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On the Space of Coefficients in the Intuitionistic Fuzzy Normed Spaces

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Abstract. Intuitionistic fuzzy normed space is defined using the concepts of t-norm and t-conorm. The concepts of fuzzy completeness, fuzzy minimality, fuzzy biorthogonality, fuzzy basicity and fuzzy space of coefficients are introduced. Weak completeness of fuzzy space of coefficients with regard to fuzzy norm and weak basicity of canonical system in this space are proved. Weak basicity criterion in fuzzy Banach space is presented in terms of coefficient operator.

Key Words and Phrases: fuzzy basicity; fuzzy completeness; fuzzy minimality; fuzzy biorthogonality; fuzzy space of coefficients

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1. Introduction

The concept of the space of coefficients belongs to the theory of bases. As is known, every basis in a Banach space has a Banach space of coefficients which is isomorphic to an initial one (see, e.g., [1;2]). Every nondegenerate system (to be defined later) in a Banach space generates the corresponding Banach space of coefficients with canonical basis (see, e.g., [2;3]). Therefore, space of coefficients plays an important role in the study of approximative properties of systems. It has very important applications in various fields of science, such as solid body physics, molecular physics, multiple production of particles, aviation, medicine, biology, data compression, etc (see, e.g., [4;5] and references therein). All these applications are closely related to wavelet analysis, and there arose a great interest in them lately [see, e.g., 5]. It is well known that many topological spaces are nonnormable. Therefore, the study of various properties of the space of coefficients in topological spaces is of special scientific interest.

Applications in various branches of mathematics and natural sciences have lately induced a strong interest toward the study of different research problems in terms of fuzzy structures. More details on this topic can be found in [6-9] and references therein. A large number of research works is appearing these days which deal with the concept of fuzzy set-numbers, and fuzzification of many classical theories has also been made. The concept of Schauder basis in *intuitionistic fuzzy normed space* and some results related to this concept have recently been studied in [10-12] where the concepts of strongly and weakly intuitionistic fuzzy (Schauder) bases in *intuitionistic fuzzy Banach spaces* (IFBS in short) have been introduced and some of their properties have been revealed. The concepts of *strongly* and *weakly intuitionistic fuzzy approximation properties* (*sif-AP* and *wif-AP* in short, respectively) are also introduced in those works. It is proved that if the intuitionistic fuzzy space has a *wif-basis*, then it has a *wif-AP*. All the results in those works are obtained on condition that IFBS admits equivalent topology using the family of norms generated by *t-norm* and *t-conorm* (we will define them later).

In our work, we define the basic concepts of classical basis theory in *intuitionistic fuzzy normed spaces* (IFNS in short). Concepts of *weak* and *strongly fuzzy spaces of coefficients* are introduced. *Weak completeness* of these spaces with regard to *fuzzy norm* and *weakly basicity* of canonical system in them are proved. *Weak basicity* criterion in fuzzy Banach space is presented in terms of coefficient operator.

In Section 2, we recall some notations and concepts. In Section 3, we state our main results. We first define *fuzzy space of coefficients* and then introduce the corresponding *fuzzy norms.* We prove that for nondegenerate system the corresponding *fuzzy space of coefficients* is *weakly fuzzy complete.* Moreover, we show that the canonical system forms a *weak basis* for this space.

2. Some preliminary notations and concepts

We will use the standard notation: N will denote the set of all positive integers, R will be the set of all real numbers, C will be the set of complex numbers and K will denote a field of scalars ($K \equiv R$, or $K \equiv C$), $R_+ \equiv (0, +\infty)$. δ_{nk} is the Kronecker symbol. Here we state some concepts and facts from IFNS theory to be used later.

One of the most important problems in fuzzy topology is to obtain an appropriate concept of intuitionistic fuzzy normed space. This problem has been investigated by Park [19]. He has introduced and studied a notion of intuitionistic fuzzy metric space. We recall it.

Definition 1. *A binary operation* $* : [0,1]^2 \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

(a) $*$ *is associative and commutative;*

(b) $*$ *is continuous;*

(c) $a * 1 = a$, $\forall a \in [0,1]$*;*

(d) $a * b \leq c * d$ *whenever* $a \leq c$ *and* $b \leq d$, $\forall a, b, c, d \in [0,1]$.

Example 1. Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min\{a; b\}$.

Definition 2. *A binary operation* $\diamondsuit : [0,1]^2 \to [0,1]$ *is a continuous t-conorm if it satisfies the following conditions:*

(α) $\diamondsuit$ *is associative and commutative;*

(β) $\diamondsuit$ *is continuous;*

(γ) $a\diamondsuit 0 = a, \forall a \in [0,1]$;
(η) $a\diamondsuit b \le c\diamondsuit d$ *whenever* $a \le c$ *and* $b \le d$, $\forall a,b,c,d \in [0,1]$.

Example 2. Two typical examples of continuous t-conorm are $a\diamondsuit b = \min\{a+b;\ 1\}$ and $a\diamondsuit b = \max\{a;b\}$.

