Journal of Contemporary Applied Mathematics
V. 8, No 2, 2018, December
ISSN 2222-5498

On Morrey type Spaces and Some Properties

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Abstract. Subspace $M_\rho^{p,\alpha}$ of the weighted Morrey -type space $L_\rho^{p,\alpha}$ is defined, it is proved that infinitely differentiable functions are dense in it. An approximation properties of the Poisson kernel is studied in $M_\rho^{p,\alpha}$. A sufficient condition for belonging of the product to the space $M_\rho^{p,\alpha}$ is obtained. It is proved that $M_\rho^{p,\alpha}$ is an invariant subspace of a singular integral operator.

Key Words and Phrases: Morrey-type classes, Minkowski inequality, Poisson kernel.

2010 Mathematics Subject Classifications: 30E25, 46E30

1. Introduction

The concept of Morrey space was introduced by C. Morrey [1] in 1938 in the study of qualitative properties of the solutions of elliptic type equations with BMO (Bounded Mean Oscillations) coefficients (see also [2, 3]). This space provides a large class of weak solutions to the Navier-Stokes system [4]. In the context of fluid dynamics, Morrey-type spaces have been used to model the fluid flow in case where the vorticity is a singular measure supported on some sets in R^n [5]. There appeared lately a large number of research works which considered many problems of the theory of differential equations, potential theory, maximal and singular operator theory, approximation theory, etc in Morrey-type spaces (for more details see [2-26]). It should be noted that the matter of approximation in Morrey-type spaces has only started to be studied recently (see, e.g., [11, 12, 16, 17]), and many problems in this field are still unsolved. This work is just dedicated to this field. Subspace $M_\rho^{p,\alpha}$ of the weighted Morrey -type space $L_\rho^{p,\alpha}$ is defined, it is proved that infinitely differentiable functions are dense in it. An approximation properties of the Poisson kernel is studied in $M_\rho^{p,\alpha}$. A sufficient condition for belonging of the product to the space $M_\rho^{p,\alpha}$ is obtained. It is proved that $M_\rho^{p,\alpha}$ is an invariant subspace of a singular integral operator .

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2. Needful Information

We will need some facts about the theory of Morrey-type spaces. Let Γ be some rectifiable Jordan curve on the complex plane C. By $|M|_{\Gamma}$ we denote the linear Lebesgue measure of the set $M \subset \Gamma$. All the constants throughout this paper (can be different in different places) will be denoted by c.

The expression $f(x) \sim g(x)$, $x \in M$, means

$$\exists \delta > 0 : \delta \leq \left| \frac{f(x)}{g(x)} \right| \leq \delta^{-1}, \forall x \in M.$$

Similar meaning is intended by the expression $f(x) \sim g(x)$, $x \to a$.

By Morrey-Lebesgue space $L^{p,\alpha}(\Gamma)$, $0 < \alpha \leq 1$, $p \geq 1$, we mean the normed space of all measurable functions $f(\cdot)$ on Γ with the finite norm

$$\|f\|_{L^{p,\alpha}(\Gamma)} = \sup_{B} \left(\left| B \bigcap \Gamma \right|_{\Gamma}^{\alpha-1} \int_{B \bigcap \Gamma} |f(\xi)|^p |d\xi| \right)^{1/p} < +\infty.$$

$L^{p,\alpha}(\Gamma)$ is a Banach space with $L^{p,1}(\Gamma) = L_p(\Gamma)$, $L^{p,0}(\Gamma) = L_{\infty}(\Gamma)$. Similarly we define the weighted Morrey-Lebesgue space $L^{p,\alpha}_{\mu}(\Gamma)$ with the weight function $\mu(\cdot)$ on Γ equipped with the norm

$$\|f\|_{L^{p,\alpha}_{\mu}(\Gamma)} = \|f\mu\|_{L^{p,\alpha}(\Gamma)}, f \in L^{p,\alpha}_{\mu}(\Gamma).$$

The inclusion $L^{p,\alpha_1}(\Gamma) \subset L^{p,\alpha_2}(\Gamma)$ is valid for $0 < \alpha_1 \leq \alpha_2 \leq 1$. Thus, $L^{p,\alpha}(\Gamma) \subset L_1(\Gamma)$, $\forall \alpha \in (0, 1]$, $\forall p \geq 1$. For $\Gamma = [-\pi, \pi]$ we will use the notation $L^{p,\alpha}(-\pi, \pi) = L^{p,\alpha}$.

More details on Morrey-type spaces can be found in [2-26].

In the sequel, we will need some auxiliary facts. Recall Minkowski's (integral) inequality.

