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The Cauchy problem for an infinite systems of parabolic convolution differential equations

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Abstract. Used sufficient conditions that guarantee the separability of linear problems in weighted L_p spaces we have maximal regularity properties of the Cauchy problem for parabolic convolution differential equation. In application, the Cauchy problem for infinite systems of parabolic convolution differential equations in mixed weighted L_p norms are obtained.

Key Words and Phrases: **:** weighted spaces, operator-valued multipliers, convolution differential equations, **p**−summable functions, mixed norm.

1. Introduction

Maximal L_p−regularity is important for linear and non-linear problems and has been studied extensively in the last years. In recent years, regularity properties of differential operator equations, especially elliptic and parabolic type have been studied e.g., in [1], [2], [4], [6], [7] , [10] , [11] , [15-17] , [20] , [21] and the references therein. Moreover, convolution-differential equations (CDEs) have been treated e.g., in [4], [8] , [14] (for comprehensive references see [14]). Convolution operators in Banach-valued function spaces studied e.g., in [3] , [8] , [12], [15] , [16] , [17]. However, the convolution-differential operator equations (CDOEs) are relatively less investigated subject. In [13] and [17] the maximal L_p−regularity properties of the linear CDOEs

$$\sum_{|\alpha|\leq l} a_\alpha * D^\alpha u + (A+\lambda) * u = f(x),\ x\in R^n, \tag{1.1}$$

are studied, where l is a natural number, $a_\alpha = a_\alpha(x)$ are complex-valued functions, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$, α_k are nonnegative integers, λ is a complex parameter and $A = A(x)$ is a linear operator in a Banach space E for $x \in R^n$.

In a series of recent publications operator-valued multiplier theorems have had many applications in the theory of convolution differential equations in particular case, maximal regularity of parabolic convolution differential-operator equations.

The Cauchy problem for parabolic CDOE, i.e.,

$$\frac{\partial u}{\partial t}+\sum_{|\alpha|\leq l}a_{\alpha}*D^{\alpha}u+A*u=f\left(t,x\right), \tag{1.2}$$

$$u\left(0,x\right)=0,\ t\in\mathbb{R}_{+},\ x\in R^{n},$$

in $E-$valued mixed $L_{\mathbf{p},\gamma}$ spaces has been investigated in [13], [15], where the author used the theory of operator-valued Fourier multipliers and the representation formula for the solution of (1.1). Here, the convolutions $a_{\alpha}*D^{\alpha}u,\ \ A*u$ are defined in the distribution sense (see e.g. [2]), $\gamma=\gamma\left(x\right)$ is a positive measurable function on $\Omega\subset R^{n}$.

$L_{\mathbf{p},\gamma}\left(R^{n+1};E\right)$ denotes the space of all $\mathbf{p}-$summable complex-valued functions with mixed norm, i.e., the space of all measurable functions f defined on R^{n+1}, for which

$$\|f\|_{L_{\mathbf{p},\gamma}(R^{n+1};E)}=\left(\int\limits_{R^{n}}\left(\int\limits_{\mathbb{R}_{+}}\|f(x)\|_{E}^{p}\,\gamma(x)dx\right)^{\frac{p_{1}}{p}}dt\right)^{\frac{1}{p_{1}}}<\infty,\ R^{n+1}=R^{n}\times\mathbb{R}_{+},\ \mathbf{p}=\left(p,p_{1}\right).$$

In this paper the author establish operator-valued multipliers in $L_{p,\gamma,}\left(R^{n};E\right)$ and apply them to study maximal regularity of the following Cauchy problem for an infinite systems of parabolic convolution differential equations.

$$\frac{\partial u_{m}}{\partial t}+\sum_{|\alpha|\leq l}a_{\alpha}*D^{\alpha}u_{m}+\sum_{m=1}^{\infty}d_{m}*u_{m}=f_{m}, \tag{1.3}$$

where l is a natural number, $a_{\alpha}=a_{\alpha}\left(x\right)$ are complex-valued functions, $d_{m}=d_{m}\left(x\right)$, $u_{m}=u_{m}\left(t,x\right),\ f_{m}=f_{m}\left(t,x\right),\ \alpha=\left(\alpha_{1},\alpha_{2},...,\alpha_{n}\right),\ \alpha_{k}$ are nonnegative integers, for $x\in R^{n}$, $t\in\mathbb{R}_{+}$.

In this paper, we establish the uniform separability properties of the problem (1.1) and the uniform maximal regularity of the Cauchy problem for parabolic CDOE (1.2). Moreover, we prove that the operator generated by problem (1.1) is $\varphi-$sectorial. This fact is derived by using the representation formula for the solution of (1.1) and operator valued multipliers in $L_{p,\gamma}\left(R^{n};E\right)$. Note that, one of main features of the present work is that the convolution equation (1.1) has a variable operator coefficient.

The present paper is organized as follows. The first section of the paper contains an introduction. Section 2 collects definitions, some notations and basic properties of vector-valued function spaces, in particular, weighted mixed $L_{\mathbf{p},\gamma}$ spaces. Section 3 contains basic coercive estimates on $E-$valued weighted $L_{p,\gamma}$ spaces. In section 4, using the results obtained in section 2, we establish the maximal regularity of (1.2) and section 4 prepares for the proof of the main result of this paper, in particular, we obtain that this corresponding operator H is a negative generator of an analytic semigroup in $L_{p,\gamma}\left(R^{n};E\right)$. Finally, in Section 5 by applying this results the maximal regularity of problem (1.3) is proved. In each case, the appropriate coercive uniform estimate are established.