Definition 3. *Let X be a linear space over a field K. Functions $\mu;\ \nu : X \times R \to [0\,,\,1]$ are called fuzzy norms on X if the following conditions hold:*

1. $\mu(x;t) = 0\,,\ \forall t \le 0\,, \forall x \in X$;
2. $\mu(x;t) = 1\,,\ \forall t > 0 \Rightarrow\ x = 0$;
3. $\mu(cx;t) = \mu\left(x;\ \frac{t}{|c|}\right)\,,\ \forall c \ne 0\,;$
4. $\mu(x;\,\cdot\,) : R \to [0,1]$ *is a non-decreasing function of t for* $\forall x \in X$ *and* $\lim\limits_{t\to\infty}\mu(x;t) = 1$, $\forall x \in X$*;*
5. $\mu(x;\ s) * \mu(y;t) \le \mu(x+y;\ s+t)\ ,\ \forall x,y \in X\,,\ \forall s,t \in R$;
6. $\nu(x;t) = 1\,,\ \forall t \le 0\,,\ \forall x \in X$;
7. $\nu(x;t) = 0,\ \forall t < 0 \Rightarrow\ x = 0$;
8. $\nu(cx;t) = \nu\left(x;\ \frac{t}{|c|}\right)\,,\ \forall c \ne 0\,;$
9. $\nu(x;\,\cdot\,) : R \to [0,1]$ *is a non-increasing function of t for* $\forall x \in X$ *and* $\lim\limits_{t\to\infty}\nu(x;t) = 0$, $\forall x \in X$*;*
10. $\nu(x;\ s)\diamondsuit\nu(y;\ t) \ge \nu(x+y;s+t)\ ,\forall x,y \in X\,,\ \forall s,t \in R$;
11. $\mu(x;\ t) + \nu(x;t) \le 1,\ \forall x \in X\,,\ \forall t \in R.$

Then the 5-tuple $(X;\mu;\ \nu;\ *;\ \diamondsuit)$ is said to be an intuitionistic fuzzy normed space (shotly IFNS).

Example 3. Let $(X\,;\|\cdot\|)$ be a normed space. Denote $a*b = ab$ and $a\diamondsuit b = \min\{a+b;\ 1\}$**,** for $\forall a,b \in [0,1]$ and define μ and ν as follows:

$$\mu(x;t) = \begin{cases} \frac{t}{t+\|x\|},\ t > 0, \\ 0\ ,\ t \le 0\,, \end{cases}$$

$$\nu(x;t) = \begin{cases} \frac{\|x\|}{t+\|x\|},\ t > 0, \\ 1\ ,\ t \le 0\,. \end{cases}$$

Then $(X;\mu;\ \nu;\ *;\ \diamondsuit)$ is an IFNS.

The above concepts allow to introduce the following kinds of convergence (or topology) in IFNS:

Definition 4. *Let $(X;\mu;\nu)$ be a fuzzy normed space and let $\{x_n\}_{n\in N}\subset X$ be some sequence. Then it is said to be strongly intuitionistic fuzzy convergent to $x\in X$ (denoted by $x_n \overset{s}{\to} x,\ n\to\infty$, or $s\text{-}\lim\limits_{n\to\infty} x_n = x$ in short) if and only if for $\forall\varepsilon>0,\ \exists n_0=n_0(\varepsilon)$: $\mu(x_n-x;\,t)\geq 1-\varepsilon,\ \nu(x_n-x;\,t)\leq\varepsilon\,,\ \forall n\geq n_0\,,\ \forall t\in R$.*

Definition 5. *Let $(X;\mu;\nu)$ be a fuzzy normed space and $\{x_n\}_{n\in N}\subset X$ be some sequence. Then it is said to be weakly intuitionistic fuzzy convergent to $x\in X$ (denoted by $x_n \overset{w}{\to} x,\ n\to\infty$, or $w\text{-}\lim\limits_{n\to\infty} x_n = x$ in short) if and only if for $\forall t\in R_+\,,\ \forall\varepsilon>0,\ \exists n_0 = n_0(\varepsilon;t):\ \mu(x_n-x;\,t)\geq 1-\varepsilon,\ \nu(x_n-x;\,t)\leq\varepsilon\,,\ \forall n\geq n_0$.*

More details on these concepts can be found in [13-22].

Let $(X;\mu;\nu)$ be an IFNS, and let $M\subset X$ be some set. By $L[M]$ we denote the linear span of M in X. The weakly (strongly) intuitionistic fuzzy convergent closure of $L[M]$ will be denoted by $\overline{L_w[M]}$ ($\overline{L_s[M]}$). If X is complete with respect to the weakly (strongly) intuitionistic fuzzy convergence, then we will call it intuitionistic fuzzy weak (strong) Banach space: (IFB_wS or X_w (IFB_sS or X_s) in short). Let X be an IFB_wS (IFB_sS). We denote by X_w^* (X_s^*) the linear space of linear and continuous in IFB_wS (IFB_sS) functionals over the same field K .

Now we define the corresponding concepts of basis theory for IFNS. Let $\{x_n\}_{n\in N}\subset X$ be some system.

Definition 6. *System $\{x_n\}_{n\in N}$ is said w-complete (s-complete) in X_w (in X_s), if $\overline{L_w\left[\{x_n\}_{n\in N}\right]}\equiv X_w$ ($\overline{L_s\left[\{x_n\}_{n\in N}\right]}\equiv X_s$).*

Definition 7. *System $\{x_n^*\}_{n\in N}\subset X_w^*$ ($\{x_n^*\}_{n\in N}\subset X_s^*$) is called w-biorthogonal (s-biorthogonal) to the system $\{x_n\}_{n\in N}$, if $x_n^*(x_k)=\delta_{nk}\,,\ \forall n,k\in N$, where δ_{nk} is the Kronecker symbol.*

Definition 8. *System $\{x_n\}_{n\in N}\subset X_w$ ($\{x_n\}_{n\in N}\subset X_s$) is called w-linearly (s-linearly) independent in X, if $\sum_{n=1}^{\infty}\lambda_n x_n=0$ in X_w (in X_s) implies $\lambda_n=0\,,\ \forall n\in N$.*

Definition 9. *System $\{x_n\}_{n\in N}\subset X_w$ ($\{x_n\}_{n\in N}\subset X_s$) is called w-basis (s-basis) for X_w (for X_s) if $\forall x\in X,\ \exists!\ \{\lambda_n\}_{n\in N}\subset K:\sum_{n=1}^{\infty}\lambda_n x_n = x$ in X_w (in X_s).*

We will also need the following concept.