Let $(X; A_x; \nu)$ and $(Y; A_y; \mu)$ be measurable spaces with $\sigma-$finite measures ν and μ, respectively. If $F(x; y)$ is $\nu \times \mu-$measurable, then we have

$$\left\| \int_X F(\cdot; y)\, d\nu(x) \right\|_{L^p(d\mu)} \leq \int_X \|F(x; \cdot)\|_{L^p(d\mu)}\, d\nu(x), 1 \leq p < +\infty,$$

where

$$\|g(y)\|_{L^p(d\mu)} = \left(\int_Y |g(y)|^p\, d\mu \right)^{1/p}.$$

Now let $Y \equiv R$ and $\mu(\cdot)$ be a Borel measure on R. We have

$$\left(\int_I \left| \int_X F(x; y)\, d\nu(x) \right|^p d\mu(y) \right)^{1/p} \leq$$

$$\leq \int_X \left(\int_I |F(x;\, y)|^p \, d\mu(y) \right)^{1/p} d\nu(x) .$$

Thus

$$\left(\frac{1}{|I|^{1-\alpha}} \int_I \left| \int_X F(x;\, y) \, d\nu(x) \right|^p d\mu(y) \right)^{1/p} \leq$$

$$\leq \int_X \left(\frac{1}{|I|^{1-\alpha}} \int_I |F(x;\, y)|^p \, d\mu(y) \right)^{1/p} d\nu(x) \leq$$

$$\leq \int_X \|F(x;\, y)\|_{L^{p,\,\alpha}(d\mu)} \, d\nu(x) \, , \forall I \subset R.$$

Taking sup over $I \subset R$, we get

$$\left\| \int_X F(x;\, y) \, d\nu(x) \right\|_{L^{p,\,\alpha}(d\mu)} \leq \int_X \|F(x;\, y)\|_{L^{p,\,\alpha}(d\mu)} \, d\nu(x) . \tag{1}$$

So the Minkowski inequality (1) holds in the Morrey-type space $L^{p,\,\alpha}(d\mu)$.

Thus, the following Minkowski's inequality regarding Morrey type spaces is true.

Statement 1. *Let* $(X;\, \mathrm{A}_x;\, \nu)$ *be a measurable space with a* $\sigma-$*finite measure* ν *and* $(R;\, \mathrm{B};;\, \mu\,)$ *be a measurable space with a Borel measure* μ *on* $\sigma-$*algebra of Borel sets* B *of* R. *Then, for* $\nu \times \mu-$ *measurable function* $F(x;\, y)$, *the following analog of Minkowski's inequality is valid*

$$\left\| \int_X F(x;\, \cdot) \, d\nu(x) \right\|_{L^{p,\,\alpha}(d\mu)} \leq \int_X \|F(x;\, \cdot)\|_{L^{p,\,\alpha}(d\mu)} \, d\nu(x) .$$

By S_Γ we denote the following singular integral operator

$$(S_\Gamma f)(\tau) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta) \, d\zeta}{\zeta - \tau}, \quad \tau \in \Gamma,$$

where $\Gamma \subset C$ is some rectifiable curve on complex plane C. Let $\omega = \{z \in C : |z| < 1\}$ be the unit disk on C and $\partial\omega = \gamma$ be its boundary. Define the Morrey-Hardy space $H_+^{p,\,\alpha}$ of analytic functions $f(z)$ inside ω equipped with the following norm

$$\|f\|_{H_+^{p,\,\alpha}} = \sup_{0<r<1} \left\| f\left(re^{it}\right) \right\|_{L^{p,\,\alpha}} .$$

In what follows, we assume that the function $f(\cdot)$ periodically continued on the whole axis R .

We will also use the following concepts. Let $\Gamma \subset C$ be some bounded rectifiable curve, $t = t(\sigma)$, $0 \leq \sigma \leq 1$, be its parametric representation with respect to the arc length σ, and l be the length of Γ. Let $d\mu(t) = d\sigma$, i.e. let $\mu(\cdot)$ be a linear measure on Γ. Let

$$\Gamma_t(r) = \{\tau \in \Gamma : |\tau - t| < r\} \, , \Gamma_{t(s)}(r) = \{\tau(\sigma) \in \Gamma : |\sigma - s| < r\} \, .$$

It is absolutely clear that $\Gamma_{t(s)}(r) \subset \Gamma_t(r)$.

Definition 1. *Curve Γ is said to be Carleson if $\exists c > 0$:*

$$\sup_{t\in\Gamma} \mu\left(\Gamma_t\left(r\right)\right) \leq cr, \forall r > 0.$$

Curve Γ is said to satisfy the chord-arc condition at the point $t_0 = t\left(s_0\right) \in \Gamma$ if there exists a constant $m > 0$ independent of t such that $\left|s - s_0\right| \leq m\left|t\left(s\right) - t\left(s_0\right)\right|$, $\forall t\left(s\right) \in \Gamma$. Γ satisfies a chord-arc condition uniformly on Γ if $\exists m > 0 : \left|s - \sigma\right| \leq m\left|t\left(s\right) - t\left(\sigma\right)\right|$, $\forall t\left(s\right), t\left(\sigma\right) \in \Gamma$.

Let's state the following lemma of [10] which is of independent interest.