2. Definitions and notation

Let us first some notions. Denote the set of natural numbers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$ and the set of complex numbers by $\mathbb{C}$.

For $1 < 1 < \infty$ we set

$$l_q = \left\{ \xi; \xi = \{\xi_i\}_{i=1}^{\infty}; \ \|\xi\|_{l_q} = \left(\sum_{i=1}^{\infty} |\xi_i|^q \right)^{1/q} < \infty, \quad \xi_i - \text{complex numbers} \right\}.$$

Let E be a Banach space and $\gamma = \gamma(x)$, $x = (x_1, x_2, ..., x_n)$ be a positive measurable weighted function on a measurable subset $\Omega \subset R^n$. Let $L_{p,\gamma}(\Omega; E)$ denote the space of strongly $E-$valued functions that are defined on Ω with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int_{\Omega} \|f(x)\|_E^p \gamma(x) dx \right)^{1/p}, \ 1 \leq p < \infty,$$

$$\|f\|_{L_{\infty,\gamma}(\Omega;E)} = ess \sup_{x \in \Omega} \left[\gamma(x) \|f(x)\|_E \right].$$

For $\gamma(x) \equiv 1$, the space $L_{p,\gamma}(\Omega, E)$ will be denoted by $L_p = L_p(\Omega; E)$.

Moreover, if $p \in (1, \infty)$ and $\gamma(x)$ is a positive measurable function, then

$$L_{p,\gamma}(R^n; l_q) = \left\{ f; f = \{f_i(x)\}_{i=1}^{\infty}, \ \|f\|_{L_{p,\gamma}(R^n;l_q)} = \left(\int_{R^n} \|\{f_i(x)\}\|_{l_q}^p \gamma(x) dx \right)^{1\backslash p} < \infty \right\}.$$

Clearly, $L_{p,\gamma}(R^n; l_q)$ is a Banach space.

The weight $\gamma = \gamma(x)$ satisfy an A_p condition, i.e., $\gamma \in A_p$, $p \in (1, \infty)$ if there is a positive constant C such that

$$\sup_Q \left(\frac{1}{|Q|} \int_Q \gamma(x) dx \right) \left(\frac{1}{|Q|} \int_Q \gamma^{-\frac{1}{p-1}}(x) dx \right)^{p-1} \leq C$$

for all cubes $Q \subset R^n$ (see e.g [9, Ch.9]).

The result [19] implies that the space l_q for $q \in (1, \infty)$ satisfies multiplier condition with respect to $p \in (1, \infty)$ and the weight functions $\gamma(x) = \prod_{k=1}^{n} |x_k|^{\nu}$ for $-\frac{1}{n} < \nu < \frac{1}{n}(p-1)$.

Suppose that $S_\varphi = \{\lambda \in \mathbb{C}, \quad |\arg \lambda| \leq \varphi\} \cup \{0\}, \ 0 \leq \varphi < \pi$.

Let E_1, E_2 be two Banach spaces and $B(E_1, E_2)$ denote the space of bounded linear operators from E_1 to E_2. For $E_1 = E_2 = E$ we denote $B(E, E)$ by $B(E)$.

Let $D(A)$, $R(A)$ and Ker A denote the domain, range and null space of the linear operator in E, respectively.

A closed linear operator A is said to be $\varphi-$ sectorial (or sectorial for $\varphi = 0$) in a Banach space E with bound $M > 0$ if Ker $A = \{0\}$, $D(A)$ and $R(A)$ are dense on E, and $\left\|(A + \lambda I)^{-1}\right\|_{B(E)} \leq M |\lambda|^{-1}$ for all $\lambda \in S_{\varphi}$, $\varphi \in [0, \ \pi)$, where I is an identity operator in E. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and will be denoted by A_{λ}. It is known (see e.g. [18, §1.15.1]) that the fractional powers of the operator A are well defined.

A closed linear operator $A = A(x)$ is said to be uniformly $\varphi-$sectorial in a Banach space E, with bound $M > 0$ if Ker $A = \{0\}$, $D(A)$ and $R(A)$ dense on E, and does not depend on x,

$$\left\|(A(x) + \lambda I)^{-1}\right\|_{B(E)} \leq M |\lambda|^{-1} \text{ for all } \lambda \in S_{\varphi}, \ \varphi \in [0, \pi).$$

Let $E(A^{\theta})$ denote the space $D(A^{\theta})$ with the graph norm

$$\|u\|_{E(A^{\theta})} = \left(\|u\|_E^p + \left\|A^{\theta} u\right\|_E^p\right)^{\frac{1}{p}}, \quad 1 \leq p < \infty, \quad -\infty < \theta < \infty.$$

Note that the above norms are equivalent for $p \in [1, \infty)$.

Here, $S = S(R^n; E)$ denotes the $E-$valued Schwartz class, i.e. the space of $E-$valued rapidly decreasing smooth functions on R^n, equipped with its usual topology generated by seminorms. $S(R^n; \mathbb{C})$ will be denoted by just S.

Let $S'(R^n; E)$ denote the space of all continuous linear operators, $L : S \to E$, equipped with topology of bounded convergence. Recall $S(R^n; E)$ is norm dense in $L_{p,\gamma}(R^n; E)$ when $1 < p < \infty$, $\gamma \in A_p$.

