Journal of Contemporary Applied Mathematics
V. 8, No 2, 2018, December
ISSN 2222-5498

On the Noetherian Properties of Reduction System of Words

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Abstract. For any set Σ of symbols, Σ^* denotes the set of all words of symbols over Σ, including the empty string ϵ. The set Σ^* denotes the free monoid generated by Σ under the operation of concatenation with the empty string serving as identity.
Let $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$ be a finite set. We define the binary relation $\underset{\mathcal{R}}{\Rightarrow}$ as follows, where $u, v \in \Sigma^* : u \underset{\mathcal{R}}{\Rightarrow} v$ if there exist $x, y \in \Sigma^*$ and $(l, m) \in \mathcal{R}$ with $u = xly$ and $v = xmy$. The structure $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is a reduction system of words and the relation $\underset{\mathcal{R}}{\Rightarrow}$ is the reduction relation. Let $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ be a reduction system of words. The relation $\underset{\mathcal{R}}{\Rightarrow}$ is Noetherian if there is no infinite sequence $w_0, w_1, ... \in \Sigma^*$ such that for all $i \geq 0, w_i \underset{\mathcal{R}}{\Rightarrow} w_{i+1}$.
In this paper, we study properties of reduction systems of words and give conditions under which a reduction system of word is Noetherian.

Key Words and Phrases: Free monoid, morphism of monoids, closure of a binary relation, reduction system of words.
2010 Mathematics Subject Classifications: 68Q42, 20M05

A reduction system of words is a pair $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ where Σ is an alphabet and $\mathcal{R}$ is a non-empty finite binary on Σ^*, we write $xly \underset{\mathcal{R}}{\Rightarrow} xmy$ whenever $x, y \in \Sigma^*$ and $(l, m) \in \mathcal{R}$. We write $u \overset{*}{\underset{\mathcal{R}}{\Rightarrow}} v$ if there exists a words $u_0, u_1, ..., u_n \in \Sigma^*$ such that,

$$u_0 = u, u_i \underset{\mathcal{R}}{\Rightarrow} u_{i+1}, \forall 0 \leq i \leq n-1 \text{ and } u_n = v.$$

If $n = 0$, we get $u = v$, and if $n = 1$, we get $u \underset{\mathcal{R}}{\Rightarrow} v$. Where $\overset{*}{\underset{\mathcal{R}}{\Rightarrow}}$ is the reflexive transitive closure of $\underset{\mathcal{R}}{\Rightarrow}$ [7].

The reduction system of words $(\Sigma, \mathcal{R})$ is Noetherian if there does not exist an infinite chain $w_1 \underset{\mathcal{R}}{\Rightarrow} w_2 \underset{\mathcal{R}}{\Rightarrow} w_3 \underset{\mathcal{R}}{\Rightarrow} ...$ in Σ^*.

However, this property is said to be undecidable. It is not possible to find an algorithm taking as input a reduction system of words and rendering true if and only if this reduction system of words is Noetherian [8, 10].

The remainder of this paper is organized as follows. In Section 2, some mathematical preliminaries. In Section 3, we give some propositions on the noetherian propertie of reductions systems of words.

1. Preliminaries

We formally define an alphabet as a non-empty finite set. A word over an alphabet Σ is a finite sequence of symbols of Σ. Although one writes a sequence as $(\sigma_1, \sigma_2, ..., \sigma_n)$, in the present context, we prefer to write it as $\sigma_1\sigma_2...\sigma_n$. The set of all words on the alphabet Σ is denoted by $\Sigma^*$and is equipped with the associative operation defined by the concatenation of two sequences. The concatenation of two sequences $\alpha_1\alpha_2...\alpha_n$ and $\beta_1\beta_2...\beta_m$ is the sequence $\alpha_1\alpha_2...\alpha_n\beta_1\beta_2...\beta_m$ $[1, 4]$.