Lemma 1. *[10] Let Γ be a bounded rectifiable curve. If the power function $\left|t - t_0\right|^{\gamma}$, $t_0 \in \Gamma$, belongs to the space $L^{p,\alpha}\left(\Gamma\right)$, $1 \leq p < \infty$, $0 < \alpha < 1$, then the inequality $\gamma \geq -\frac{\alpha}{p}$ holds. If Γ is a Carleson curve, then this condition is also sufficient.*

We will extensively use the following theorem of N.Samko [10].

Theorem 1. *[10] Let the curve Γ satisfy the chord-arc condition and the weight $\rho\left(\cdot\right)$ be defined by*

$$\rho\left(t\right) = \prod_{k=1}^{m} \left|t - t_k\right|^{\alpha_k}; \left\{t_k\right\}_1^m \subset \Gamma, t_i \neq t_j, i \neq j. \tag{2}$$

A singular operator S_Γ is bounded in the weighted space $L_\rho^{p,\alpha}\left(\Gamma\right)$, $1 < p < +\infty$, $0 < \alpha \leq 1$, if the following inequalities are satisfied

$$-\frac{\alpha}{p} < \alpha_k < -\frac{\alpha}{p} + 1, k = \overline{1, m}. \tag{3}$$

Moreover, if Γ is smooth in some neighborhoods of the points t_k, $k = \overline{1, m}$, then the validity of inequalities (3) is necessary for the boundedness of the operator S_Γ in $L_\rho^{p,\alpha}\left(\Gamma\right)$.

In what follows, as Γ we will consider a boundary of unit disk: $\gamma = \partial\omega$. Consider the weighted space $L_\rho^{p,\alpha}\left(\gamma\right) =: L_\rho^{p,\alpha}$ with the weight $\rho\left(\cdot\right)$. In an absolutely similar way to the non-weighted case, we define the space $M_\rho^{p,\alpha}$ with the weight $\rho\left(\cdot\right)$. Denote by $\tilde{M}_\rho^{p,\alpha}$ the set of functions whose shifts are continuous in $L_\rho^{p,\alpha}$, i.e.

$$\left\|S_\delta f - f\right\|_{p,\alpha;\rho} = \left\|f\left(\cdot + \delta\right) - f\left(\cdot\right)\right\|_{p,\alpha;\rho} \to 0, \delta \to 0,$$

where S_δ is a shift operator: $\left(S_\delta f\right)\left(x\right) = f\left(x + \delta\right)$ and we will consider that the function $f\left(\cdot\right)$ (in sequel also) periodically continued to the whole real axis R. It is not difficult to see that $\tilde{M}_\rho^{p,\alpha}$ is a linear subspace of $L_\rho^{p,\alpha}$. Denote the closure of $\tilde{M}_\rho^{p,\alpha}$ in $L_\rho^{p,\alpha}$ by $M_\rho^{p,\alpha}$.

Consider the following class

$$\tilde{L}_\rho^{p,\alpha} =: \left\{f \in L_\rho^{p,\alpha} : \left\|T_\delta f - f\right\|_{p,\alpha;\rho} \to 0,\, \delta \to 0\right\}.$$

It is evident that the class $\tilde{L}_\rho^{p,\alpha}$ is a linear subspace of $L_\rho^{p,\alpha}$. Let us denote by $M_\rho^{p,\alpha}$ the closure of $\tilde{L}_\rho^{p,\alpha}$ in $L_\rho^{p,\alpha}$.

Let us remember the following properties of Poisson kernel $P_r(\varphi)$:

$$P_r(\varphi) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos\varphi + r^2}\ ,\ 0 < r < 1.$$

(a) $\sup\limits_{|t|<\delta} P_r(t) \to 0$ as $|r| \to 1$;

(b) $\int_{|t|>\delta} P_r(t)\,dt \to 0$ as $|r| \to 1$ for $\forall\delta > 0$.

These properties directly follows from the expression for $P_r(\varphi)$. We have

$$\|(P_r * f)(\cdot) - f(\cdot)\|_{p,\alpha;\rho} = \left\| \frac{1}{2\pi} \int_{-\pi} P_r(t) f(t-s)\,dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) f(s)\,dt \right\|_{p,\alpha;\rho} \leq$$

$$\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt =$$

$$= \frac{1}{2\pi} \left[\int_{|t|>\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt + \int_{|t|<\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt \right].$$

Regarding the second integral in the right-hand side, we have

$$\frac{1}{2\pi} \int_{|t|<\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt \leq$$

$$\leq \sup_{|t|<\delta} \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,, \quad \text{as} \quad \delta \to 0.$$

To estimate the first integral, consider

$$\|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho} \leq \|f\|_{p,\alpha;\rho} + \|f(t-\cdot)\|_{p,\alpha;\rho} = 2\|f\|_{p,\alpha;\rho}\,.$$

