1 * U(\beta(1 - x)) =$$

$$= \frac{\beta^2}{2m_1}\left((1 - x)^2 + 2m_{21}(1 - x)\frac{1}{\beta} - 2\left(1 * \bar{G}_1(\beta(1 - x))\right)\frac{1}{\beta^2} + O\left(\frac{1 * \bar{F}_1(\beta(1 - x))}{\beta^2}\right)\right). \tag{16}$$

Taking into account (15) and (16) in (6) can be obtained (8).
This completes the proof of Theorem 3.2.

Now investigate the asymptotic behavior as $\beta \to \infty$ of n-order moment of the ergodic distribution of $W_\beta(t)$. For this aim, give the following lemma:
Lemma 3.1. *Let the conditions of Theorem 3.2 be satisfied. Then as* $\beta \to \infty$

$$U_n(\beta) = \frac{1}{m_1 n(n + 1)}\beta^{n+1} + \frac{m_2}{2m_1^2 n}\beta^n - \frac{1}{m_1}\left(\beta^{n-1} * \bar{G}_1(\beta)\right) + O\left(\beta^{n-1} * \bar{F}_1(\beta)\right), \tag{17}$$

whre $\beta^{n-1}*\bar{G}_1(\beta) \equiv \int_0^\beta (\beta-t)^{n-1}\bar{G}_1(t)\,dt$, $n = 1,\, 2,\, \ldots$, $m_k = E\eta_1^k$.

$$\bar{G}_1(\beta) = \int_\beta^\infty \bar{F}_1(u)du, F_1(x) = \frac{1}{m_1}\int_0^x \bar{F}(u)du, \bar{F}_1(x) = \frac{1}{m_1}\int_x^\infty \bar{F}(u)du, \bar{F}(u) = 1-F(u).$$

Proof. Using asymptotic expansion (6) for the each $n = 1,\, 2,\, \ldots$ can be written:

$$U_n(\beta) \equiv \beta^{n-1}*U(\beta) \equiv \int_0^\beta (\beta-x)^{n-1}U(x)\,dx =$$

$$= \frac{1}{m_1}\int_0^\beta (\beta-x)^{n-1}xdx + \frac{m_2}{2m_1^2}\int_0^\beta (\beta-x)^{n-1}dx - \frac{1}{m_1}\int_0^\beta (\beta-x)^{n-1}\bar{G}_1(x)dx + J_n(\beta) =$$

$$= \frac{1}{m_1 n(n+1)}\beta^{n+1} + \frac{m_2}{2m_1^2 n}\beta^n - \frac{1}{m_1}\left(\beta^{n-1}*\bar{G}_1(\beta)\right) + J_n(\beta), \tag{18}$$

where $J_n(\beta) = \int_0^\beta (\beta-x)^{n-1}g(x)\,dx$ and $g(x) = O\left(\bar{F}_1(x)\right)$.
It is not difficult to see that

$$|J_n(\beta)| \le \int_0^\beta (\beta-x)^{n-1}|g(x)|\,dx \le C\int_0^\beta (\beta-x)^{n-1}\bar{F}_1(x)dx. \tag{19}$$

From (19) for the each $n = 1,\, 2,\, \ldots$

$$J_n(\beta) = O\left(\int_0^\beta (\beta-x)^{n-1}\bar{F}_1(x)dx\right) = O\left(\beta^{n-1}*\bar{F}_1(\beta)\right). \tag{20}$$

Taking into account (20) from (18) can be obtained (14).
This completes the proof of Lemma 3.1.

Now, we can state the second result of this study as follows:
Theorem 3.3. *Let the conditions of Theorem 3.2 be satisfied. Then as* $\beta \to \infty$ *the following asymptotic expansion for the n-order moment of the ergodic distribution of can be written:*

$$E(W_\beta^n) = \frac{2}{(n+1)(n+2)} + \frac{2m_{21}}{(n+1)}\beta^{-1} - \left(1*\left(\beta^{n-1}*\bar{G}_1(\beta)\right)\right)\frac{2n}{\beta^{2-n}} + O\left(\frac{1*\left(\beta^{n-1}*\bar{F}_1(\beta)\right)}{\beta^{2-n}}\right). \tag{21}$$

Proof. Taking into account (17) as $\beta \to \infty$ can be obtained:

$$1*U_n(\beta) = \frac{1}{m_1}\left(\frac{1}{n(n+1)(n+2)}\beta^{n+2} + m_{21}\frac{1}{n(n+1)}\beta^{n+1} - \right.$$

$$\left. -1*\left(\beta^{n-1}*\bar{G}_1(\beta)\right) + O\left(1*\left(\beta^{n-1}*\bar{F}_1(\beta)\right)\right)\right), \tag{22}$$

Taking (15) and (22) into account, from (2) as $\beta \to \infty$ can be obtained:

$$E(W_\beta^n) = \frac{n\beta^{-n}\left[1 * U_n(\beta)\right]}{1 * U(\beta) + K\beta} =$$

$$= \frac{n\beta^{-n}}{m_1}\left(\frac{1}{n(n+1)(n+2)}\beta^{n+2} + m_{21}\frac{1}{n(n+1)}\beta^{n+1} - \right.$$
$$\left. -1 * \left(\beta^{n-1} * \bar{G}_1(\beta)\right) + O\left(1 * \left(\beta^{n-1} * \bar{F}_1(\beta)\right)\right)\right)$$

$$\frac{2m_1}{\beta^2}\left(1 - 2\left(m_{21} + Km_1\right)\frac{1}{\beta^2} + 2\left(1 * \bar{G}_1(\beta)\right)\frac{1}{\beta^2} + O\left(\frac{1 * \bar{F}_1(\beta)}{\beta^2}\right)\right) =$$

$$= \frac{2}{(n+1)(n+2)} + \frac{2m_{21}}{(n+1)}\beta^{-1} - \left(1 * \left(\beta^{n-1} * \bar{G}_1(\beta)\right)\right)\frac{2n}{\beta^{2-n}} + O\left(\frac{1 * \left(\beta^{n-1} * \bar{F}_1(\beta)\right)}{\beta^{2-n}}\right).$$

This completes the proof of Theorem 3.2.

