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On embedding theorems in Sobolev-Morrey type spaces $\bigcap\limits_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$

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Abstract. In this paper it is constructed a new generalized Sobolev-Morrey type spaces and untilizing integral representation of weak derivatives of functions defined on n-dimensional domains satisfying flexible φ-horn condition it is proved theorems on differential properties functions from in this spaces is established.

Key Words and Phrases: generalized Sobolev-Morrey type spaces with dominant mixed derivatives, integral representation, embedding theorems.

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1. Introduction

In this paper we introduce a generalized Sobolev-Morrey type spaces with dominant mixed derivatives

$$\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G), \tag{1}$$

where $G \subset R^n$, $1 \leq p^i < \infty$, $i = 1, 2, ..., 2^n$, $e_n = \{1, 2, ..., n\}$, $e^i \subseteq e_n(\quad e^i \quad 2^n)$; let $e^i = \left(e_1^i, ..., e_n^i\right)$, $e_j^i > 0$ are entire $j \in e^i$; $e_j^i \geq 0$ are entire $j \in e_n \backslash e^i$; $\beta \in [0, 1]^n$; $\varphi(t) = (\varphi_1(t_1), ..., \varphi_n(t_n))$, $\varphi_j(t_j) > 0$ $(t_j > 0,\ j \in e_n)$ by Lebesgue measurable functions $\lim\limits_{t_j \to +0} \varphi_j(t_j) = 0$ and $\lim\limits_{t_j \to +\infty} \varphi_j(t_j) = L \leq \infty$. We denote by A the set of all vector functions. Further, using the integral representation method we study differential and differential-difference properties of functions from this spaces.

Definition. Denote by $\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$ the spaces locally summable functions f on G having the weak derivatives $D^{e^i} f$ with the finite norm

$$\|f\|_{\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)} = \sum_{i=1}^{2^n} \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta,G}, \tag{2}$$

where

$$\|f\|_{p^i,\varphi,\beta,G} = \|f\|_{L_{p,\varphi,\beta}(G)} = \sup_{\substack{x \in G, \\ t>0}} \left(|\varphi([t]_1)|^{-\beta} \|f\|_{p,G_{\varphi(t)}(x)} \right), \quad (3)$$

$|\varphi([t]_1)|^{-\beta} = \prod_{j \in e_n} \left(\varphi_j\left([t_j]_1\right)\right)^{-\beta_j}$, $[t_j]_1 = \min\{1, t_j\}$, $j \in e_n$. For any $x \in R^n$

$$G_{\varphi(t)}(x) = G \bigcap \left\{ y : |y_j - x_j| < \frac{1}{2}\varphi_j(t_j), \, j \in e_n \right\}.$$

The space $\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)$ in the case when $\beta_j = 0$ $(j \in e_n)$ coincides with the $\bigcap_{i=1}^{2^n} L^{l^i}_{p^i,\varphi,\beta}(G)$ introduced and studied [3], in the case $p^i = p$, $l^i = l = (l_1, ..., l_n)$, $l_j \in N$, $j \in e_n$ and let $l^e = \left(l_1^2, ..., l_n^e\right)$, $l_j^e = l_j$ for $j \in e_n \backslash e$ coincides with the spaces $S^l_{p,\varphi,\beta}W(G)$ introduced and studied [12].

The spaces of such type with different norms were introduced and studied in [1, 4-13] and others.

Let for any $t_j > 0$ $(j \in e_n)$, there exists a positive constant $C > 0$ such that $|\varphi[t]_1| \le C$, then the embeddings $L_{p,\varphi,\beta}(G) \hookrightarrow L_p(G)$, $\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G) \hookrightarrow \bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i}(G)$ hold i.e.

$$\|f\|_{p,G} \le c\|f\|_{p,\varphi,\beta,G} \quad \text{and} \quad \|f\|_{\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i}(G)} \le c\|f\|_{\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)} \quad (4)$$

The spaces $L_{p,\varphi,\beta}(G)$ and $\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i}(G)$ are complete.