We have

$$\|f(t-\cdot)\|_{p,\alpha;\rho}^p = \sup_B \frac{1}{|B\bigcap\gamma|^{1-\alpha}} \int_{B\bigcap\gamma} |f(t-s)|^p\,ds =$$

$$= \sup_B \frac{1}{|B\bigcap\gamma|^{1-\alpha}} \int_{(B\bigcap\gamma)_t} |f(s)|^p\,ds,$$

where $(B\bigcap\gamma)_t \equiv \{s : t - s \in B\bigcap\gamma\}$. It is clear that

$$\left|B\bigcap\gamma\right| = \left|\left(B\bigcap\gamma\right)_t\right|,$$

holds. Therefore

$$\|f(t-\cdot)\|_{p,\alpha;\rho} = \|f\|_{p,\alpha;\rho}\,.$$

As a result

$$\int_{|t|>\delta} P_r(t) \|f(t-\cdot) - f(\cdot)\|_{p,\alpha;\rho}\,dt \leq$$

$$\leq 2\|f\|_{p,\alpha;\rho} \int_{|t|>\delta} P_r(t)\,dt \to 0 \text{ as } |r| \to 1.$$

So we have proved the following

Theorem 2. *If* $f \in M_\rho^{p,\alpha}$, $1 \le p < +\infty \wedge 0 \le \alpha \le 1$, *then* $\|P_r * f - f\|_{p,\alpha;\rho} \to 0$ *as* $|r| \to 1$.

From Theorem 2 we immediately get the validity of the following

Theorem 3. *Let* $f \in M_\rho^{p,\alpha}$, $0 < \alpha \le 1$, $1 < p < +\infty$. *Then it holds*

$$\left\|(Kf)(r\xi) - f^+(\xi)\right\|_{p,\alpha;\rho} \to 0,\ r \to 1-0.$$

Similar assertion is true in case of $f^-(\xi)$ when $r \to 1+0$.

3. Subspace $M_\rho^{p,\alpha}$

Let $\rho : [-\pi, \pi] \to (0, +\infty)$ be some weight function and consider the space $M_\rho^{p,\alpha}$. It is easy to see that if $\rho \in L^{p,\alpha}$, then $C[-\pi, \pi] \subset M_\rho^{p,\alpha}$ is true. Indeed, let $f \in C[-\pi, \pi]$. Without loss of generality, we assume that the function f periodically continued on the whole axis. We have

$$|f(x+\delta) - f(x)| \le \|f(\cdot+\delta) - f(\cdot)\|_\infty \to 0, \delta \to 0.$$

Consequently

$$\|f(\cdot+\delta) - f(\cdot)\|_{p,\alpha;\rho} = \|(f(\cdot+\delta) - f(\cdot))\,\rho(\cdot)\|_{p,\alpha} \le$$

$$\le \|f(\cdot+\delta) - f(\cdot)\|_\infty \|\rho(\cdot)\|_{p,\alpha} \to 0, \delta \to 0.$$

Hence, we have $f \in M_\rho^{p,\alpha}$.

Let us show that the set of infinitely differentiable functions is dense in $M_\rho^{p,\alpha}$. Consider the following averaged function

$$\omega_\varepsilon(t) = \begin{cases} c_\varepsilon \exp\left(-\frac{\varepsilon^2}{\varepsilon^2 - |t|^2}\right), & |t| < \varepsilon, \\ 0, & |t| \ge \varepsilon, \end{cases}$$

where

$$c_\varepsilon \int_{-\infty}^{+\infty} \omega_\varepsilon(t)\,dt = 1.$$

Take $\forall f \in M_\rho^{p,\alpha}$ and consider the convolution $f * g$:

$$(f * g)(t) = \int_{-\infty}^{+\infty} f(t-s)\,g(s)\,ds,$$

and let

$$f_\varepsilon(t) = (\omega_\varepsilon * f)(t) = (f * \omega_\varepsilon)(t).$$

It is clear that f_ε is infinitely differentiable on $[-\pi, \pi]$. We have

$$\|f_\varepsilon - f\|_{p,\alpha;\rho} = \left\| \int_{-\infty}^{+\infty} \omega_\varepsilon(s) f(\cdot - s)\, ds - f(\cdot) \right\|_{p,\alpha;\rho} =$$

$$= \left\| \int_{-\infty}^{+\infty} \omega_\varepsilon(s) \left[f(\cdot - s) - f(\cdot) \right] ds \right\|_{p,\alpha;\rho}.$$

Applying Minkowski inequality (1) to this expression, we obtain

$$\|f_\varepsilon - f\|_{p,\alpha;\rho} \leq \int_{-\infty}^{+\infty} \omega_\varepsilon(s) \|f(\cdot - s) - f(\cdot)\|_{p,\alpha;\rho}\, ds =$$

$$= \int_{-\varepsilon}^{\varepsilon} \omega_\varepsilon(s) \|f(\cdot - s) - f(\cdot)\|_{p,\alpha;\rho}\, ds \leq$$

$$= \sup_{|s| \leq \varepsilon} \|f(\cdot - s) - f(\cdot)\|_{p,\alpha;\rho} \to 0, \varepsilon \to 0.$$

Thus, the following theorem is true.