4. Special cases: Pareto distribution

In this section we will consider special case, when random variable η_1 has Pareto distribution. It is known that distribution function of the Pareto distribution with parameters $(\alpha, \lambda)$is

$$F(x) = 1 - \left(\frac{\lambda}{x}\right)^\alpha, \ \alpha > 0, \ \ x \geq \lambda.$$

In case $0 < \alpha \leq 2$ variance of the Pareto distribution does not exists. Therefore, we assume that $\alpha > 2$, such that in this case Pareto distributon belong to class L^*. For the simplicity assume that $\lambda = 1$. Consequently, in our case tail function of the Pareto distribution has the following form:

$$\bar{F}(x) = x^{-\alpha}, \ \alpha > 2, \ \ x \geq 1.$$

Now, we can state the following corollaries:

Corollary 4.1. *Let the conditions of Proposition 2.1 be satisfied. Suppose $F(x)$ has the Pareto distribution with parameters $(\alpha, 1)$, $\alpha > 2$. Then for the each $x \in (0,1)$ as$\beta \to \infty$:*

$$Q_{W_\beta}(x) = x(2-x) + B(x)\frac{1}{\beta} + O\left(\frac{1}{\beta^2}\right), \tag{23}$$

where $B(x) = 2(1-x)\left(\frac{K\alpha(1-x)}{\alpha-1} - \frac{(\alpha-1)x}{2(\alpha-2)}\right)$.

Proof. It is not difficult to see that tail of the equilibrium distribution of Pareto distribution is

$$\bar{F}_1(t) = \frac{1}{m_1}\int_t^{\infty}\bar{F}(u)du = \frac{\alpha t^{1-\alpha}}{(\alpha-1)^2},\ \alpha > 2.$$

Therefore,

$$\left|\bar{G}_1(\beta)\right| \leq \int_\beta^{\infty}\left|\bar{F}_1(t)\right|dt = \frac{1}{m_1}\int_\beta^{\infty}\frac{t^{1-\alpha}}{\alpha-1}dt = \frac{\alpha\beta^{2-\alpha}}{(\alpha-1)^2(\alpha-2)},\ \alpha > 2.$$

Consequently,

$$\bar{G}_1(\beta) = O(\beta^{2-\alpha}),\ \alpha > 2.$$

Analogically, it is not difficult to see that for the each$x \in (0,1)$ as $\beta \to \infty$

$$1 * \bar{G}_1(\beta(1-x)) = O(1))\ \text{ and} 1 * \bar{F}_1(\beta(1-x) = O(1)). \tag{24}$$

Asymptotic expansion (23) can be obtained from Theorem 3.2, by using (24).
Corollary 4.2. *Let the conditions of Proposition 2.1 be satisfied. Suppose* $F(x)$*has the Pareto distribution with parameters* $(\alpha, 1)$, $\alpha > 2$. *Then as*$\beta \to \infty$:

$$E(W_\beta^n) = \frac{2}{(n+1)(n+2)} + \frac{(\alpha-1)}{(n+1)(\alpha-2)}\beta^{-1} + O\left(\beta^{-1}\right). \tag{25}$$

Proof. Analogically, it is not difficult to see that as $\beta \to \infty$

$$1 * \left(\beta^{n-1} * \bar{G}_1(\beta)\right) = O\left(\beta^{n+1}\right)\ \text{ and }\ 1 * \left(\beta^{n-1} * \bar{F}_1(\beta)\right) = O\left(\beta^{n+1}\right). \tag{26}$$

Asymptotic expansion (25) can be obtained from Theorem 3.3, by using (26).

5. Conclusions

In present study, renewal reward process $X(t)$ with a heavy tailed distributed rewards having finite variance is considered. Exact expressions for the ergodic distribution and its n-order moment are obtained. But it is very difficult to solve real world problems by using exact formulas because of the complexity of their mathematical structure. For an alternative solving method, asymptotic method can be used.The asymptotic expansions as $\beta \to \infty$ for the ergodic distribution and n-order moment of the process $W_\beta(t) \equiv \beta^{-1}(X(t) - s)$ are obtained. The obtained results show that for the large values of parameter$\beta = S - s$ the probability characteristics of the process $W_\beta(t)$, tends to respective characteristics of the distribution function $G(x) = x(2-x),\ x \in (0,1)$. In other words, $Q_{W_\beta}(x) \to G(x) \equiv x(2-x),\ x \in (0,1)$ and $E\left(W_\beta^n\right) \to \frac{1}{(n+1)(n+2)}$, $n \geq 1$ as $\beta \to \infty$. Estimation of the rate of convergence in this asymptotic results can be investigated in future studies.

Acknowledgements: The author express his thanks to Professor T.A.Khaniyev, TOBB University of Economics and Technology (Turkey), for his supports and valuable advices.

The paper was written within the scientific research program "Mathematical modeling and optimization of continuous and discrete processes with the help of modern computer programs" funded by the decision of Presidium of National Academy of Sciences of Azerbaijan number 7/3March 14, 2018.

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Rovshan Aliyev
Department of Operation Research and Probability Theory, Baku State University,
*Email:*aliyevrovshan@yahoo.com