Lemma 1.1. Let $G \subset R^n$, $1 \le p^i < \infty$ and $f \in \bigcap_{i=1}^{n} L^{<l^i>}_{p^i}(G)$. Then we can construct the sequence $h_s = h_s(x)$ $(s = 1, 2, ...)$ of infinitely differentiable finite in R^n functions for which

$$\lim_{s \to \infty} \|f - h_s\|_{\bigcap_{i=1}^{n} L^{<l^i>}_{p^i}(G)} = 0. \quad (5)$$

Proof. Let $G = \bigcup_{\lambda=1}^{M} G^\lambda$. For obtaining equality (**??**) we estimate the

$$\|f - h_s\|_{\bigcap_{i=1}^{n} L^{<l^i>}_{p^i}(G)} = \sum_{i=1}^{2^n} \left\| D^{le^i}(f - h_s) \right\|_{p^i,G}. \quad (6)$$

The sequence $h_s(x)$ $(s = 1, 2, ...)$ is determined by the equalities

$$h_s(x) = F(x, \varphi(t))|_{t=\frac{1}{s}} = \sum_{\lambda=1}^{M} \eta_\lambda(x) f_{\varphi^\lambda(t)}(x),$$

here the averaging functions are determined as follows:

$$f_{\varphi^\lambda(t)}(x) = \int_{R^n} f\left(x + \varphi^\lambda(t) y\right) K_\lambda(y) \, dy,$$

where $K_\lambda(y) \in C_0^\infty(R^n)$ $(\lambda = 1,2,...,M)$, $\operatorname{supp} K_\lambda(\,\cdot\,) \subset [-1,\,1]$ and

$$\int_{R^n} K_\lambda(y)\,dy = 1,$$

the functions $\eta_\lambda = \eta_\lambda(x)$ $(\lambda = 1,2,...,M)$ determine the expansion of a unit in the domain G, i.e.

1. $0 \le \eta_\lambda(x) \le 1$ in R^n;
2. $\eta_\lambda(x) = 0$ in $G \backslash G_\lambda$ for all $\lambda = 1,2,...,M$;
3. $\sum_{\lambda=1}^M \eta_\lambda(x) = 1$ in G;
4. $|D^\alpha \eta_\lambda(x)| \le C_\lambda$, $C_\lambda = const$ for all $\lambda = 1,2,...,M$ and $\alpha \ge 0$.

Obviously,

$$f(x) - h_s(x) = \sum_{\lambda=1}^M \eta_\lambda(x)\left(f(x) - f_{\varphi^\lambda(t)}(x)\right).$$

Consequently,

$$\|f(\,\cdot\,) - h_s(\,\cdot\,)\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)} \le \sum_{\lambda=1}^M \left\|\eta_\lambda(\,\cdot\,)\left(f(\,\cdot\,) - f_{\varphi^\lambda(t)}(\,\cdot\,)\right)\right\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)} \le$$

$$\le C\sum_{\lambda=1}^M \left\|\left(f(\,\cdot\,) - f_{\varphi^\lambda(t)}(\,\cdot\,)\right)\right\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)}. \tag{7}$$

As much as small for rather small t, as a consequence of continuity of L_p-average functions, belonging to the space $L_p\left(G^\lambda\right)$, from (**??**) it follows

$$\|f(\,\cdot\,) - h_s(\,\cdot\,)\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)} < \varepsilon,$$

in other words,

$$\lim_{s\to\infty} \|f - h_s\|_{\bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)} = 0.$$

Assuming that $\varphi_j(t_j)(j \in e_n)$ are also differentiable on $[0,\,T_j]$ $(j \in e_n)$, we can show that for $f \in \bigcap_{i=1}^n L_{p^i}^{<l^i>}(G)$ determined in n- dimensional domains, satisfying the condition of flexible φ- horn defined in [Naj] it holds the following integral representation $(f_x \in U \subset G)$

$$D^\nu f(x) = \sum_{i=1}^{2^n} (-1)^{|\nu|+|l^i|} \prod_{j\in e_n \backslash e^i} (\varphi_j(T_j))^{l^i_{j-1-\nu_j}} \times$$

$$\times \int_{0^i}^{T^i} \int_{R^n} M_i^{(\nu)}\left(\frac{y}{\varphi(t+T)^i}, \frac{\rho\left(\varphi(t+T)^i, x\right)}{\varphi(t+T)^i}, \rho'\left(\varphi(t+T)^i, x\right)\right) \times$$