Theorem 4. *Let $\rho \in L^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$. Then infinitely differentiable functions are dense in $M_\rho^{p,\alpha}$.*

Consider the singular operator $S(\cdot)$:

$$Sf(t) = \frac{1}{\pi} \int_\gamma \frac{f(\tau)\, d\tau}{\tau - t}, t \in \gamma.$$

Applying Theorem 1 [10] to the operator S we obtain the following result.

Theorem 5. *Let the weight $\rho(\cdot)$ be defined by the expression (2), where $\Gamma = \gamma$. Then the operator S is bounded in $L_\rho^{p,\alpha}$,$1 < p < +\infty$, $0 < \alpha \leq 1$, i.e. the following inequality holds*

$$\|Sf\|_{p,\alpha;\rho} \leq c \|f\|_{p,\alpha;\rho}, \forall f \in L_\rho^{p,\alpha},$$

if and only if the following inequalities are fulfilled

$$-\frac{\alpha}{p} < \alpha_k < -\frac{\alpha}{p} + 1, k = \overline{1, m}. \quad (4)$$

Let us show that the subspace $M_\rho^{p,\alpha}$ is an invariant with respect to the operator S. It is sufficient to prove that the shift operator S is continuous in $M_\rho^{p,\alpha}$. So, let $f \in M_\rho^{p,\alpha}$ and $\delta \in R$. Consider the shift operator S:

$$(Sf)\left(te^{i\delta}\right) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau)\, d\tau}{\tau - te^{i\delta}}, t \in \gamma.$$

We have

$$(Sf)\left(te^{i\delta}\right) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau)\, d\tau e^{-i\delta}}{\tau e^{-i\delta} - t} =$$

$$= \frac{1}{2\pi i} \int_{\gamma} \frac{f\left(\tau e^{-i\delta} e^{i\delta}\right) d\tau e^{-i\delta}}{\tau e^{-i\delta} - t} = \frac{1}{2\pi i} \int_{\gamma} \frac{f\left(\xi e^{i\delta}\right) d\xi}{\xi - t}.$$

Consequently

$$(Sf)\left(te^{i\delta}\right) - (Sf)(t) = \frac{1}{2\pi i} \int_{\gamma} \frac{f\left(\xi e^{i\delta}\right) - f(\xi)}{\xi - t} d\xi =$$

$$= \left(S\left(f\left(\cdot e^{i\delta}\right) - f(\cdot)\right)\right)(t).$$

Paying attention to Theorem 5, hence we immediately obtain

$$\left\|(Sf)\left(te^{i\delta}\right) - (Sf)(t)\right\|_{p,\,\alpha;\,\rho} \leq$$

$$\leq c \left\|f\left(e^{i\delta}\right) - f(\cdot)\right\|_{p,\,\alpha;\,\rho} \to 0, \delta \to 0,$$

as $f \in M_{\rho}^{p,\,\alpha}$. Thus, the following theorem is true.

Theorem 6. *Let the weight ρ be defined by the expression*

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}, t \in \gamma, \tag{5}$$

where $\{t_k\}_{k=\overline{1,\,m}} \subset \gamma-$are different points. If the inequalities (4) hold, then the operator S boundedly acts in $M_{\rho}^{p,\,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$.

Remark 1. *In previous statements and in their proofs the spaces $L_{\rho}^{p,\,\alpha}$, $M_{\rho}^{p,\,\alpha}(-\pi,\,\pi)$ and $L_{\rho}^{p,\,\alpha}$, $M_{\rho}^{p,\,\alpha}$, are naturally identified, respectively, i.e. $L_{\rho}^{p,\,\alpha} = L_{\rho}^{p,\,\alpha}(-\pi, \pi) = L_{\rho}^{p,\,\alpha}(\gamma)$ & $M_{\rho}^{p,\,\alpha} = M_{\rho}^{p,\,\alpha}(-\pi, \pi) = M_{\rho}^{p,\,\alpha}(\gamma)$.*

Consider the following Cauchy integral

$$(Kf)(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(\xi)\, d\xi}{\xi - z}, z \notin \gamma.$$

Let $f \in L_{\rho}^{p,\,\alpha}$, where the weight $\rho(\cdot)$ is defined by the expression (5). Applying Holder's inequality we obtain

$$\|f\|_{L_1} = \left\|f\rho\rho^{-1}\right\|_{L_1} \leq c\, \|f\rho\|_{p,\,\alpha} \left\|\rho^{-1}\right\|_{q,\,\alpha} =$$

$$= c\, \|f\|_{p,\,\alpha;\,\rho} \left\|\rho^{-1}\right\|_{q;\,\alpha}. \tag{6}$$