Definition 10. *System $\{x_n\}_{n\in N}\subset X$ is called nondegenerate if $x_n\neq 0,\ \forall n\in N$.*

3. Main results

3.1. Space of coefficients. Let X be an IFNS and let $\{x_n\}_{n\in N}\subset X$ be some system. Assume that

$$\mathscr{K}_{\bar{x}}^{w}\equiv\left\{\{\lambda_n\}_{n\in N}\subset C:\sum_{n=1}^{\infty}\lambda_n x_n \text{ converges in } X_w\right\};$$

$$\mathscr{K}_{\bar{x}}^{s} \equiv \left\{ \{\lambda_n\}_{n\in N} \subset C : \sum_{n=1}^{\infty} \lambda_n x_n \quad converges \ in \quad X_s \right\}.$$

It is not difficult to see that $\mathscr{K}_{\bar{x}}^{w}$ and $\mathscr{K}_{\bar{x}}^{s}$ are linear spaces with regard to component-specific summation and component-specific multiplication by a scalar. Take $\forall \lambda \equiv \{\lambda_n\}_{n\in N} \in \mathscr{K}_{\bar{x}}^{w}$ and assume

$$\mu_K\left(\bar{\lambda};t\right) = \inf_m \mu\left(\sum_{n=1}^{m} \lambda_n x_n; t\right); \nu_K\left(\bar{\lambda};t\right) = \sup_m \nu\left(\sum_{n=1}^{m} \lambda_n x_n; t\right).$$

Let's show that μ_K and ν_K satisfy the conditions 1)-11).

1) It is clear that $\mu_K\left(\bar{\lambda};t\right) = 0$, $\forall t \le 0$.

2) Let $\mu_K\left(\bar{\lambda};t\right) = 1$, $\forall t > 0$. Hence, $\mu\left(\sum_{n=1}^{m} \lambda_n x_n; t\right) = 1$, $\forall m \in N$, $\forall t > 0$. Suppose that the system $\{x_n\}_{n\in N}$ is nondegenerate. It follows from above-stated relations that for $m = 1$ we have $\mu(\lambda_1 x_1; t) = 1$, $\forall t > 0$. Hence, $\lambda_1 x_1 = 0 \Rightarrow \lambda_1 = 0$. Continuing this way, we get at the end of this process that $\lambda_n = 0$, $\forall n \in N$, i.e. $\bar{\lambda} = 0$;

3) The validity of relation $\mu_K\left(c\bar{\lambda};t\right) = \mu_K\left(\bar{\lambda}; \frac{t}{|c|}\right)$, $\forall c \neq 0$, is beyond any doubt.

4) As $\mu(x; \cdot)$ is a non-decreasing function on R, it is not difficult to see that $\mu_K\left(\bar{\lambda}; \cdot\right)$ has the same property. Let's show that $\lim_{t\to\infty} \mu_K\left(\bar{\lambda};t\right) = 1$.Take $\forall \varepsilon > 0$. Let $S_m = \sum_{n=1}^{m} \lambda_n x_n$ and $w\text{-}\lim_{m\to\infty} S_m = S \in X_w$. It is clear that $\exists t_0 > 0 : \mu(S; t_0) \ge 1 - \varepsilon$. Then it follows from the definition of $\lim_{m} {}_w$ that $\exists m_0(\varepsilon; t_0) : \mu(S_m - S; t_0) \ge 1 - \varepsilon$, $\forall m \ge m_0(\varepsilon; t_0)$. Property 4) implies

$$\mu(S_m; 2t_0) = \mu(S_m - S + S;\ t_0 + t_0) \ge \mu(S_m - S; t_0) * \mu(S; t_0).$$

As a result we get

$$\mu(S_m; t_0) \ge 1 - \varepsilon,\ \forall m \ge m_0(\varepsilon; t_0). \tag{1}$$

As $\mu(x; \cdot)$ is a non-decreasing function of t, it follows from (1) that

$$\mu(S_m; t) \ge 1 - \varepsilon,\ \forall m \ge m_0(\varepsilon; t_0),\ \forall t \ge t_0. \tag{2}$$

We have

$$\mu_K\left(\bar{\lambda};t\right) = \inf_m \mu(S_m; t) = \min\left\{ \mu(S_1; t)\ ; ...; \mu(S_{m_0-1}; t)\ ;\ \inf_{m\ge m_0} \mu(S_m; t) \right\}, \tag{3}$$

where $m_0 = m_0(\varepsilon; t_0)$. As $\lim_{t\to\infty} \mu(S_k; t) = 1$, for $\forall k \in N$, we have $\exists t_k(\varepsilon)\ ; \forall t \ge t_k(\varepsilon) :$ $\mu(S_k; t) \ge 1 - \varepsilon$, $k = \overline{1, m_0 - 1}$. Let $t_\varepsilon^0 = \max\left\{t_k(\varepsilon)\ ,\ k = \overline{1,\ m_0 - 1}\right\}$. Then it is clear that

$$\mu(S_k; t) \ge 1 - \varepsilon, \forall t \ge t_\varepsilon^0. \tag{4}$$

It follows from (2) and (3) that

$$\inf_{m\geq m_0}\mu\left(S_m;t\right)\geq 1-\varepsilon,\ \forall t\geq t_0.$$

Let $t_\varepsilon=\max\left\{t_0;\,t_\varepsilon^0\right\}$. Hence we obtain from (3) and (4):

$$\mu_K\left(\bar{\lambda};t\right)\geq 1-\varepsilon,\ \forall t\geq t_\varepsilon.$$

Thus, $\lim_{t\to\infty}\mu_K\left(\bar{\lambda};t\right)=1\,,\ \forall\bar{\lambda}\in\mathscr{K}_{\bar{x}}^w$.