$$\times D^{l^i} f(x+y) \prod_{j\in e^i} (\varphi_j(t_j))^{l^i_j-\nu_j-2} \times$$

$$\times \prod_{j\in e^i} \left(\varphi'_j(t_j)\right) dt^i dy, \tag{8}$$

where

$$\int_{0^i}^{T^i} f(x)\,dx^i = \left(\prod_{j\in e^i}\int_0^{T_j} dx_j\right) f(x),$$

i.e. integration is carried out only with respect to the variables x_j whose indices belong to e^i, $(t+T)^i = t_j\ (j\in e^i)$; $(t+T)^i = T_j\ (j\in e_n\backslash e^i)$; $\nu=(\nu_1,...,\nu_n)$, $\nu_j\ge 0 (j\in e_n)$ are integers. Recall that the flexible φ-horn $x+\bigcup_{\substack{0<t_j\le T_j\\ j\in e_n}} [\beta(\varphi(t),)+\varphi(T)\theta_I]$ is the support of the representation (**??**).

Let $M_i(\cdot,y,z)\in C_0^\infty(R^n)$ $(i=1,2,\dots,2^n)$, be such that

$$S(M_i)\subset I_{\varphi(T)} = \left\{y : |y_j| < \frac{1}{2}\varphi_j(T_j),\quad j=e_n\right\}.$$

and assume that for any $0<T_j\le 1$ $(j\in e_n)$

$$V = \bigcup_{0<t_j\le T_j, j\in e_n} \left\{y : \frac{y}{\varphi((t+T)^i)} \in S(M_i)\right\}.$$

It is clear that $V\subset I_{\varphi(T)}$ and suppose that $U+V\subset G$.

Lemma 1.2. Let $1\le pi\le p\le r\le\infty$; $0<\eta_j,\ t_j<T_j\le 1$ $(j\in e_n)$, $\nu = (\nu_1,\nu_2,\dots,\nu_n)$, $\nu_j\ge 0$ be entire, $(j\in e_n)$; $\Phi\in L_{p,\varphi,\beta}(G)$ and

$$\varepsilon_j = l^i_j - \nu_j - (1-\beta_j p)\left(\frac{1}{p^i}-\frac{1}{p}\right),$$

$$R^i_\eta(x) = \prod_{j=e_n\backslash e'} (\varphi_j(T_j))^{l^i_j-1-\nu_j} \int_{0^i}^{T^i} L_i(x;t,T)\Phi(x+y)$$

$$\times \prod_{j\in e^i} (\varphi_j(t_j))^{l^i_j-\nu_j-2} \prod_{j\in e^i} \varphi'_j(t_j)\,dt^i, \tag{9}$$

$$R^i_{\eta,T}(x) = \prod_{j=e_n\backslash e'} (\varphi_j(T_j))^{l^i_j-\nu_j-2} \int_{0^i}^{T^i} L_i(x;t,T)\Phi(x+y)$$

$$\times \prod_{j\in e^i} (\varphi_j(t_j))^{l^i_j-\nu_j-2} \prod_{j\in e^i} \varphi'_j(t_j)\,dt^i, \tag{10}$$

where

$$L_i(x;t,T)=\int_{R^n} M_i^{(\nu)}\left(\frac{y}{\varphi\left((t+T)^i\right)},\frac{\rho\left(\varphi\left((t+T)^i,x\right)\right)}{\varphi\left((t+T)^i\right)},\rho'\left(\varphi\left((t+T)^i\right),x\right)\right)dy.$$

Then for any $\overline{x}\in U$ the following inequalities are true

$$\sup_{\overline{x}\in U}\left\|R_\eta^i\right\|_{q,U_{\psi(\xi)}(\overline{x})}\leq C_1\left\|\Phi\right\|_{p^i,\varphi,\beta;G}\times$$

$$\prod_{j\in e_n\backslash e^i}\left(\varphi_j\left(T_j\right)\right)^{l_j^i-\nu_j-(1-\beta_j p^i)\left(\frac{1}{p^i}-\frac{1}{p}\right)}\prod_{j\in e_n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p^i}{p}}\prod_{j\in e^i}\left(\varphi_j\left(\eta_j\right)\right)^{-\varepsilon_j^i}\ (\varepsilon_j^i>0),\qquad(11)$$