Suppose that the following inequalities are fulfilled

$$-\frac{\alpha}{p} < \alpha_k < \frac{\alpha}{q}, k = \overline{1,\,m}, \frac{1}{p} + \frac{1}{q} = 1. \tag{7}$$

Then from (6) it follows that $f \in L_1$. As a result, according to the classical facts, the following Sokhotskii-Plemelj is true

$$f^{\pm}(\xi) = \pm\frac{1}{2} f(\xi) + (Sf)(\xi), \xi \in \gamma, \quad (8)$$

where $f^{+}(\xi)$ (respectively, $f^{-}(\xi)$) boundary values of a Cauchy integral $(Kf)(z)$ on γ inside ω (outside ω). Paying attention to Theorem 6, form (8) we obtain that if $f \in M_{\rho}^{p,\alpha}$, then $f^{\pm} \in M_{\rho}^{p,\alpha}$ and the following inequality holds.

$$\left\| f^{\pm} \right\|_{p,\alpha;\rho} \le c \left\| f \right\|_{p,\alpha;\rho}, \forall f \in M_{\rho}^{p,\alpha}. \quad (9)$$

Assume
$K_z(s) = \frac{e^{is}}{e^{is}-z}$ −Cauchy kernel; $P_z(s) = Re\frac{e^{is}+z}{e^{is}-z}$ − Poisson kernel;
$Q_z(s) = Im\frac{e^{is}+z}{e^{is}-z}$ −is the conjugate Poisson kernel,
(Re−is a real part, Im−is an imaginary part). We have

$$K_z(s) = \frac{1}{2} + \frac{1}{2}\left(P_z(s) + iQ_z(s)\right) = \frac{1}{2} + \frac{1}{2}\frac{e^{is}+z}{e^{is}-z}, z \in \omega. \quad (10)$$

Let $F(z) = u(z) + i\vartheta(z)$ be an analytic function in ω. It is clear that $F \in H_{\rho}^{p,\alpha}$ if and only if $u;\vartheta \in h_{\rho}^{p,\alpha}$. Paying attention to the relation (10) we arrive at the conclusion that many of the properties of functions from $h_{\rho}^{p,\alpha}$ transferred to the function from $H_{\rho}^{p,\alpha}$. For example, $\forall F \in H_{\rho}^{p,\alpha}$ has a.e. on γ the nontangential boundary values F^{+}, since, $\lim\limits_{r\to 1-0} F\left(re^{it}\right) = F^{+}\left(e^{it}\right)$, a.e. $t \in [-\pi, \pi]$. We have

$$F^{+}(\tau) = u^{+}(\tau) + i\vartheta^{+}(\tau), \tau \in \gamma.$$

Let all the conditions of Theorem 2 be fulfilled. Then the following representation is true.

$$u\left(re^{i\theta}\right) = \left(P_r * u^{+}\right)(\theta), \vartheta\left(re^{i\theta}\right) = \left(P_r * u^{+}\right)(\theta),$$

and, as a result

$$F\left(re^{i\theta}\right) = \left(P_r * F^{+}\right)(\theta).$$

Paying attention to Theorem 2, we obtain the following result.

Theorem 7. *Let the weight $\rho(\cdot)$ is defined by the expression* (5) *and the inequality* (7) *holds. Then the Sokhotskii-Plemelj formula* (8) *is valid and for the boundary values the inequality* (9) *holds .*

Let us prove the following

Theorem 8. *Let $f(\cdot) \in L_{\infty} \bigcap M_{\rho}^{p,1} \wedge g(\cdot) \in M_{\rho}^{p,\alpha}$ and let the weight function ρ satisfies the following condition*

$$\exists \delta_0 > 0 : \frac{1}{|I|^{1-\alpha}} \int_I \rho^p(t)\,dt \le c\,|I|^{\delta_0}, \forall I \in [-\pi, \pi].$$

Then $f(\cdot)\,g(\cdot) \in M_{\rho}^{p,\alpha}$ when $0 < \alpha \le 1$ and $p \ge 1$.

Proof. Let $f(\cdot) \in L_\infty \bigcap M_\rho^{p,1}$ and $g(\cdot) \in M_\rho^{p,\alpha}$, $0 < \alpha \le 1$, $p \ge 1$. For $\alpha = 1$ it is evident that $M_\rho^{p,1} = L_{p,\rho}$ and the following estimation is true

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|^p \rho^p(t)\, dt \le c \int_{-\pi}^{\pi} |g(t)|^p \rho^p(t)\, dt = c\, \|g\|_{p,\rho}^p < +\infty.$$

So, $f(\cdot)\, g(\cdot) \in M_\rho^{p,\alpha}$ when $\alpha = 1$.