5) Let $\bar{\lambda},\bar{\mu}\in\mathscr{K}_{\bar{x}}^w\left(\bar{\lambda}\equiv\{\lambda_n\}_{n\in N}\,;\,\bar{\mu}\equiv\{\mu_n\}_{n\in N}\right)$ and $s,t\in R$. We have

$$\mu_K\left(\bar{\lambda}+\bar{\mu};s+t\right)=\inf_m\mu\left(\sum_{n=1}^m\left(\lambda_n+\mu_n\right)x_n;s+t\right)=\inf_m\mu\left(\sum_{n=1}^m\lambda_nx_n+\right.$$

$$\left.+\sum_{n=1}^m\mu_nx_n;s+t\right)\geq\inf_m\left[\mu\left(\sum_{n=1}^m\lambda_nx_n;s\right)*\mu\left(\sum_{n=1}^m\mu_nx_n;t\right)\right]=$$

$$=\left[\inf_m\mu\left(\sum_{n=1}^m\lambda_nx_n;s\right)\right]*\left[\inf_m\mu\left(\sum_{n=1}^m\mu_nx_n;t\right)\right]=\mu\left(\bar{\lambda};\,s\right)*\mu\left(\bar{\mu};t\right).$$

6) As $\nu\left(x;t\right)=1$, $\forall t\leq 0$, it is clear that $\nu_K\left(\bar{\lambda};t\right)=1\,,\ \forall t\leq 0\,,\ \forall\bar{\lambda}\in\mathscr{K}_{\bar{x}}^w$.

7) Let the system $\{x_n\}_{n\in N}$ be nondegenerate. Assume that $\nu_K\left(\bar{\lambda};t\right)=0$, $\forall t>0$. Then $\nu\left(\sum_{n=1}^m\lambda_nx_n\,;\,t\right)=0$, $\forall t>0, \forall m\in N$. For $m=1$ we have $\nu\left(\lambda_1x_1;t\right)=0$, $\forall t>0\Rightarrow\lambda_1x_1=0\Rightarrow\lambda_1=0$. Continuing this process , we get $\lambda_n=0,\ \forall n\in N\Rightarrow\bar{\lambda}=0$.

8) Clearly, $\nu_K\left(c\bar{\lambda};t\right)=\nu_K\left(\bar{\lambda};\frac{t}{|c|}\right)$, $\forall c\neq 0$.

9) It follows from the property 9) that $\nu\left(x\,;\,\cdot\right)$ is a non-increasing function on R. Therefore, $\nu_K\left(\bar{\lambda}\,;\,\cdot\right)$ is a non-increasing function on R. Let us show that $\lim_{t\to\infty}\nu_K\left(\bar{\lambda}\,;\,t\right)=0$. Let $S_m=\sum_{n=1}^m\lambda_nx_n$ and $w\text{-}\lim_{m\to\infty}S_m=S\in X$. Take $\forall\varepsilon>0$. It is clear that $\exists t_0>0\,:\ \nu\left(S;\,t_0\right)\leq\varepsilon$. Then it follows from the definition of $\lim_{m}{}_w$ that $\exists m_0=m_0\left(\varepsilon;t_0\right):$ $\nu\left(S_m-S;t_0\right)\leq\varepsilon$, $\forall m\geq m_0$. We have

$$\nu\left(S_m;t_0\right)=\nu\left(S_m-S+S;\,t_0+t_0\right)\leq\nu\left(S_m-S;t_0\right)\diamondsuit\,\nu\left(S;t_0\right)\leq\varepsilon,\forall m\geq m_0.$$

As $\nu\left(x\,;\,\cdot\right)$ is a non-increasing function, it is clear that

$$\nu\left(S_m;t\right)\leq\varepsilon,\forall m\geq m_0,\forall t\geq t_0. \tag{5}$$

We have

$$\nu_K\left(\bar{\lambda};t\right)=\sup_m\nu\left(S_m;t\right)=\max\left\{\nu\left(S_1;t\right)\,;...;\nu\left(S_{m_0-1};t\right)\,;\,\sup_{m\geq m_0}\nu\left(S_m;t\right)\right\}.$$

As $\lim\limits_{t\to\infty}\nu(S_k;t)=0$ for $\forall k\in N$, we have $\exists t_k(\varepsilon)$; $\forall t\geq t_k(\varepsilon)$: $\nu(S_k;t)\leq\varepsilon$, $k=\overline{1,m_0-1}$. Let $t_\varepsilon^0=\max\left\{t_k(\varepsilon)\ ,\ k=\overline{1,\,m_0-1}\right\}$. It is clear that $\nu(S_k;t)\leq\varepsilon$, $\forall t\geq t_\varepsilon^0$. It follows from (5) that $\sup\limits_{m\geq m_0}\nu(S_m;t)\leq\varepsilon$, $\forall t\geq t_0$. Let $t_\varepsilon=\max\left\{t_0;\,t_\varepsilon^0\right\}$. Then it is clear that $\nu_K\left(\bar\lambda;t\right)\leq\varepsilon$, $\forall t\geq t_\varepsilon\Rightarrow\lim\limits_{t\to\infty}\nu_K\left(\bar\lambda;t\right)=0$.