Let Ω be a domain in R^n. $C(\Omega, E)$ and $C^m(\Omega; E)$ will denote the spaces of $E-$valued uniformly bounded strongly continuous and $m-$times continuously differentiable functions on Ω, respectively.

An $E-$valued generalized function $D^{\alpha} f$ is called a generalized derivative in the sense of Schwartz distributions of the function $f \in S(R^n; E)$ if

$$\langle D^{\alpha} f, \varphi \rangle = (-1)^{|\alpha|} \langle f, D^{\alpha} \varphi \rangle$$

holds for all $\varphi \in S$.

Let F denote the Fourier transform. Throughout this section the Fourier transformation of a function f will be denoted by $\widehat{f}$ and $F^{-1} f = \check{f}$. It is known that

$$F(D_x^{\alpha} f) = (i\xi_1)^{\alpha_1} ... (i\xi_n)^{\alpha_n} \widehat{f}, \quad D_{\xi}^{\alpha}(F(f)) = F\left[(-ix_1)^{\alpha_1} ... (-ix_n)^{\alpha_n} f\right]$$

for all $f \in S'(R^n; E)$.

A function $\Psi \in L_{\infty}(R^n; B(E_1, E_2))$ is called a Fourier multiplier from $L_{p,\gamma}(R^n; E_1)$ to $L_{p,\gamma}(R^n; E_2)$ for $p \in (1, \infty)$ if the map $u \to Tu = F^{-1} \Psi(\xi) F u$, $u \in S(R^n; E_1)$ is well defined and extends to a bounded linear operator

$$T : L_{p,\gamma}(R^n; E_1) \rightarrow L_{p,\gamma}(R^n; E_2).$$

The space of all Fourier multipliers from $L_{p,\gamma}(R^n; E_1)$ to $L_{p,\gamma}(R^n; E_2)$ will be denoted by $M_{p,\gamma}^{p,\gamma}(E_1, E_2)$. For $E_1 = E_2 = E$ we denote $M_{p,\gamma}^{p,\gamma}(E_1, E_2)$ by $M_{p,\gamma}^{p,\gamma}(E)$.

A Banach space E is called a UMD space (see e.g [7], [8]) if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \rightarrow 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is initially defined on $S(R; E)$ and is bounded in $L_p(R; E)$, $p \in (1, \infty)$. UMD spaces include e.g. L_p, l_p spaces and Lorentz spaces L_{pq}, $p, q \in (1, \infty)$.

A set $K \subset B(E_1, E_2)$ is called $R-$bounded (see e.g [6], [20]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_m \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$\int\limits_0^1 \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int\limits_0^1 \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy,$$

where $\{r_j\}$ is a sequence of independent symmetric $\{-1; 1\}$ $-$valued random variables on $[0, 1]$. The smallest C for which the above estimate holds is called the $R-$bound of K and denoted by $R(K)$.

A Banach space E is said to be a space satisfying the multiplier condition with respect to weighted function γ and $p \in (1, \infty)$ (or multiplier condition with respect to $p \in (1, \infty)$ when $\gamma(x) \equiv 1$) if for any $\Psi \in C^{(n)}(R^n \backslash \{0\}; B(E))$ the $R-$boundedness of the set

$$\left\{ |\xi|^{|\beta|} D_\xi^\beta \Psi(\xi) : \xi \in R^n \backslash \{0\}, \beta = (\beta_1, \beta_2, ..., \beta_n), \ \beta_k \in \{0, 1\} \right\}$$

implies that Ψ is a Fourier multiplier in $L_{p,\gamma}(R^n; E)$, i.e., $\Psi \in M_{p,\gamma}^{p,\gamma}(E)$ (or Ψ is a Fourier multiplier in $L_p(R^n; E)$, i.e., $\Psi \in M_p^p(E)$).

Note that, if E is UMD space then it satisfies the multiplier condition with respect to $p \in (1, \infty)$ (see e.g. [6] , [8] , [20]).

A sectorial operator $A(x)$, $x \in R^n$ is said to be uniformly $R-$sectorial in a Banach space E if there exists a $\varphi \in [0, \pi)$ such that

$$\sup_{x \in R^n} R\left(\left\{\left[A(x)(A(x) + \xi I)^{-1}\right] : \xi \in S_\varphi\right\}\right) \leq M.$$

Note that, in Hilbert spaces every norm bounded set is $R-$bounded. Therefore, in Hilbert spaces all sectorial operators are $R-$sectorial.

The Fourier transformation of $A(x)$ is a linear operator with the domain $D(A)$ defined as

$$\hat{A}(\xi) u(\varphi) = A(x) u(\hat{\varphi}) \text{ for } u \in S'(R^n; D(A)), \ \varphi \in S(R^n).$$

(For details see e.g [2, Section 3]).

Let $h \in \mathbb{R}$, $m \in \mathbb{N}$ and e_k, $k = 1, 2, ..., n$ be the standard unit vectors of R^n,

$$\Delta_k (h) f (x) = f (x + he_k) - f (x).$$

Let $A = A(x)$, $x \in R^n$ be closed linear operator in E with domain $D(A)$ independent of x.

$A(x)$ is differentiable if there is the limit

$$\left(\frac{\partial A}{\partial x_k}\right) u = \lim_{h \to 0} \frac{\Delta_k (h) A (x) u}{h}, \; k = 1, 2, ...n, \; u \in D(A)$$

in the sense of $E-$norm.