The concatenation is an associative operation. The string consisting of zero letters is called the empty word, written ϵ. Thus, $\epsilon, \alpha, \beta, \alpha\alpha\beta\alpha, \alpha\alpha\alpha\beta\alpha$ are words over the alphabet $\{\alpha, \beta\}$. Thus the set Σ^* of words is equipped with the structure of a monoid. The monoid Σ^* is called the free monoid on Σ. The length of a word w, denoted $|w|$, is the number of letters in w when each letter is counted as many times as it occurs. Again by definition, $|\epsilon| = 0$. For example $|\alpha\alpha\beta\alpha| = 4$ and $|\alpha\alpha\alpha\beta\alpha| = 5$. Let w be a word over an alphabet Σ. For $\sigma \in \Sigma$, the number of occurrences of σ in w shall be denoted by $|w|_\sigma$. For example $|\alpha\alpha\beta\alpha|_\beta = 1$ and $|\alpha\alpha\alpha\beta\alpha|_\alpha = 4$.

A mapping $h : \Sigma^* \longrightarrow \Delta^*$, where Σ and Δ are alphabets, satisfying the condition

$$h(uv) = h(u)h(v), \text{ for all words } u \text{ and } v,$$

is called a morphism. To define a morphism h, it suffices to list all the words $h(\sigma)$, where a ranges over all the (finitely many) letters of Σ. If M is a monoid, then any mapping $f : \Sigma \longrightarrow M$ extends to a unique morphism $h : \Sigma^* \longrightarrow M$. For instance, if M is the additive monoid $\mathbb{N}$, and f is defined by $f(\sigma) = 1$ for each $\sigma \in \Sigma$, then $h(u)$ is the length $|u|$ of the word u $[6, 7]$.

A binary reation on Σ^* is a subset $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$. If $(x, y) \in \mathcal{R}$, we say that x is related to y by $\mathcal{R}$, denoted $x\mathcal{R}y$. The relation $I_{\Sigma^*} = \{(x, x), x \in \Sigma^*\}$ is called the identity relation. The relation $(\Sigma^*)^2$ is called the complete relation.

Let $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$ and $S \subseteq \Sigma^* \times \Sigma^*$ binary relations. The composition of $\mathcal{R}$ and S is a binary relation $S \circ \mathcal{R} \subseteq \Sigma^* \times \Sigma^*$ defined by

$$x (\mathcal{S} \circ \mathcal{R}) z \Longleftrightarrow \exists y \in \Sigma^* \text{ such that } x\mathcal{R}y \text{ and } y\mathcal{S}z.$$

A binary relation $\mathcal{R}$ on a set Σ^* is said to be

- reflexive if $x\mathcal{R}x$ for all x in Σ^*.
- transitive if $x\mathcal{R}y$ and $y\mathcal{R}z$ imply $x\mathcal{R}z$.

Let $\mathcal{R}$ be a relation on a set Σ^*. The reflexive closure of $\mathcal{R}$ is the smallest reflexive relation $\mathcal{R}^r$ on Σ^* that contains $\mathcal{R}$, that is,

• $\mathcal{R} \subseteq \mathcal{R}^r$

• if S is a reflexive relation on Σ^* and $\mathcal{R} \subseteq \mathcal{S}$, then $\mathcal{R}^r \subseteq S$.

The transitive closure of $\mathcal{R}$ is the smallest transitive relation $\mathcal{R}^+$ on Σ^* that contains $\mathcal{R}$; that is,

• $\mathcal{R} \subseteq \mathcal{R}^+$

• if S is a transitive relation on Σ^* and $\mathcal{R} \subseteq S$, then $\mathcal{R}^+ \subseteq S$.

The reflexive transitive closure of $\mathcal{R}$ is the smallest reflexive transitive relation $\mathcal{R}^*$ on Σ^* that contains $\mathcal{R}$; that is,

• $\mathcal{R} \subseteq \mathcal{R}^*$

• if S is a reflexive transitive relation on Σ^* and $\mathcal{R} \subseteq S$, then $\mathcal{R}^* \subseteq S$.

Let $\mathcal{R}$ be a relation on a set Σ^*. Then

$$\mathcal{R}^0 = \mathcal{R} \cup I_{\Sigma^*},\ \mathcal{R}^+ = \bigcup_{k=1}^{k=+\infty} \mathcal{R}^k,\ \mathcal{R}^* = \bigcup_{k=0}^{k=+\infty} \mathcal{R}^k \text{ [6]}.$$

Where $\mathcal{R}^k = \mathcal{R}^0 \circ \mathcal{R}^{k-1}$, $\mathcal{R}^0$ is the identity relation, and $\circ$ denote composition of relations.