$$\sup_{\overline{x}\in U}\left\|R_{\eta,T}^i\right\|_{q,U_{\psi(\xi)}(\overline{x})}\leq C_2\left\|\Phi\right\|_{p^i,\varphi,\beta;G}\times$$

$$\prod_{j\in e_n\backslash e^i}\left(\varphi_j\left(T_j\right)\right)^{l_j^i-\nu_j-(1-\beta_j p^i)\left(\frac{1}{p^i}-\frac{1}{p}\right)}\prod_{j\in e_n}\left(\psi_j\left([\xi]_1\right)\right)^{\beta_j\frac{p^i}{p}}\times$$

$$\begin{cases}\prod_{j\in e^i}\left(\varphi_j(T_j)\right)^{-\varepsilon_j^i} & \text{for } \varepsilon_j^i>0,\\ \prod_{j\in e^i}\ln\frac{\varphi_j(T_j)}{\varphi_j(\eta_j)} & \text{for } \varepsilon_j^i=0\\ \prod_{j\in e^i}\left(\varphi_j\left(\eta_j\right)\right)^{-\varepsilon_j^i} & \text{for } \varepsilon_j^i<0,\end{cases}\qquad(12)$$

$$\left\|R_\eta^i\right\|_{p,\psi,\beta_1,U}\leq C\left\|\Phi\right\|_{p^i,\psi,\beta;G}\qquad(13)$$

where $U_{\psi(\xi)}\left(\overline{x}\right)=\left\{x:\left|x_j-\overline{x}_j\right|<\frac{1}{2}\psi_j\left(\xi_j\right),j=e_n\right\}$ and $\psi\in N$, C_1, C_2 are the constants independent of φ, ξ, η and T.

2. Main results

Prove two theorems on the properties of the functions from the space $\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$.

Theorem 2.1. Let $G\subset R^n$ satisfy the condition of flexible φ-horn, $1\leq p^i\leq p\leq\infty$, $\nu=(\nu_1,\nu_2,\ldots,\nu_n)$, $\nu_j\geq 0$ be entire $j\in e_n$, $\varepsilon_j^i>0(j\in e_n, i=1,2,\ldots,2^n)$ and let $f\in\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$. Then the following embeddings hold

$$D^\nu:\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)\to L_{p,\psi,\beta^1}\left(G\right)$$

i.e. for $f\in\bigcap_{i=1}^{2^n} L_{p^i,\varphi,\beta}^{<l^i>}(G)$ there exists a generalized derivative $D^\nu f$ and the following inequalities are true

$$\left\|D^\nu f\right\|_{p,G}\leq C_1\sum_{i=1}^{2^n}\prod_{j\in e_n}\left(\varphi_j(T_j)\right)^{\varepsilon_j}\left\|D^{l_e}f\right\|_{p^i,\varphi,\beta;G}\qquad(14)$$

$$\|D^{\nu}f\|_{p,\psi,\beta^1;G}\le C_2\,\|f\|_{\bigcap\limits_{i=1}^{2n}L^{<l^i>}_{p^i,\varphi,\beta}(G)}\,,\ (p^i\le p<\infty),\tag{15}$$

In particular, if

$$\varepsilon^i_{j,0}=l^i-\nu_j-(1-\beta_j p^i)\frac{1}{p}>0, j\in e_n, i=1,2,\ldots,2^n$$

then $D^{\nu}f\,(x)$ is continuous on G, i.e.

$$\sup_{x\in G}|D^{\nu}f\,(x)|\le C_1\sum_{i=1}^{2^n}\prod_{j\in e_n}(\varphi_j(T_j))^{\varepsilon^i_{j,0}}\left\|D^{l^i}f\right\|_{p^i,\varphi,\beta;G}\tag{16}$$

where $0<T_j\le\min\{1,T_{0j}\}$, T_0 is a fixed number; C_1, C_2 are the constants independent of f, C_1 are independent also on T.