Let E_0 and E be two Banach spaces, where E_0 is continuously and densely embedded into E. Let l be a natural number. $W^l_{p,\gamma}(R^n; E_0, E)$ denotes the space of all functions from $S'(R^n; E_0)$ such that $u \in L_{p,\gamma}(R^n; E_0)$ and the generalized derivatives $D^l_k u = \frac{\partial^l u}{\partial x^l_k} \in L_{p,\gamma}(R^n; E)$ with the norm

$$\|u\|_{W^l_{p,\gamma}(R^n;E_0,E)} = \|u\|_{L_{p,\gamma}(R^n;E_0)} + \sum_{k=1}^{n} \left\|D^l_k u\right\|_{L_{p,\gamma}(R^n;E)} < \infty.$$

It is clear that

$$W^l_{p,\gamma}(R^n; E_0, E) = W^l_{p,\gamma}(R^n; E) \cap L_{p,\gamma}(R^n; E_0).$$

Note that if $l \geq 2$, E is a space satisfying the multiplier condition with respect to weighted function γ and $p \in (1, \infty)$, then the above definitions are equivalent with usual definitions, i.e.

$$\|u\|_{W^l_{p,\gamma}(R^n;E_0,E)} \simeq \|u\|_{L_{p,\gamma}(R^n;E_0)} + \sum_{|\alpha| \leq l} \|D^\alpha u\|_{L_{p,\gamma}(R^n;E)},$$

In a similar way as [6, Theorem 3.25] we obtain:

Proposition A. Let E be a UMD space and $\gamma \in A_p$. Assume Ψ_h is a set of operator functions in $C^{(n)}(R^n \setminus \{0\}; B(E))$ depending on the parameter $h \in Q \in \mathbb{R}$ and there exists a positive constant K such that

$$\sup_{h \in Q} R\left(\left\{|\xi|^{|\beta|} D^\beta \Psi_h(\xi) : \xi \in R^n \setminus \{0\}, \; \beta_k \in \{0, 1\}\right\}\right) \leq K.$$

Then the set Ψ_h is a uniformly bounded collection of Fourier multipliers in $L_{p,\gamma}(R^n; E)$.

3. Convolution-elliptic equation

Consider the following CDOE

$$\sum_{|\alpha| \leq l} a_\alpha * D^\alpha u + (A + \lambda) * u = f, \tag{3.1}$$

where λ are parameter, a_α are complex-valued functions defined in (1.1) and $A = A(x)$ is a linear operator in a Banach space E.

We find sufficient conditions that guarantee the separability of the problem (3.1).

In this section we rely on the recent papers by [15] and [17] where the above problem is studied extensively.

Condition 3.1. Suppose the following are satisfied:

(1) $L(\xi) = \sum\limits_{|\alpha|\leq l} \hat{a}_\alpha(\xi)(i\xi)^\alpha \in S_{\varphi_1}$, $\varphi_1 \in [0,\ \pi)$ for $\xi \in R^n$,

$|L(\xi)| \geq C \sum\limits_{k=1}^{n} \left|\hat{a}_{\alpha(l,k)}\right| |\xi_k|^l$, $\alpha(l,k) = (0,0,...,l,0,0,...,0)$, i.e. $\alpha_i = 0$, $i \neq k$, $\alpha_k = l$;

(2) $\hat{a}_\alpha \in C^{(n)}(R^n)$ and

$$|\xi|^{|\beta|} \left|D^\beta \hat{a}_\alpha(\xi)\right| \leq C_1,\ \beta_k \in \{0,1\},\ 0 \leq |\beta| \leq n;$$

(3) for $0 \leq |\beta| \leq n$, ξ, $\xi_0 \in R^n \setminus \{0\}$,

$$\left[D^\beta \hat{A}(\xi)\right] \hat{A}^{-1}(\xi_0) \in C(R^n; B(E)),\ \ |\xi|^{|\beta|} \left\| \left[D^\beta \hat{A}(\xi)\right] \hat{A}^{-1}(\xi_0) \right\|_{B(E)} \leq C_2.$$

In a similar way as in [13] we have:

Theorem 3.1. Assume that Condition 3.1 holds and E is a Banach space satisfying the multiplier condition with respect to weighted function $\gamma \in A_p$ and $p \in (1,\infty)$. Let $\hat{A}$ be a uniformly $R-$sectorial operator in E with $\varphi \in [0,\ \pi)$, $\lambda \in S_{\varphi_2}$ and $0 \leq \varphi + \varphi_1 + \varphi_2 < \pi$. Then, problem (3.1) has a unique solution u and the coercive uniform estimate holds

$$\sum_{|\alpha|\leq l} |\lambda|^{1-\frac{|\alpha|}{l}} \|a_\alpha * D^\alpha u\|_{L_{p,\gamma}(R^n;E)} + \|A * u\|_{L_{p,\gamma}(R^n;E)} + |\lambda| \|u\|_{L_{p,\gamma}(R^n;E)} \leq C \|f\|_{L_{p,\gamma}(R^n;E)} \tag{3.2}$$

for all $f \in L_{p,\gamma}(R^n; E)$ and $\lambda \in S_\varphi$.