Let $\mathcal{R} \subseteq \Sigma^* \times \Sigma^*$be a finite set. We define the binary relation $\underset{\mathcal{R}}{\Rightarrow}$ as follows, where $u, v \in \Sigma^* : u \underset{\mathcal{R}}{\Rightarrow} v$ if there exist $x, y \in \Sigma^*$ and $(l, m) \in \mathcal{R}$ with $u = xly$ and $v = xmy$. The structure $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is a reduction system of words and the relation $\underset{\mathcal{R}}{\Rightarrow}$ is the reduction relation. If $u \in \Sigma^*$ and there is no $v \in \Sigma^*$ suth that $u \underset{\mathcal{R}}{\Rightarrow} v$, then u is irreducible; otherwise, u is reducible. The set of all irreducible elements of Σ^* with respect to $\underset{\mathcal{R}}{\Rightarrow}$ is denoted $IRR\left(\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)\right)$ [2]. Let $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ be a reduction system of words, we write $u \overset{*}{\underset{\mathcal{R}}{\Rightarrow}} v$ if there words $u_0, u_1, ..., u_n \in \Sigma^*$ such that,

$$u_0 = u, u_i \underset{\mathcal{R}}{\Rightarrow} u_{i+1}, \forall 0 \leq i \leq n-1 \text{ and } u_n = v.$$

If $n = o$, we get $u = v$, and if $n = 1$, we get $u \underset{\mathcal{R}}{\Rightarrow} v$. Where $\overset{*}{\underset{\mathcal{R}}{\Rightarrow}}$ is the reflexive transitive closure of $\underset{\mathcal{R}}{\Rightarrow}$.

We say that $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian if there does not exist an infinite sequence of words $w_i \in \Sigma^*$ $(i \in \mathbb{N})$ with $w_0 \underset{\mathcal{R}}{\Rightarrow} w_1 \underset{\mathcal{R}}{\Rightarrow} w_2 \underset{\mathcal{R}}{\Rightarrow} \ldots$. For example $(\mathbb{N}, >)$ is Noetherian [5].

2. Results

In this section, we study reduction systems and we provide conditions under which a reduction system of word is Noetherian.

Theorem 1. [2, *Theorem* 2.2.4, *p*.42] *Let* $\left(\Sigma_1, \underset{\mathcal{R}_1}{\Rightarrow}\right)$ *be a reduction system of words. Then the following two statements are equivalent :*

1. $\left(\Sigma_1, \underset{\mathcal{R}_1}{\Rightarrow}\right)$ is Noetherian;

2. There exists another reduction system of words $\left(\Sigma_2, \underset{\mathcal{R}_2}{\Rightarrow}\right)$ that is Noetherian and the morphism $\psi : \Sigma_1^* \longrightarrow \Sigma_2^*$ such that $\psi(\underset{\mathcal{R}_1}{\Rightarrow}) \subseteq \underset{\mathcal{R}_2}{\overset{+}{\Rightarrow}}$. $\underset{\mathcal{R}_2}{\overset{+}{\Rightarrow}}$ is the transitive closure of $\underset{\mathcal{R}_2}{\Rightarrow}$.

Proposition 2. *Let* $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ *be a reduction system of words and* $\varphi : \Sigma^* \longrightarrow \mathbb{N}$ *the morphism of monoids. Consider the mapping* $P : \Sigma^* \longrightarrow \mathbb{N}$ *defined by :*
$P(w) = \sum\limits_{i=1}^{i=|w|} n^i \times \varphi(w(i))$, $n \in \mathbb{N} - \{0\}$ *where* $w\,(i)$ *is the* $i-th$ *letter of* w.

If for all $(l, m) \in \mathcal{R}$, $\begin{cases} |l| = |m| & (C_1) \\ \text{and} & \\ P(l) > P(m) & (C_2) \end{cases}$, then $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian.