10) Let $\bar\lambda,\bar\mu\in\mathscr{K}_{\bar x}^w$ $\left(\bar\lambda\equiv\{\lambda_n\}_{n\in N}\,;\,\bar\mu\equiv\{\mu_n\}_{n\in N}\right)$ and $s,t\in R$. We have

$$\nu_K\left(\bar\lambda+\bar\mu;s+t\right)=\sup_m\nu\left(\sum_{n=1}^m(\lambda_n+\mu_n)\,x_n;s+t\right)\leq\sup_m\left[\nu\left(\sum_{n=1}^m\lambda_nx_n;s\right)\diamondsuit\right.$$

$$\left.\nu\left(\sum_{n=1}^m\mu_nx_n;t\right)\right]=\left[\sup_m\nu\left(\sum_{n=1}^m\lambda_nx_n;s\right)\right]\diamondsuit\left[\sup_m\nu\left(\sum_{n=1}^m\mu_nx_n;t\right)\right]=$$

$$=\nu_K\left(\bar\lambda;\,s\right)\diamondsuit\,\nu_K\left(\bar\mu;t\right).$$

11) $\mu_K\left(\bar\lambda;t\right)+\nu_K\left(\bar\lambda;t\right)=\inf\limits_m\mu\left(\sum_{n=1}^m\lambda_nx_n;t\right)+\sup\limits_m\nu\left(\sum_{n=1}^m\lambda_nx_n;t\right)\leq$

$$\leq\sup_m\left[\mu\left(\sum_{n=1}^m\lambda_nx_n;t\right)+\nu\left(\sum_{n=1}^m\lambda_nx_n;t\right)\right]\leq1,\forall\bar\lambda\in\mathscr{K}_{\bar x}^w,\forall\lambda\in R.$$

Thus, we have proved the validity of the following

Theorem 1. *Let $(X;\mu;\nu)$ be a fuzzy normed space and let $\{x_n\}_{n\in N}\subset X$ be a nondegenerate system. Then the space of coefficients $(\mathscr{K}_{\bar x}^w;\mu_K;\nu_K)$ is also weakly fuzzy normed space.*

The following theorem is proved in absolutely the same way.

Theorem 2. *Let $(X;\mu;\nu)$ be a fuzzy normed space and let $\{x_n\}_{n\in N}\subset X$ be a nondegenerate system. Then the space of coefficients $(\mathscr{K}_{\bar x}^s;\mu_K;\nu_K)$ is also strongly fuzzy normed space.*

3.2. Completeness of the space of coefficients. In the sequel, we assume that $(X;\mu;\nu)$ is *IFBS*. Let us show that $(\mathscr{K}_{\bar x}^w;\mu_K;\nu_K)$ is a weakly fuzzy complete normed space. First we prove the following

Lemma 1. *Let $x_0\neq0\,,x_0\in X$, and let $\{\lambda_n\}_{n\in N}\subset R$ be some sequence. If w-$\lim\limits_{n\to\infty}(\lambda_nx_0)=0$, i.e. for $\forall\varepsilon>0$, $\exists n_0=n_0(\varepsilon;t)$: $\mu(\lambda_nx_0\,;\,t)>1-\varepsilon$ $(\nu(\lambda_nx_0\,;\,t)<\varepsilon)$, $\forall n\geq n_0$; then $\lambda_n\to0\,,\,n\to\infty$.*

Proof. As $x_0\neq0$, it is clear that $\exists t_0>0:\mu(x_0\,;\,t_0)<1$ (this follows from the Property 1). For $\forall t>0$ we have $\mu(\lambda_nx_0\,;\,t)=\mu\left(x_0\,;\frac{t}{|\lambda_n|}\right)$, if $\lambda_n\neq0$. Let the $\{\lambda_n\}_{n\in N}$ does not converge to zero. Then $\exists\{\lambda_{n_k}\}_{k\in N}$ and $\exists\delta>0:|\lambda_{n_k}|\geq\delta\,,\,\forall k\in N\Rightarrow\frac{t}{|\lambda_{n_k}|}\leq\frac{t}{\delta}$. As $\mu(x_0\,;\,\cdot)$ is a nondecreasing function of t, then $\mu\left(x_0;\,\frac{t}{|\lambda_{n_k}|}\right)\leq\mu\left(x_0;\,\frac{t}{\delta}\right)$, $\forall t\in R_+$. Take

$\tilde{t} = \delta\, t_0$. We have $\mu\left(\lambda_{n_k} x_0\, ;\, \tilde{t}\right) \le \mu\left(x_0\, ;\, t_0\right) < 1, \forall k \in N$. So we came upon a contradiction which proves the Lemma. □

In the sequel, we also assume that the following condition is fulfilled.

12) *The functions* $\mu(x\, ;\, \cdot),\ \nu(x\, ;\, \cdot) : R \to [0,1]$ *are continuous for* $\forall x \in X$.