Consider the case $0 < \alpha < 1$. We have

$$\sup_I \left(\frac{1}{|I|^{1-\alpha}} \int_I |f(t)\, g(t)|^p \rho^p(t)\, dt \right)^{\frac{1}{p}} \le$$

$$\le c \sup_I \left(\frac{1}{|I|^{1-\alpha}} \int_I |g(t)|^p \rho^p(t)\, dt \right)^{\frac{1}{p}} = c\, \|g\|_{p,\alpha;\rho} < +\infty.$$

Let us consider

$$\Delta_\delta = \|f(\cdot + \delta)\, g(\cdot + \delta) - f(\cdot)\, g(\cdot)\|_{p,\alpha;\rho}.$$

For any $\varepsilon > 0$ and $m > 0$ there is a $\varphi(\cdot) \in C[-\pi;\pi]$ such that $\|g(\cdot) - \varphi(\cdot)\|_{p,\alpha;\rho} < \frac{\varepsilon}{m}$, as $g \in M_\rho^{p,\alpha}$. We have

$$\Delta_\delta = \|f(\cdot + \delta)[g(\cdot + \delta) - \varphi(\cdot + \delta) + \varphi(\cdot + \delta)] - f(\cdot)[g(\cdot) - \varphi(\cdot) + \varphi(\cdot)]\|_{p,\alpha;\rho} \le$$

$$\le c_f\, \|g(\cdot + \delta) - \varphi(\cdot + \delta)\|_{p,\alpha;\rho} + \|f(\cdot + \delta)\, \varphi(\cdot + \delta) - f(\cdot)\, \varphi(\cdot)\|_{p,\alpha;\rho} + c_f\, \|g(\cdot) - \varphi(\cdot)\|_{p,\alpha;\rho},$$

where $c_f = \|f(\cdot)\|_{L_\infty}$. From $\|g(\cdot) - \varphi(\cdot)\|_{p,\alpha;\rho} < \frac{\varepsilon}{m}$ it follows

$$\|g(\cdot + \delta) - \varphi(\cdot + \delta)\|_{p,\alpha;\rho} \le \|g(\cdot + \delta) - g(\cdot)\|_{p,\alpha;\rho} +$$

$$+ \|g(\cdot) - \varphi(\cdot)\|_{p,\alpha;\rho} + \|\varphi(\cdot) - \varphi(\cdot + \delta)\|_{p,\alpha;\rho}.$$

It is obvious that $\|g(\cdot + \delta) - g(\cdot)\|_{p,\alpha;\rho} \to 0$, $\delta \to 0$.

Let the weight function ρ satisfies the following condition

$$\exists \delta_0 > 0 : \frac{1}{|I|^{1-\alpha}} \int_I \rho^p(t)\, dt \le c\, |I|^{\delta_0}, \forall I \in [-\pi;\pi],$$

where $c > 0$ is some constant.

It follows from uniformly continuity that for $\forall m > 0$, $\varepsilon > 0$ there exists $\delta_1 > 0 : \forall \delta \in (-\delta_1, \delta_1)$:

$$\|\varphi(\cdot) - \varphi(\cdot + \delta)\|_{p,\alpha;\rho} = \sup_I \left(\frac{1}{|I|^{1-\alpha}} \int_I |\varphi(t) - \varphi(t + \delta)|^p \rho^p(t) \right)^{\frac{1}{p}} <$$

$$< \frac{\varepsilon}{m} \sup_I \left(\frac{1}{|I|^{1-\alpha}} \int_I \rho^p(t) \right)^{\frac{1}{p}} < \frac{\varepsilon}{m} \sup_I \left(c\, |I|^{\delta_0} \right)^{\frac{1}{p}} = c \frac{\varepsilon}{m} (2\pi)^{\frac{\delta_0}{p}}.$$

Then the previous inequality implies

$$\Delta_\delta \le c_f \frac{\varepsilon}{m}\left(2 + c\,(2\pi)^{\frac{\delta_0}{p}}\right) + \left\| f\,(\cdot+\delta)\,\varphi\,(\cdot+\delta) - f\,(\cdot)\,\varphi\,(\cdot)\right\|_{p,\,\alpha;\,\rho}.$$

Thus, it is suffices to prove that for $\varphi\,(\cdot) \in C\,[-\pi;\,\pi]$ it is true

$$\lim_{\delta\to 0}\left\| f\,(\cdot+\delta)\,\varphi\,(\cdot+\delta) - f\,(\cdot)\,\varphi\,(\cdot)\right\|_{p,\,\alpha;\,\rho} = 0.$$