Proof. By applying the Fourier transform to equation (3.1) we get

$$\hat{u}(\xi) = \left[\hat{A}(\xi) + L(\xi) + \lambda\right]^{-1} \hat{f}(\xi),$$

where $\hat{a}_\alpha(\xi)$, $\hat{A}(\xi)$, $\hat{u}(\xi)$ and $\hat{f}(\xi)$ denote the Fourier transforms of $a_\alpha(x)$, $A(x)$, $u(x)$ and $f(x)$ respectively.

Since $L(\xi) \in S_{\varphi_1}$ for all $\xi \in R^n$ and $\hat{A}(\xi)$ is a sectorial operator in E, the operator $\hat{A}(\xi) + L(\xi) + \lambda$ is invertible in E, i.e., $\left(\hat{A}(\xi) + L(\xi) + \lambda\right)^{-1} \in B(E)$. Hence, the solution of (3.1) can be represented as

$$u(x) = F^{-1}\left[\hat{A}(\xi) + L(\xi) + \lambda\right]^{-1} \hat{f}. \tag{3.3}$$

In a similar manner as in [17] one can easily show that operator-valued functions, arising in the solution of equation (3.1), is uniformly $R-$bounded multiplier in $L_{p,\gamma}(R^n; E)$.

Then, by definition of $R-$boundednesss we obtain that corresponding operator-valued functions are Fourier multipliers from $L_{p,\gamma}(R^n;E)$ to $L_{p,\gamma}(R^n;E)$. Thus, for each function $f\in L_{p,\gamma}(R^n;E)$ equation (3.1) has a unique solution of the form (3.3). It implies that

$$|\lambda|\,\|u\|_{L_{p,\gamma}(R^n;E)}\leq C_0\,\|f\|_{L_{p,\gamma}(R^n;E)}\,,\ \ \|A*u\|_{L_{p,\gamma}(R^n;E)}\leq C_1\,\|f\|_{L_{p,\gamma}(R^n;E)}\,,$$

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\,\|a_\alpha*D^\alpha u\|_{L_{p,\gamma}(R^n;E)}\leq C_2\,\|f\|_{L_{p,\gamma}(R^n;E)}$$

for all $f\in L_{p,\gamma}(R^n;E)$. Hence, we get the assertion.

By using a similar technique as in [13] and [17] we obtain the, for $f\in L_{p,\gamma}(R^n;E)$ and $\lambda\in S_\varphi$ equation (3.1) has a unique solution $u\in W^l_{p,\gamma}(R^n;E(A),E)$ and the following coercive uniform estimate holds

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\,\|D^\alpha u\|_{L_{p,\gamma}(R^n;E)}+\|Au\|_{L_{p,\gamma}(R^n;E)}\leq C\,\|f\|_{L_{p,\gamma}(R^n;E)}\,.$$

Let H be an operator in $L_{p,\gamma}(R^n;E)$ generated by problem (3.1) for $\lambda=0$, i.e.

$$D(H)=Y,\ Hu=\sum_{|\alpha|\leq l}a_\alpha*D^\alpha u+A*u.$$

From Theorem 3.1 we have:

Result 3.1. Assume that the all conditions of Theorem 3.1 hold. Then, for all $\lambda\in S_{\varphi_2}$ the following uniform coercive estimate holds

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha*D^\alpha(H+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;E))}+$$

$$\left\|A*(H+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;E))}+\left\|\lambda(H+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;E))}\leq C.$$

Result 3.2. Theorem 3.1, particularly implies that the operator H is uniformly sectorial in $L_{p,\gamma}(R^n;E)$; moreover, if $\hat{A}$ is uniformly $R-$sectorial for $\varphi\in\left(\frac{\pi}{2},\pi\right)$, then the operator H is a negative generator of an analytic semigroup in $L_{p,\gamma}(R^n;E)$ (see e.g. [18, §1.14.5]).

From Theorems 3.1 and Proposition A we obtain:

Result 3.3. Let conditions of Theorems 3.1 hold for $E\in UMD$. Then the assertions of Theorems 3.1 are valid.

4. The Cauchy problem for parabolic CDOE

In this section, we shall consider the Cauchy problem for the convolution parabolic equation

$$\frac{\partial u}{\partial t}+\sum_{|\alpha|\leq l}a_{\alpha}*D^{\alpha}u+A*u+du=f\left(t,x\right), \tag{4.1}$$

$$u\left(0,x\right)=0,\ t\in\mathbb{R}_{+},\ x\in R^{n},$$

where parameter $d>0$, a_{α} are complex-valued functions defined in (1.1) and A is a linear operator in a Banach space E. For $R_{+}^{n+1}=R^{n}\times\mathbb{R}_{+}$, $\mathbf{p}=\left(p,\,p_{1}\right)$, $Z=L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};E\right)$ will denote the space of all $\mathbf{p}$-summable $E-$valued functions with mixed norm (see e.g. [5, § 4] for complex-valued case), i.e., the space of all measurable $E-$valued functions f defined on R_{+}^{n+1} for which

$$\|f\|_{Z}=\left(\int\limits_{R^{n}}\left(\int\limits_{\mathbb{R}_{+}}\|f\left(t,x\right)\|_{E}^{p}\gamma\left(x\right)dx\right)^{\frac{p_{1}}{p}}dt\right)^{\frac{1}{p_{1}}}<\infty.$$