Proof. At first note that in the conditions of our theorem there exists a generalized derivative $D^{\nu}f$ on G. Indeed, from the condition $\varepsilon^i_j>0,(j\in e_n)$ it follows that for $f\in\bigcap\limits_{i=1}^{2^n}L^{<l^i>}_{p^i,\varphi,\beta}(G)\to\bigcap\limits_{i=1}^{2^n}L^{<l^i>}_{p^i}(G)$, there exists $D^{\nu}f\in L_{p^i}(G)$ and for it integral representation (**??**) with the same kernels is valid.

Based around the Minkowsky inequality, from identities (**??**) we get

$$\|D^{\nu}f\|_{p,G}\le\sum_{i=1}^{2^n}\left\|R^i_T\right\|_{p,G}.\tag{17}$$

By means of inequality (**??**) for $U=G,D^{l^i}f=\Phi\ \eta_j=T_j,(j\in e_n)\ \xi\to\infty$ we get inequality (**??**). By means of inequality (**??**) for $\eta_j=T_j,(j\in e_n)$ we get inequality (**??**).

Now let conditions $\varepsilon^i_j>0,(j\in e_n,i=1,2,\ldots,2^n)$ be satisfied, then based around identities (**??**) from inequality (**??**) we get

$$\left\|D^{\nu}f-D^{\nu}f_{\varphi(T)}\right\|_{\infty,G}\le C\sum_{\varnothing\ne e\subseteq e_n}\prod_{j\in e}(\varphi_j(T_j))^{\varepsilon^i_{j,0}}\left\|D^{l^i}f\right\|_{p^i,\varphi,\beta;G}.$$

As $T_j\to 0$, the left side of this inequality tends to zero, since $f_{\varphi(T)}\,(x)$ is continuous on G and the convergence on $L_{\infty}\,(G)$ coincides with the uniform convergence. Then the limit function $D^{\nu}f$ is continuous on G. Theorem 1 is proved.

Let γ be an n-dimensional vector.

Theorem 2.2. Let all the conditions of theorem 1 be fulfilled. Then for $\varepsilon^i_j>0$ $(j\in e_n,i=1,2,\ldots,2^n)$ the generalized derivatives $D^{\nu}f$ satisfies on G the generalized Holder condition, i.e. the following inequality is valid:

$$\|\Delta\,(\gamma,G)\,D^{\nu}f\|_{p,G}\le C\,\|f\|_{\bigcap\limits_{i=1}^{2n}L^{<l^i>}_{p^i,\varphi,\beta}(G)}\prod_{j\in e_n}(\varphi_j(|\gamma_j|))^{\sigma_j},\tag{18}$$

where σ_j $(j \in e_n)$ any number, satisfying the inequality:

$$\begin{cases} 0 \le \sigma_j \le 1, & \text{if } \varepsilon^i_j > 1 \\ 0 \le \sigma_j < 1, & \text{if } \varepsilon^i = 1 \\ 0 \le \sigma_j \le \mu_j, & \text{if } \varepsilon^i < 1 \end{cases} \tag{19}$$

If $\varepsilon^i_{j,0} > 0$ $(j \in e_n, i = 1, 2, \ldots, 2^n)$, then

$$\sup_{x \in G} |\Delta(\gamma, G) D^\nu f(x)| \le C \|f\|_{\bigcap_{i=1}^{2^n} L^{<l^i>}_{p^i,\varphi,\beta}(G)} \prod_{j \in e_n} (\varphi_j(|\gamma_j|))^{\sigma^0_j}. \tag{20}$$

where σ^0_j satisfy the same conditions as σ_j, but replaced ε^i_j in $\varepsilon^i_{j,0}$.

Proof. According to lemma 8.6 from [?] there exists a domain

$$G_u \subset G \left(u_j = \zeta_j k(x), \, \zeta_j > 0, j \in e_n, \; k(x) = \rho(x, \partial G), \; x \in G \right)$$

and assume that $|\gamma| < u$, then for any $x \in G_u$ the segment connecting the points $x, x+\gamma$ is contained in G. Consequently, for all the points of this segment, identies (**??**) with the same kernels are valid. After same transformations, from (**??**) we get

$$|\Delta(\gamma, G) D^\nu f(x)| \le C_1 \sum_{i=1}^{2^n} \prod_{j \in e_n \setminus e^i} (\varphi_j(T_j))^{l^i_j - 1 - \nu_j}$$