We have

$$\left\| f\,(\cdot+\delta)\,\varphi\,(\cdot+\delta) - f\,(\cdot)\,\varphi\,(\cdot)\right\|_{p,\,\alpha;\,\rho} \le \left\| f\,(\cdot+\delta)\,[\varphi\,(\cdot+\delta) - \varphi\,(\cdot)]\right\|_{p,\,\alpha;\,\rho} +$$

$$+ \left\| [f\,(\cdot+\delta) - f\,(\cdot)]\,\varphi\,(\cdot)\right\|_{p,\,\alpha;\,\rho} \le c_f \left\|\varphi\,(\cdot+\delta) - \varphi\,(\cdot)\right\|_{p,\,\alpha;\,\rho} + c_\varphi \left\| f\,(\cdot+\delta) - f\,(\cdot)\right\|_{p,\,\alpha;\,\rho}.$$

where $c_\varphi = \left\|\varphi\,(\cdot)\right\|_{L_\infty}$. Let us take

$$\Delta_\delta\,(f) = \left\| f\,(\cdot+\delta) - f\,(\cdot)\right\|_{p,\,\alpha;\,\rho}.$$

Let $\vartheta > 0$ be an arbitrary number. We have

$$\Delta_\delta\,(f) = \max\left\{\Delta_\delta^{(1)}\,(f)\,,\,\Delta_\delta^{(1)}\,(f)\right\},$$

where

$$\Delta_\delta^{(1)}\,(f) = \sup_{I:|I|\le\vartheta}\left(\frac{1}{|I|^{1-\alpha}}\int_I |f\,(t+\delta) - f\,(t)|^p\,\rho^p\,(t)\,dt\right)^{\frac{1}{p}},$$

$$\Delta_\delta^{(21)}\,(f) = \sup_{I:|I|\le\vartheta}\left(\frac{1}{|I|^{1-\alpha}}\int_I |f\,(t+\delta) - f\,(t)|^p\,\rho^p\,(t)\,dt\right)^{\frac{1}{p}}.$$

Regarding $\Delta_\delta^{(1)}\,(f)$, we have

$$\Delta_\delta^{(1)}\,(f) \le 2c_f \sup_{|I|\le\vartheta}\left(\frac{1}{|I|^{1-\alpha}}\int_I \rho^p\,(t)\,dt\right)^{\frac{1}{p}} \le$$

$$\le 2c_f \sup_{|I|\le\vartheta}\,(|I|)^{\frac{\delta_0}{p}} = \tilde{c}\vartheta^{\frac{\delta_0}{p}}.$$

Regarding $\Delta_\delta^{(2)}\,(f)$, we have

$$\Delta_\delta^{(2)}\,(f) \le \vartheta_\varepsilon^{\frac{\alpha-1}{p}} \sup_{|I|\ge\vartheta}\left(\int_I |f\,(t+\delta) - f\,(t)|^p\,\rho^p\,(t)\,dt\right)^{\frac{1}{p}} \le$$

$$\le \vartheta_\varepsilon^{\frac{\alpha-1}{p}}\left(\int_{-\pi}^{\pi} |f\,(t+\delta) - f\,(t)|^p\,\rho^p\,(t)\,dt\right)^{\frac{1}{p}} \le$$

$$\vartheta_{\varepsilon}^{\frac{\alpha-1}{p}} \left\| f\left(\cdot+\delta\right)-f\left(\cdot\right)\right\|_{p,\,\rho}.$$

where $\vartheta=\left(\frac{\varepsilon}{\tilde{c}}\right)^{\frac{p}{\delta_0}}:=\vartheta_\varepsilon$. It is clear that $\exists\delta_2>0$:

$$\left\| f\left(\cdot+\delta\right)-f\left(\cdot\right)\right\|_{p,\,\rho}<\vartheta_{\varepsilon}^{\frac{1}{p}},\forall\delta\in\left(-\delta_2,\,\delta_2\right),$$

where we can choose $\varepsilon_1=\vartheta_{\varepsilon}^{\frac{1}{p}}$ for any $\varepsilon_1>0$. Hence we get $\Delta_{\delta}^{(2)}\left(f\right)\leq\left(\frac{\varepsilon}{\tilde{c}}\right)^{\frac{\alpha}{\delta_0}}$.
Now let us take $\Delta_\delta\left(f\right)\leq\max\left\{\varepsilon_2,\,\left(\frac{\varepsilon_2}{\tilde{c}}\right)^{\frac{\alpha}{\delta_0}}\right\}$, where $\varepsilon_2=\frac{\varepsilon}{m}$ for any $m>0$.
Consequently

$$\Delta_\delta\leq\frac{\varepsilon}{m}\left(c_f\left(2+2c\left(2\pi\right)\right)^{\frac{\delta_0}{p}}+c_\varphi\right),\forall\delta\in\left(-\delta_3,\,\delta_3\right),$$

where $\delta_3=\min\left\{\delta_1,\,\delta_2\right\}$. By taking $m=\left(c_f\left(2+2c\left(2\pi\right)\right)^{\frac{\delta_0}{p}}+c_\varphi\right)$, we get $\Delta_\delta\leq\varepsilon$, $\forall\delta\in\left(-\delta_3,\delta_3\right)$. It follows that $\Delta_\delta\to0$ as $\delta\to0$. ◄ □

Acknowledgements

The authors would like to express their profound gratitude to Prof. Bilal Bilalov for his valuable guidance on writing this article.

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