Let E_{0} and E be two Banach spaces, where E_{0} is continuously and densely embedded into E. Suppose l is an integer and $Z_{0}=W_{\mathbf{p},\gamma}^{1,l}\left(R_{+}^{n+1};E_{0},E\right)$ denotes the space of all functions $u\in Z$ such that the generalized derivatives $D_{t}u$, $D_{k}^{l}u\in Z$, with the norm

$$\|u\|_{Z_{0}}=\|u\|_{Z\left(E_{0}\right)}+\|D_{t}u\|_{Z}+\sum_{k=1}^{n}\left\|D_{k}^{l}u\right\|_{Z},$$

where

$$Z\left(E_{0}\right)=L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};E_{0}\right).$$

Applying Theorem 3.1 we establish the maximal regularity of (4.1) in Z. For this purpose, by using a similar technique as in [13] and [17] we obtain the operator H is uniformly $R-$sectorial in $L_{p,\gamma}\left(R^{n};E\right)$.

By Fubini's theorem we have

$$\begin{aligned}\|f\|_{L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};E\right)} &= \left(\int\limits_{R^{n}}\left(\int\limits_{\mathbb{R}_{+}}\|f\left(t,x\right)\|_{E}^{p}\gamma(x)dx\right)^{\frac{p_{1}}{p}}dt\right)^{1/p_{1}} = \\ &= \left(\int\limits_{\mathbb{R}_{+}}\left(\int\limits_{R^{n}}\|f\left(t,x\right)\|_{E}^{p}\gamma(x)dx\right)^{\frac{p_{1}}{p}}dt\right)^{1/p_{1}} =\end{aligned}$$

$$= \left(\int_{\mathbb{R}_+} \|f(t,x)\|^{p_1}_{L_{p,\gamma}(R^n;E)} \, dt \right)^{1/p_1} = \|f(t,x)\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(R^n;E))}$$

Now, we are ready to state the main result of this section.

Theorem 4.1. Assume that all the conditions of Theorem 3.1 are satisfied for $\varphi \in \left(\frac{\pi}{2}, \pi\right)$. Then the equation (4.1) has a unique solution $u \in W^{1,l}_{\mathbf{p},\gamma}\left(R^{n+1}_+; E(A), E\right)$. Moreover, for sufficiently large d, the following coercive uniform estimate holds

$$\left\| \frac{\partial u}{\partial t} \right\|_Z + \sum_{|\alpha| \leq l} \|a_\alpha * D^\alpha u\|_Z + \|A * u\|_Z \leq C \|f\|_Z . \tag{4.2}$$

Proof. It is known that $Z = L_{p_1}(\mathbb{R}_+; L_{p,\gamma}(R^n; E))$. Moreover, by definition of spaces $W^l_{p,\gamma}(R^n; E(A), E)$ and Z_0 for $E_0 = E(A)$ we obtain

$$\|u\|_{Z_0} = \|u\|_{Z(A)} + \left\| \frac{du}{dt} \right\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(R^n;E))} + \|Hu\|_{L_{p_1}(\mathbb{R}_+:L_{p,\gamma}(R^n;E))} \simeq$$

$$\left\| \frac{\partial u}{\partial t} \right\|_{L_{\mathbf{p},\gamma}\left(R^{n+1}_+:E\right)} + \|Hu\|_{L_{\mathbf{p},\gamma}\left(R^{n+1}_+;E\right)} \simeq$$

$$\|Au\|_{L_{\mathbf{p},\gamma}\left(R^{n+1}_+;E\right)} + \left\| \frac{\partial u}{\partial t} \right\|_{L_{\mathbf{p},\gamma}\left(R^{n+1}_+;E\right)} + \sum_{k=1}^{n} \left\| D^l_k u \right\|_{L_{\mathbf{p},\gamma}\left(R^{n+1}_+;E\right)} \simeq \|u\|_{Z_0} ,$$

where

$$Z(A) = L_{p_1}(\mathbb{R}_+; L_{p,\gamma}(R^n; E(A))) .$$

Hence, we get

$$Z_0 = W^1_{p_1}(\mathbb{R}_+; D(H), L_{p,\gamma}(R^n; E)) .$$

Therefore, the problem (4.1) can be expressed as

$$\frac{du}{dt} + Hu(t) = f(t), \; u(0) = 0, \; t \in \mathbb{R}_+. \tag{4.3}$$

By virtue of [2, Theorem 4.5.2] and [8], $L_{p,\gamma}(R^n; E)$ is a Banach space satisfying the multiplier condition with respect to $p \in (1, \infty)$. Then due to $R-$sectoriality of H with $\varphi \in \left(\frac{\pi}{2}, \pi\right)$, by virtue of [20, Theorem 4.2] and [2, Propositions 8.10] we obtain that for $f \in L_{p_1}(\mathbb{R}_+; L_{p,\gamma}(R^n; E))$ the problem (4.3) has a unique solution $u \in W^1_{p_1}(\mathbb{R}_+; D(H), L_{p,\gamma}(R^n; E))$ satisfying

$$\left\| \frac{du}{dt} \right\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(R^n;E))} + \|Hu\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(R^n;E))} \leq C \|f\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(R^n;E))} .$$

Using the results obtained from Theorem 3.1 and from the above estimate we obtain inequality (4.2).