Proof. First, we show that, we have : $\forall x, y \in \Sigma^* : P(xy) = P(x) + n^{|x|} \times P(y)$. We have
$P(xy) = \sum\limits_{i=1}^{i=|xy|} n^i \times \varphi((xy)\,(i)) = \sum\limits_{i=1}^{i=|x|} n^i \times \varphi((xy)\,(i)) + \sum\limits_{i=|x|+1}^{i=|x|+|y|} n^i \times \varphi((xy)\,(i))$
$= \sum\limits_{i=1}^{i=|x|} n^i \times \varphi((x)\,(i)) + \sum\limits_{i=1}^{i=|y|} n^{|x|+i} \times \varphi((xy)\,(|x|+i))$
$= \sum\limits_{i=1}^{i=|x|} n^i \times \varphi((x)\,(i)) + \sum\limits_{i=1}^{i=|y|} n^{|x|+i} \times \varphi((y)\,(i)) = P(x) + n^{|x|} \times P(y)$.
Let $(l, m) \in \mathcal{R}$ and $x, y \in \Sigma^*$, we show that $P(xly) > P(xmy)$.
We have $P(xly) = P(x(ly)) = P(x) + n^{|x|} \times P(ly) = P(x) + n^{|x|}\left(P(l) + n^{|l|} \times P(y)\right)$
$= P(x) + n^{|x|} \times P(l) + n^{|x|+|l|} \times P(y)$. A similar argument, we have $P(xmy) = P(x(my))$
$= P(x) + n^{|x|} \times P(my) = P(x) + n^{|x|}\left(P(m) + n^{|m|} \times P(y)\right)$
$= P(x) + n^{|x|} \times P(m) + n^{|x|+|m|} \times P(y)$. According to the conditions (C_1), (C_2) described above, we have $P(xly) > P(xmy)$. Consequently $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian. □

Example 3. *Consider the reduction system of words* $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ *with* $\Sigma = \{\alpha, \beta, \gamma\}$ *and* $\mathcal{R} = \{(\beta\alpha, \alpha\beta)\,;(\gamma\beta, \beta\gamma)\}$. *Let the morphism* $\varphi : \Sigma^* \longrightarrow \mathbb{N}$, *defined by* $\varphi\,(\alpha) = 3$,

$\varphi(\beta) = 2, \varphi(\gamma) = 1$ and the mapping $P : \Sigma^* \longrightarrow \mathbb{N}$, where $P(w) = \sum_{i=1}^{i=|w|} 2^i \times \varphi(w(i))$. For the condition (C_1), we have $|\beta\alpha| = |\alpha\beta| = 2$ and $|\gamma\beta| = |\beta\gamma| = 2$. For the condition (C_2), we show that $P(\beta\alpha) > P(\alpha\beta)$ and $P(\gamma\beta) > P(\beta\gamma)$.
We have $P(\beta\alpha) = \sum_{i=1}^{i=2} 2^i \times \varphi(\beta\alpha(i)) = 2 \times \varphi(\beta) + 2^2 \times \varphi(\alpha) = 16$.
Similarly, $P(\alpha\beta) = \sum_{i=1}^{i=2} 2^i \times \varphi(\alpha\beta(i)) = 2 \times \varphi(\alpha) + 2^2 \times \varphi(\beta) = 14$, then $P(\beta\alpha) > P(\alpha\beta)$.
We have $P(\gamma\beta) = \sum_{i=1}^{i=2} 2^i \times \varphi(\gamma\beta(i)) = 2 \times \varphi(\gamma) + 2^2 \times \varphi(\beta) = 10$.
Similarly, $P(\beta\gamma) = \sum_{i=1}^{i=2} 2^i \times \varphi(\beta\gamma(i)) = 2 \times \varphi(\beta) + 2^2 \times \varphi(\gamma) = 8$, then $P(\gamma\beta) > P(\beta\gamma)$.
Consequently $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian.

Proposition 4. *Let $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ be a reduction system of words and $\varphi : (\Sigma^*, \cdot) \longrightarrow (\mathbb{N}, +)$ with a morphism of monoids, the mapping $P : \Sigma^* \longrightarrow \mathbb{N}$ defined by : $P(w) = \sum_{i=1}^{i=|w|} i \times \varphi(w(i))$, where $w(i)$ is the $i-th$ letter of w.*

$$\text{If for all } (l, m) \in \mathcal{R}, \begin{cases} |l| = |m| & (C_1) \\ \quad\quad \text{and} & \\ P(l) > P(m) & (C_2) \\ \quad\quad \text{and} & \\ \varphi(l) > \varphi(m) & (C_3) \end{cases}, \text{ then } \left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right) \text{ is Noetherian.}$$