Take *w-fundamental* sequence $\left\{\bar{\lambda}_n\right\}_{n \in N} \subset \mathscr{K}_{\bar{x}}^{w}$, $\bar{\lambda}_n \equiv \left\{\lambda_k^{(n)}\right\}_{k \in N}$. Then $\lim\limits_{n,m \to \infty} \mu_K\left(\bar{\lambda}_n - \bar{\lambda}_m\, ;\, t\right) = 1, \forall t \in R$, i.e.

$$\lim_{n,m \to \infty} \inf_r \mu\left(\sum_{k=1}^{r}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right) x_k;\, t\right) = 1, \forall t \in R.$$

Take $\forall k_0 \in N$ and fix it. We have

$$\left(\lambda_{k_0}^{(n)} - \lambda_{k_0}^{(m)}\right) x_{k_0} = \sum_{k=1}^{k_0}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right) x_k - \sum_{k=1}^{k_0-1}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right) x_k.$$

Then from Property 5) we get

$$\mu\left(\left(\lambda_{k_0}^{(n)} - \lambda_{k_0}^{(m)}\right) x_{k_0}\, ;\, t\right) \ge \mu\left(\sum_{k=1}^{k_0}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right) x_k;\, \frac{t}{2}\right) * \mu\left(\sum_{k=1}^{k_0-1}\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right) x_k\, ;\, \frac{t}{2}\right).$$

It follows directly from this relation that

$$\lim_{n,m \to \infty} \mu\left(\left(\lambda_{k_0}^{(n)} - \lambda_{k_0}^{(m)}\right) x_{k_0};\, t\right) = 1, \forall t \in R.$$

As $x_{k_0} \neq 0$, Lemma 1 implies $\lim\limits_{n,m \to \infty}\left|\lambda_{k_0}^{(n)} - \lambda_{k_0}^{(m)}\right| = 0$, i.e. sequence $\left\{\lambda_{k_0}^{(n)}\right\}_{n \in N}$ is fundamental in R. Let $\lambda_{k_0}^{(n)} \to \lambda_{k_0}$, as $n \to \infty$. Denote $\bar{\lambda} \equiv \left\{\lambda_n\right\}_{n \in N}$. Let us show that $\lim\limits_{n \to \infty} \mu_K\left(\bar{\lambda}_n - \bar{\lambda};\, t\right) = 1, \forall t \in R$. Take $\forall \varepsilon > 0, \forall t > 0$. It is clear that

$$\exists n_0 = n_0(\varepsilon; t) : \mu_K\left(\bar{\lambda}_n - \bar{\lambda}_{n+p};\, t\right) > 1 - \varepsilon, \forall n \ge n_0\, , \forall p \in N.$$

Consequently

$$\inf_r \mu\left(\sum_{k=1}^{r}\left(\lambda_k^{(n)} - \lambda_k^{(n+p)}\right) x_k; t\right) > 1 - \varepsilon, \forall n \ge n_0\, , \forall p \in N.$$

Hence

$$\mu\left(\sum_{k=1}^{r}\left(\lambda_k^{(n)} - \lambda_k^{(n+p)}\right) x_k; t\right) > 1 - \varepsilon, \forall n \ge n_0\, , \forall r, p \in N. \tag{6}$$

As shown above, $\lim\limits_{n,m \to \infty} \mu\left(\left(\lambda_k^{(n)} - \lambda_k^{(m)}\right) x_k; t\right) = 1, \forall t \in R$. Now let's take into account the fact that $\lim\limits_{m \to \infty} \mu\left(\lambda_k^{(m)} x_k; t\right) = \mu\left(\lambda_k x_k; t\right), \forall t \in R_+$. Indeed, if $\lambda_k = 0$, then $\mu(0\, ;\, t) =$

1, $\forall t \in R_+$, and clearly, $\lim_{m\to\infty} \mu\left(\lambda_k^{(m)} x_k; t\right) = 1$, $\forall t \in R_+$. If $\lambda_k \neq 0$, then for sufficiently large values of m we have $\lambda_k^{(m)} \neq 0$, and as a result

$$\mu\left(\lambda_k^{(m)} x_k; t\right) = \mu\left(x_k; \frac{t}{\left|\lambda_k^{(m)}\right|}\right) \overset{m\to\infty}{\rightarrow} \mu\left(x_k; \frac{t}{|\lambda_k|}\right) = \mu\left(\lambda_k x_k; t\right), \forall t \in R_+.$$

Passage to the limit in the inequality (6) as $p \to \infty$ yields

$$\mu\left(\sum_{k=1}^{r}\left(\lambda_k^{(n)} - \lambda_k\right) x_k; t\right) \geq 1 - \varepsilon, \forall n \geq n_0\,, \forall r \in N. \tag{7}$$

We have

$$\mu\left(\sum_{k=r}^{r+p}\left(\lambda_k^{(n)} - \lambda_k\right) x_k; t\right) = \mu\left(\sum_{k=1}^{r+p}\left(\lambda_k^{(n)} - \lambda_k\right) x_k - \sum_{k=1}^{r-1}\left(\lambda_k^{(n)} - \lambda_k\right) x_k;\, t\right) \geq$$

$$\geq \mu\left(\sum_{k=1}^{r+p}\left(\lambda_k^{(n)} - \lambda_k\right) x_k\,;\, \frac{t}{2}\right) * \mu\left(\sum_{k=1}^{r-1}\left(\lambda_k^{(n)} - \lambda_k\right) x_k\,;\, \frac{t}{2}\right) \geq 1 - \varepsilon,$$

$$\forall n \geq n_0\,, \forall r, p \in N.$$

As $\bar{\lambda}_n \in \mathscr{K}_{\bar{x}}^w$, it is clear that $\exists m_0^{(n)} : \forall m \geq m_0^{(n)}\,, \forall p \in N$:

$$\mu\left(\sum_{k=m}^{m+p} \lambda_k^{(n)} x_k; t\right) > 1 - \varepsilon.$$

We have

$$\mu\left(\sum_{k=m}^{m+p} \lambda_k x_k; t\right) = \mu\left(\sum_{k=m}^{m+p}\left(\lambda_k - \lambda_k^{(n)}\right) x_k + \sum_{k=m}^{m+p} \lambda_k^{(n)} x_k\,;\, t\right) \geq$$

$$\geq \mu\left(\sum_{k=m}^{m+p}\left(\lambda_k - \lambda_k^{(n)}\right) x_k\,;\, \frac{t}{2}\right) * \mu\left(\sum_{k=m}^{m+p} \lambda_k^{(n)} x_k\,;\, \frac{t}{2}\right) \geq 1 - \varepsilon, \forall m \geq m_0^{(n)}\,, \forall p \in N.$$

It follows that the series $\sum_{k=1}^{\infty} \lambda_k x_k$ is weakly fuzzy convergent in X_w, i.e. $\exists\, w\text{-}\lim_{m\to\infty} \sum_{k=1}^{m} \lambda_k x_k$. Consequently, $\bar{\lambda} \in \mathscr{K}_{\bar{x}}^w$ and the relation (7) implies that $\lim_{n\to\infty} \mu_K\left(\bar{\lambda}_n - \bar{\lambda}; t\right) = 1, \forall t \in R_+$. It can be proved in similar way that $\lim_{n\to\infty} \nu_K\left(\bar{\lambda}_n - \bar{\lambda}; t\right) = 0$, $\forall t \in R_+$. As a result we obtain that the space $(\mathscr{K}_{\bar{x}}^w; \mu_K; \nu_K)$ is weakly fuzzy complete. Thus, we have proved the following

Theorem 3. *Let* $(X;\mu;\nu)$ *be a fuzzy Banach space with condition 12) and let* $\{x_n\}_{n\in N}\subset X$ *be a nondegenerate system. Then the space of coefficients* $(\mathscr{K}_{\bar{x}}^{w};\mu_K;\nu_K)$ *is a weakly fuzzy complete normed space.*

Consider operator $T:\mathscr{K}_{\bar{x}}^{w}\to X$, defined by

$$T\bar{\lambda}=\sum_{n=1}^{\infty}\lambda_n x_n,\bar{\lambda}\equiv\{\lambda_n\}_{n\in N}\in\mathscr{K}_{\bar{x}}^{w}.$$

Let $w\text{-}\lim_{n\to\infty}\bar{\lambda}_n=\bar{\lambda}$, in $\mathscr{K}_{\bar{x}}^{w}$, where $\bar{\lambda}_n\equiv\left\{\lambda_k^{(n)}\right\}_{k\in N}\in\mathscr{K}_{\bar{x}}^{w}$. We have

$$\mu\left(T\bar{\lambda}_n-T\bar{\lambda}\,;\,t\right)=\mu\left(\sum_{k=1}^{\infty}\left(\lambda_k^{(n)}-\lambda_k\right)\,x_k\,;\,t\right)\geq\inf_m\mu\left(\sum_{k=1}^{m}\left(\lambda_k^{(n)}-\lambda_k\right)\,x_k\,;\,t\right)=$$

$$=\mu\left(\bar{\lambda}_n-\bar{\lambda}\,;\,t\right).$$

It follows directly that $w\text{-}\lim_{n\to\infty}T\,\bar{\lambda}_n=T\bar{\lambda}$, i.e. the operator T is weakly fuzzy continuous. Let $\bar{\lambda}\in KerT$, i.e. $T\bar{\lambda}=0\Rightarrow\sum_{n=1}^{\infty}\lambda_n x_n=0$, where $\bar{\lambda}\equiv\{\lambda_n\}_{n\in N}\in\mathscr{K}_{\bar{x}}^{w}$. It is clear that if the system $\{x_n\}_{n\in N}$ is *w-linearly independent*, then $\lambda_n=0$, $\forall n\in N$, and, as a result, $KerT=\{0\}$. In this case $\exists T^{-1}:ImT\to\mathscr{K}_{\bar{x}}^{w}$. If, in addition, ImT is *w-close* in X, then T^{-1} is also continuous.

Denote by $\{\bar{e}_n\}_{n\in N}\subset\mathscr{K}_{\bar{x}}^{w}$ a canonical system in $\mathscr{K}_{\bar{x}}^{w}$, where $\bar{e}_n=\{\delta_{nk}\}_{k\in N}\in\mathscr{K}_{\bar{x}}^{w}$. Obviously, $T\bar{e}_n=x_n$, $\forall n\in N$. Let us prove that $\{\bar{e}_n\}_{n\in N}$ forms an *w-basis* for $\mathscr{K}_{\bar{x}}^{w}$. Take $\forall\bar{\lambda}\equiv\{\lambda_n\}_{n\in N}\in\mathscr{K}_{\bar{x}}^{w}$ and show that the series $\sum_{n=1}^{\infty}\lambda_n\bar{e}_n$ is weakly fuzzy convergent in $\mathscr{K}_{\bar{x}}^{w}$. In fact, the existence of $w\text{-}\lim_{m\to\infty}\sum_{n=1}^{m}\lambda_n x_n$ in X_w implies that $\forall\varepsilon>0$, and $\forall t>0$, $\exists m_0=m_0(\varepsilon;t)\in N$:

$$\mu\left(\sum_{n=m}^{m+p}\lambda_n x_n;t\right)>1-\varepsilon,\forall m\geq m_0\,,\,\forall p\in N.$$