5. The Cauchy problem for infinite system of parabolic convolution differential equations

Consider the following Cauchy problem for infinite system of parabolic convolution differential equations

$$\frac{\partial u_m}{\partial t}+\sum_{|\alpha|\leq l} a_\alpha * D^\alpha u_m+\sum_{m=1}^{\infty} d_m * u_m=f_m, \tag{5.1}$$

$$u_m\left(0,x\right)=0,\ x\in R^n, t\in\mathbb{R}_+.$$

in $L_{p,\gamma}\left(R^n;l_q\right)$, where l is a natural number, $a_\alpha=a_\alpha\left(x\right)$ are complex-valued functions, $d_m=d_m\left(x\right)$, $u_m=u_m\left(t,x\right)$, $f_m=f_m\left(t,x\right)$, $\alpha=\left(\alpha_1,\alpha_2,...,\alpha_n\right)$, α_k are nonnegative integers, $\gamma\left(x\right)=\left|x\right|^\gamma$, $-1<\gamma<p-1$.

For $1<1<\infty$ we set

$$l_q=\left\{\xi;\xi=\left\{\xi_i\right\}_{i=1}^{\infty};\ \left\|\xi\right\|_{l_q}=\left(\sum_{i=1}^{\infty}\left|\xi_i\right|^q\right)^{1/q}<\infty,\quad \xi_i-\text{complex numbers}\right\}.$$

Moreover, if $p\in(1,\infty)$ and $\gamma(x)$ is a positive measurable function, then

$$L_{p,\gamma}\left(R^n;l_q\right)=\left\{f;f=\left\{f_i(x)\right\}_{i=1}^{\infty},\ \left\|f\right\|_{L_{p,\gamma}\left(R^n;l_q\right)}=\left(\int_{R^n}\left\|\left\{f_i(x)\right\}\right\|_{l_q}^p\gamma\left(x\right)dx\right)^{1\backslash p}<\infty\right\}.$$

Clearly, $L_{p,\gamma}\left(R^n;l_q\right)$ is a Banach space. In this section using the results obtained in section 4, we establish the regularity Cauchy problem for infinite systems.

Let

$$d=\left\{d_m(x)\right\},\ d_m>0,\ u=\left\{u_m\right\},\ d*u=\left\{d_m*u_m\right\},\ l_q(d)=$$

$$\left\{u\in l_q,\ \left\|u\right\|_{l_q(d)}=\left(\sum_{m=1}^{\infty}\left|d_m\left(x\right)*u_m\right|^q\right)^{\frac{1}{q}}<\infty\right\},\ 1<q<\infty.$$

Let H_0 denote the differential operator in $L_{p,\gamma}\left(R^n;l_q\right)$ generated by the following elliptic CDE

$$\sum_{|\alpha|\leq l} a_\alpha * D^\alpha u_m+\sum_{m=1}^{\infty} d_m * u_m=f_m,$$

$\mathbf{p}=\left(p,p_1\right)$, $L_{\mathbf{p},\gamma}\left(R_+^{n+1};l_q\right)$ will denote the space of all $\mathbf{p}-$summable l_q- valued functions with mixed norm, i.e.,

$$\|f\|_{L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};l_q\right)}=\left(\int\limits_{R^n}\left(\int\limits_{\mathbb{R}_+}\|\{f_m(t,x)\}\|_{l_q}^p\gamma(x)dx\right)^{\frac{p_1}{p}}dt\right)^{1/p_1}<\infty,$$

where $f=\{f_m(t,x)\}_{m=1}^{\infty}$.

Analogously, $W_{p,\gamma}^{1,l}\left(R_{+}^{n+1};l_q(d),l_q\right)$ denotes the space of all functions $u=\{u_m\}\in L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};l_q\right)$ such that the generalized derivatives $D_t u$, $D_k^l u\in L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};l_q\right)$ with the norm

$$\|u\|_{W_{\mathbf{p},\gamma}^{1,l}\left(R_{+}^{n+1};l_q(d),l_q\right)}=\|u\|_{L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};l_q(d)\right)}+\|D_t u\|_{L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};l_q\right)}+\sum_{k=1}^{n}\left\|D_k^l u\right\|_{L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};l_q\right)}.$$

It is clear to see that

$$L_{\mathbf{p},\gamma}\left(R_{+}^{n+1};l_q\right)=L_{p_1}\left(\mathbb{R}_+;L_{p,\gamma}\left(R^n;l_q\right)\right)\text{ and}$$

$$W_{\mathbf{p},\gamma}^{1,l}\left(R_{+}^{n+1};l_q(d),l_q\right)=W_{p_1}^{1}\left(\mathbb{R}_+;D\left(H_0\right),L_{p,\gamma}\left(R^n;l_q\right)\right).$$

Condition 5.1. Assume there are positive constants C_1 and C_2 such that for $\{d_m(x)\}_1^{\infty}\in l_q$, for all $x\in R^n$ and some $x_0\in R^n$,

$$C_1\left|d_m\left(x_0\right)\right|\leq\left|d_m(x)\right|\leq C_2\left|d_m\left(x_0\right)\right|.$$

Suppose $\hat{a}_\alpha$, $\hat{d}_m\in C^{(n)}\left(R^n\right)$ and there are positive constants M_i, $i=1,2$ such that

$$|\xi|^{|\beta|}\left|D^\beta\hat{a}_\alpha(\xi)\right|\leq M_1,\ |\xi|^{|\beta|}\left|D^\beta\hat{d}_m(\xi)\right|\leq M_2,\ \xi\in R^n\backslash\{0\},\ \beta_k\in\{0,1\},\ 0\leq|\beta|\leq n.$$