$$\times \int_{0^i}^{|\gamma^i_1|} \cdots \int_{0^i}^{|\gamma^i_n|} \prod_{j \in e^i} (\varphi_j(t_j))^{l^i_j - \nu_j - 2} \prod_{j \in e^i} \varphi'_j(t_j) dt_j$$

$$\times \int_{R^n} \left| M^{(\nu)}_i \left(\frac{y}{\varphi(t+T)^i}, \frac{\rho\left(\varphi(t+T)^i, x\right)}{\varphi(t+T)^i}, \rho'\left(\varphi(t+T)^i, x\right) \right) \right|$$

$$\times \left| \Delta(\gamma, G) D^{l^i} f(x+y) \right| dy + C_2 \sum_{i=1}^{2^n} \prod_{j \in e_n \setminus e^i} (\varphi_j(T_j))^{l^i_j - 2 - \nu_j}$$

$$\times \prod_{j \in e_n} \varphi_j(|\gamma_j|) \int_{|\gamma^i_1|}^{T^i_1} \cdots \int_{|\gamma^i_n|}^{T^i_n} \prod_{j \in e^i} (\varphi_j(t_j))^{l^i_j - \nu_j - 3} \prod_{j \in e^i} \varphi'_j(t_j) dt^i_j$$

$$\times \int_{R^n} \left| M^{(\nu+1)}_i \left(\frac{y}{\varphi(t+T)^i}, \frac{\rho\left(\varphi(t+T)^i, x\right)}{\varphi(t+T)^i}, \rho'\left(\varphi(t+T)^i, x\right) \right) \right|$$

$$\times \int_0^1 \cdots \int_0^1 \left| D^{l^i} f(x + y + \gamma_1 u_1 + \ldots + \gamma_n u_n) \right| dy du$$

$$= C_1 \sum_{i=1}^{2^n} A^i_\gamma(x,t) + C_2 \sum_{i=1}^{2^n} A^i_{\gamma,T}(x,t), \tag{21}$$

where $0 < T_j \leq \{1, T_{0,j}\}$, $j \in e_n$. We also assume that $|\gamma_j| < T_j (j \in e_n)$. Consequently, $|\gamma_j| < \min(u_j, T_j)(j \in e_n)$. If $x \in G \setminus G_u$ then by definition

$$\Delta(\gamma, G) D^\nu f(x) = 0.$$

Based around (??) we have

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \leq C_1 \sum_{i=1}^{2^n} \left\| A_\gamma^i(\cdot, \gamma) \right\|_{p,G_u} +$$

$$+ C_2 \sum_{i=1}^{2^n} \left\| A_{\gamma,T}^i(\cdot, \gamma) \right\|_{p,G_u}. \tag{22}$$

By means of inequality (??), for $D^{l_i} f = \Phi$, $\eta = |\gamma_j|$ $(j \in e_n)$ we get

$$\left\| A_{|\gamma|}^i(\cdot, \gamma) \right\|_{p,G_u} \leq C_1 \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G} \prod_{j \in e_n} (\varphi_j(|\gamma_j|))^{\varepsilon_j^i}. \tag{23}$$

and by means of inequality (??) for $D^{l^i} f = \Phi$, $\eta_j = |\gamma_j|, j \in e_n$ we get

$$\left\| A_{|\gamma|,T}(\cdot, \gamma) \right\|_{p,G_u} \leq C_2 \left\| D^{l^i} f \right\|_{p^i,\varphi,\beta;G} \prod_{j \in e_n} \varphi_j(|\gamma_j|) \prod_{j \in e_n} (\varphi_j(|\gamma_j|))^{\delta_j}. \tag{24}$$

From inequalities (??) - (??) we get the required inequality.

Now suppose that $|\gamma_j| \geq \min(u_j, T_j), (j \in e_n)$. Then

$$\|\Delta(\gamma, G) D^\nu f\|_{p,G} \leq 2 \|D^\nu f\|_{p,G} \leq C(u, T) \|D^\nu f\|_{p,G} \prod_{j \in e_n} (\varphi_j(|\gamma_j|))^{\delta_j}.$$

Estimating for $\|D^\nu f\|_{p,G}$ by means of inequality (??)), in this case we get estimation (??).

Theorem Theorem 2 is proved.

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