Proof. First, we show that, we have : $\forall x, y \in \Sigma^* : P(xy) = P(x) + P(y) + |x| \times \varphi(y)$.
We have $P(xy) = \sum_{i=1}^{i=|xy|} i \times \varphi(xy(i)) = \sum_{i=1}^{i=|x|} i \times \varphi(xy(i)) + \sum_{i=|x|+1}^{i=|x|+|y|} i \times \varphi(xy(i))$
$= \sum_{i=1}^{i=|x|} i \times \varphi(x(i)) + \sum_{i=1}^{i=|y|} (|x| + i) \times \varphi((xy)(|x| + i))$
$= \sum_{i=1}^{i=|x|} i \times \varphi((x)(i)) + \sum_{i=1}^{i=|y|} (|x| + i) \times \varphi((y)(i))$
$= P(x) + P(y) + |x| \times \varphi(y)$.
Let $(l, m) \in \mathcal{R}$ and $x, y \in \Sigma^*$, we show that $P(xly) > P(xmy)$.
We have, $P(xly) = P(x(ly)) = P(x) + P(ly) + |x| \times \varphi(ly)$
$= P(x) + P(l) + P(y) + |l| \times \varphi(y) + |x| \times (\varphi(l) + \varphi(y))$
$= [P(x) + P(y) + |x| \times \varphi(y)] + [P(l) + |l| \times \varphi(y) + |x| \times \varphi(l)]$.
On the other hand, $P(xmy) = [P(x) + P(y) + |x| \times \varphi(y)] + [P(m) + |m| \times \varphi(y) + |x| \times \varphi(m)]$.
According to the conditions (C_1), (C_2), (C_1) described above,we have $P(xly) > P(xmy)$.

Finally $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ is Noetherian. $\square$

Example 5. *Let* $\Sigma = \{\alpha, \beta, \gamma\}$ *and* $\mathcal{R} = \{(\beta\alpha, \beta\gamma)\,; (\alpha\beta, \alpha\gamma)\}$. *We define the morphism of monoids* $\varphi : \Sigma^* \longrightarrow \mathbb{N}$, *by* $\varphi(\alpha) = 2, \varphi(\beta) = 1, \varphi(\gamma) = 0$. *We consider the mapping* $P : \Sigma^* \longrightarrow \mathbb{N}$, *where* $P(w) = \sum_{i=1}^{i=|w|} i \times \varphi(w(i))$.
For the condition (C_1), *we have* $|\beta\alpha| = |\beta\gamma| = 2$ *and* $|\alpha\beta| = |\alpha\gamma| = 2$.
For the condition (C_2), *we show that* $P(\beta\alpha) > P(\beta\gamma)$ *and* $P(\alpha\beta) > P(\alpha\gamma)$.
We have $P(\beta\alpha) = \sum_{i=1}^{i=2} i \times \varphi(\beta\alpha(i)) = 1 \times \varphi(\beta) + 2 \times \varphi(\alpha) = 5$.
A similar argument, we have $P(\beta\gamma) = \sum_{i=1}^{i=2} i \times \varphi(\beta\gamma(i)) = 1 \times \varphi(\beta) + 2 \times \varphi(\gamma) = 1$,
then $P(\beta\alpha) > P(\beta\gamma)$. *And* $P(\alpha\beta) = \sum_{i=1}^{i=2} i \times \varphi(\alpha\beta(i)) = 1 \times \varphi(\alpha) + 2 \times \varphi(\beta) = 4$.
Similarly, $P(\alpha\gamma) = \sum_{i=1}^{i=2} i \times \varphi(\alpha\gamma(i)) = 1 \times \varphi(\alpha) + 2 \times \varphi(\gamma) = 2.$, *then* $P(\alpha\beta) > P(\alpha\gamma)$.
For the condition (C_3), *we show that* $\varphi(\beta\alpha) > \varphi(\beta\gamma)$ *and* $\varphi(\alpha\beta) > \varphi(\alpha\gamma)$.
We have $\varphi(\beta\alpha) = 3, \varphi(\beta\gamma) = 1, \varphi(\alpha\beta) = 3, \varphi(\alpha\gamma) = 2$.
Consequently $\left(\Sigma, \underset{\mathcal{R}}{\Rightarrow}\right)$ *is Noetherian.*

3. Conclusion

In this paper, we give some give conditions under which a reduction system of word is Noetherian.

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