Theorem 5.1. Suppose the part (1) of Condition 3.1 and Condition 5.1 are satisfied. Then:

(a) for all $f(x)=\{f_m(x)\}_1^{\infty}\in L_{p,\gamma}\left(R^n;l_q\right)$, for $\lambda\in S_\varphi$, $\varphi\in[0,\pi)$ problem (5.1) has a unique solution $u=\{u_m(t,x)\}_1^{\infty}$ that belongs to $W_{p_1}^{1}\left(\mathbb{R}_+;D\left(H_0\right),L_{p,\gamma}\left(R^n;l_q\right)\right)$ and the following coercive uniform estimate holds

$$\left\|\frac{\partial u_m}{\partial t}\right\|_{L_{p_1}\left(\mathbb{R}_+;L_{p,\gamma}\left(R^n;l_q\right)\right)}+\sum_{|\alpha|\leq l}\left\|a_\alpha*D^\alpha u_m\right\|_{L_{p_1}\left(\mathbb{R}_+;L_{p,\gamma}\left(R^n;l_q\right)\right)}+$$

$$+\sum_{m=1}^{\infty}\left\|d_m*u_m\right\|_{L_{p_1}\left(\mathbb{R}_+;L_{p,\gamma}\left(R^n;l_q\right)\right)}\leq C\left\|f_m\right\|_{L_{p_1}\left(\mathbb{R}_+;L_{p,\gamma}\left(R^n;l_q\right)\right)}$$

(b) For $\lambda\in S_\varphi$ there exists a resolvent $\left(H_0+\lambda\right)^{-1}$ and

$$\sum_{|\alpha|\leq l}|\lambda|^{1-\frac{|\alpha|}{l}}\left\|a_\alpha*D^\alpha\left(H_0+\lambda\right)^{-1}\right\|_{B\left(L_{p,\gamma}\left(R^n;l_q\right)\right)}+$$

$$+\left\|d*(H_0+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;l_q))}+\left\|\lambda(H_0+\lambda)^{-1}\right\|_{B(L_{p,\gamma}(R^n;l_q))}\leq C.$$

Proof. In fact, let $E=l_q$ and $d_m=[d_m(x)\,\delta_{jm}]$, $m,\ j=1,2,...\infty$.It is easy to see that $\hat{d}_m(\xi)\,\delta_{jm}$ is uniformly R-sectorial in l_q and all conditions of Theorem 4.1 hold. Moreover, by [19] we get that the space l_q satisfies the multiplier condition with respect to weighted function $\gamma(x)=|x|^{\gamma}$. Therefore, by virtue of Theorem 4.1 and by using a similar technique as in section 4 we obtain the

$$\frac{du_m(t)}{dt}+H_0u_m(t)=f_m(t),\quad u_m(0)=0,\quad t\in\mathbb{R}_+ \tag{5.2}$$

It is easy to see that space l_q satisfies the multiplier condition. Due to R sectoriality of H_0 in l_q with $\varphi\in\left(\frac{\pi}{2},\pi\right)$, by virtue of [20, Theorem 4.2] we obtain that for $f\in L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(R^n;l_q))$ the problem (5.2) has a unique solution $u\in W^1_{p_1}(\mathbb{R}_+;D(H_0),L_{p,\gamma}(R^n;l_q))$ satisfying

$$\left\|\frac{du_m}{dt}\right\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(R^n;l_q))}+\|H_0u_m\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(R^n;l_q))}\leq C\,\|f_m\|_{L_{p_1}(\mathbb{R}_+;L_{p,\gamma}(R^n;l_q))}.$$

By using of definition spaces $L_{\mathbf{p},\gamma}\left(R^{n+1}_+;l_q\right)$ and l_q in a similar way as in section 4 we obtain following estimate

$$\left(\int\limits_{\mathbb{R}_+}\left(\int\limits_{R^n}\left(\sum_{m=1}^{\infty}\left|\frac{\partial u_m}{\partial t}\right|^q\right)^{\frac{p}{q}}\gamma(x)dx\right)^{\frac{p_1}{p}}dt\right)^{\frac{1}{p_1}}+\sum_{|\alpha|\leq l}\left(\int\limits_{\mathbb{R}_+}\left(\int\limits_{R^n}\left(\sum_{m=1}^{\infty}|a_\alpha*D^\alpha u_m|^q\right)^{\frac{p}{q}}\gamma(x)dx\right)^{\frac{p_1}{p}}dt\right)^{\frac{1}{p_1}}+$$

$$+\sum_{m=1}^{\infty}\left(\int\limits_{\mathbb{R}_+}\left(\int\limits_{R^n}\left(\sum_{m=1}^{\infty}|d_m*u_m|^q\right)^{\frac{p}{q}}\gamma(x)dx\right)^{\frac{p_1}{p}}dt\right)^{\frac{1}{p_1}}\leq C\left(\int\limits_{\mathbb{R}_+}\left(\int\limits_{R^n}\left(\sum_{m=1}^{\infty}|f_m|^q\right)^{\frac{p}{q}}\gamma(x)dx\right)^{\frac{p_1}{p}}dt\right)^{\frac{1}{p_1}}.$$

From this we get that assertion (a). Taking into account Theorem 3.1 and Result 3.1 we have for all $\lambda\in S_\varphi$ there exist the resolvent of operator H_0 and assertion (b) is obtained.

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