We have

$$\mu_K\left(\sum_{n=m}^{m+p}\lambda_n\bar{e}_n;t\right)=\inf_r\left(\sum_{n=m}^{r}\lambda_n x_n;t\right)\geq 1-\varepsilon,\forall m\geq m_0\,,\,\forall p\in N.$$

It follows that the series $\sum_{n=1}^{\infty}\lambda_n\bar{e}_n$ is weakly fuzzy convergent in $\mathscr{K}_{\bar{x}}^{w}$. Moreover,

$$\mu_K\left(\bar{\lambda}-\sum_{n=1}^{m}\lambda_n\bar{e}_n;t\right)=\mu_K\left(\{\,0;\,...;0;\,\lambda_{m+1};...\}\,\,;\,t\right)=\inf_r\mu\left(\sum_{n=m+1}^{r}\lambda_n x_n;t\right)\geq 1-\varepsilon,$$

$$\forall m\geq m_0\,,\,\forall t\in R_+.$$

Consequently, $w\text{-}\lim_{m\to\infty}\sum_{n=1}^{m}\lambda_n\bar{e}_n = \bar{\lambda}$, i.e. $\bar{\lambda}\overset{w}{=}\sum_{n=1}^{\infty}\lambda_n\bar{e}_n$. Consider the functionals $e_n^*\left(\bar{\lambda}\right)=\lambda_n$, $\forall n\in N$. Let us show that they are *w-continuous.* Let $w\text{-}\lim_{n\to\infty}\bar{\lambda}_n=\bar{\lambda}$, where $\bar{\lambda}_n\equiv\left\{\lambda_k^{(n)}\right\}_{k\in N}\in\mathscr{K}_{\bar{x}}^{w}$. As established in the proof of Theorem 3, we have $\lambda_k^{(n)}\to\lambda_k$, as $n\to\infty$, i.e. $e_k^*\left(\bar{\lambda}_n\right)\to e_k^*\left(\bar{\lambda}\right)$, as $n\to\infty$, for $\forall k\in N$. Thus, e_k^* is *w-continuous* in $\mathscr{K}_{\bar{x}}^{w}$ for $\forall k\in N$. On the other hand, it is easy to see that $e_n^*\left(\bar{e}_k\right)=\delta_{nk}$, $\forall n,k\in N$, i.e. $\{e_n^*\}_{n\in N}$ is *w-biorthogonal* to $\{\bar{e}_n\}_{n\in N}$. As a result we obtain that the system $\{\bar{e}_n\}_{n\in N}$ forms an *w-basis* for $\mathscr{K}_{\bar{x}}^{w}$. So we get the validity of the following

Theorem 4. *Let $(X;\mu;\nu)$ be a fuzzy Banach space with condition 12) and $\{x_n\}_{n\in N}\subset X$ be a nondegenerate system. Then the corresponding space of coefficients $(\mathscr{K}_{\bar{x}}^{w};\mu_K;\nu_K)$ is weakly fuzzy complete with canonical w-basis $\{\bar{e}_n\}_{n\in N}$.*

Suppose that the system $\{x_n\}_{n\in N}$ is *w- linearly independent* and ImT is closed. Then it is easily seen that $\{x_n\}_{n\in N}$ forms an *w-basis* for ImT, and, in case of its *w-completeness* in X_w, it forms an *w-basis* for X_w. In this case, $\mathscr{K}_{\bar{x}}^{w}$ and X_w are isomorphic to each other, and T is an isomorphism between them. The opposite of it is also true, i.e. if the above-defined operator T is an isomorphism between $\mathscr{K}_{\bar{x}}^{w}$ and X_w, then the system $\{x_n\}_{n\in N}$ forms an *w-basis* for X_w. We will call T a *coefficient operator.* Thus, the following theorem holds.

Theorem 5. *Let $(X;\mu;\nu)$ be a fuzzy Banach space with condition 12), $\{x_n\}_{n\in N}\subset X$ be a nondegenerate system, $(\mathscr{K}_{\bar{x}}^{w};\mu_K;\nu_K)$ be a corresponding weakly fuzzy complete normed space and T:$\mathscr{K}_{\bar{x}}^{w}\to X_w$ be a coefficient operator. System $\{x_n\}_{n\in N}$ forms an w-basis for X_w if and only if the operator T is an isomorphism between $\mathscr{K}_{\bar{x}}^{w}$ and X_w.*

4. Conclusion

We arrive at the following conclusion: Let 5-tuple $(X;\mu;\,\nu;\,*;\,\diamondsuit)$ be an IFB$_w$S. Then:

1. this structure generates the corresponding concepts for the theory of approximation;
2. every nondegenerate system $\{x_n\}_{n\in N}\subset X$ generates a corresponding weakly fuzzy complete normed space $(\mathscr{K}_{\bar{x}}^{w};\mu_K;\nu_K)$ of coefficients;
3. canonical system $\{\bar{e}_n\}_{n\in N}$ forms a weak basis for $\mathscr{K}_{\bar{x}}^{w}$;
4. system $\{x_n\}_{n\in N}$ generates a coefficient operator $T:\mathscr{K}_{\bar{x}}^{w}\to X$;
5. system $\{x_n\}_{n\in N}$ forms a weak basis for X if and only if T is an isomorphism between $\mathscr{K}_{\bar{x}}^{w}$ and X .

Note that many results of this work are new in classical case, too.

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