Journal of Contemporary Applied Mathematics
V. 9, No 1, 2019, July
ISSN 2222-5498

Fuzzy Hadamard Type Inequality for Strongly Preinvex Functions

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Abstract. In this paper, we considered initially a generalization of preinvexity for functions which is called strongly preinvexity. Moreover, we cited as evidence two examples that the well-known Hermite-Hadamard inequalities are not valid in the fuzzy context. That's why; we established an upper bound on the fuzzy Hadamard type inequality for strongly preinvex functions. In this context a specific theorem was firstly given to illustrate this result. Finally, we obtained the Hermite-Hadamard type integral inequality via fuzzy Sugeno integrals for these functions.

Key Words and Phrases: Hermite Hadamard type inequality; Fuzzy measure, Sugeno integrals; Strongly preinvex functions.

2010 Mathematics Subject Classifications: Primary 03E72, 28E10; Secondary 26A51

1. Introduction

Fuzzy measure was proposed by Sugeno [1] in 1974. The properties and applications of Sugeno-integral have been studied by lots of authors. Between these others, Ralescu and Adams [2] proposed several equivalent definitions of fuzzy integrals; Dubois [3] defined the level-continuity of fuzzy integrals and the H-continuity of fuzzy measures. Many authors generalized the Sugeno integral by using some other operators to replace the special operators $\vee$ and/or $\wedge$. Suárez García and Gil Álvarez [4] presented two families of fuzzy integrals, the so-called seminormed fuzzy integrals and semiconormed fuzzy integrals.

In recent years, some authors generalized several classical integral inequalities for fuzzy integral [5]-[13] . The book by Wang and Klir [5] contains a general overview on fuzzy measurement and fuzzy integration theory. Caballero and Sadarangani [7, 8, 9] showed some inequalities of fuzzy integrals for in 2009, 2010. Li, Song and Yue [12] served Hermite-Hadamard type inequality for Sugeno integrals in 2014. Turhan et al [13] obtained Hermite-Hadamard type inequality for strongly convex functions via Sugeno Integrals in 2017.

A significant generalization of convex functions is that of invex functions introduced by Hanson [14] in 1981. Hansons' initial result inspired a great deal of subsequent work which has greatly expanded the role and applications of invexity in nonlinear optimization and other branches of pure and applied sciences. Weir and Mond [15] introduced preinvex functions in multiobjective optimization in 1988. Noor et al [16, 17, 18] studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems in 1994, 2005, 2006. Okur [23] obtained Hermite-Hadamard type inequality for log-preinvex functions via Sugeno integrals in 2018.

In the light of these developments, our aim of this paper is to prove a Hermite-Hadamard type inequality for strongly preinvex functions [20], an extension of preinvexity in [22], using Sugeno integrals differently from these studies. Some examples are also given to illustrate the validity of the proposed inequality.

Let's see some proporties of fuzzy integral.

2. Preliminary Discussions

In this section, we remember some basic definition. For details we refer the readers to Refs [1, 17, 18].

Definition 2.1. *[1] Suppose that Σ is a σ-algebra of subsets of X and that $\mu : \Sigma \to [0, \infty)$ is a non-negative, extended real-valued set function. We say that μ is a fuzzy measure if and only if*

1. $\mu(\emptyset) = 0$;

2. $E, F \in \Sigma$ and $E \subset F$ imply $\mu(E) \leq \mu(F)$ (monotonicity);

3. $\{E_n\} \subset \Sigma, E_1 \subset E_2 \subset ...,$ imply $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right)$ (continuity from below);

4. $\{E_n\} \subset \Sigma, E_1 \supset E_2 \supset ..., \mu(E_1) < \infty,$ imply $\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right)$ (continuity from above).

Then the triple (X, Σ, μ) is called a fuzzy measure space.

Let (X, Σ, μ) be a fuzzy measure space, By M^+, we denote the set of all non-negative measurable functions with respect to Σ. If f is a non-negative real-valued function defined on X, we denote the set

$$F_\alpha = \{x \in X : f(x) \geq \alpha, \alpha \geq 0\} = \{f \geq \alpha\}$$

Note that if $\alpha \leq \beta$ then $F_\beta \subset F_\alpha$.

Definition 2.2. *[1] Let $A \in \Sigma$, $f \in M^+$ The fuzzy integral of f on A with respect to μ which is denoted by*

$$(s)\int f d\mu = \bigvee_{\alpha \geq 0} \left[\alpha \wedge \mu\left(A \cap \{f \geq \alpha\}\right)\right].$$

When $A = X$, the fuzzy integral may also be denoted by $(s)\int f d\mu$. Where $\vee$ and $\wedge$ denote the operations infimum and supremum on $(0, \infty)$, respectively.

The following properties of the Sugeno integral are well known and can be found in.

Proposition 2.3. *[1] Let (X, Σ, μ) be a fuzzy measure space, $A \in \Sigma$ and $f, g \in M^+$*

1. $(s)\int_A f d\mu \leq \mu(A)$;
2. $(s)\int_A k d\mu = k \wedge \mu(A)$, *$k$ non-negative constant;*
3. *If $f \leq g$ on A then* $(s)\int_A f d\mu \leq (s)\int_A g d\mu$;
4. *If $A \subset B$ then* $(s)\int_A f d\mu \leq (s)\int_B f d\mu$;
5. $\mu(A \cap \{f \geq \alpha\}) \geq \alpha \Rightarrow (s)\int_A f d\mu \geq \alpha$;
6. $\mu(A \cap \{f \geq \alpha\}) \leq \alpha \Rightarrow (s)\int_A f d\mu \leq \alpha$;
7. $(s)\int_A f d\mu < \alpha \Leftrightarrow$ *there exists $\gamma < \alpha$ such that* $\mu(A \cap \{f \geq \gamma\}) < \alpha$;
8. $(s)\int_A f d\mu > \alpha \Leftrightarrow$ *there exists $\gamma > \alpha$ such that* $\mu(A \cap \{f \geq \gamma\}) > \alpha$.

Remark 2.4. *Consider the distribution function F associated to f on A, that is, $F(\alpha) = \mu(A \cap \{f \geq \alpha\})$. Then, due to (5) and (6) of Proposition 1, we have that*

$$F(\alpha) = \alpha \Rightarrow (s)\int f d\mu = \alpha.$$

Thus, from a numerical point of view, the fuzzy integral can be calculated solving the equation $F(\alpha) = \alpha$.

Definition 2.5. *[17] Let $x, y \in I$.A non-empty closed subset I of $\mathbb{R}^n$ is said to be invex at x with respect to the given vector function $\eta : I \times I \longrightarrow \mathbb{R}^n$ if*

$$x + \lambda\eta(y, x) \in I, \lambda \in [0, 1].$$

I is said to be an invex set with respect to μ, if I is invex at each $x, y \in I$. The invex set I is also called μ-connected set.

Clearly, we would like to mention that Definition 2 of an invex set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point x which is contained in I. We do not require that the point y should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that y should be an end point of the path for every pair of points, $x, y \in I$, then $\mu(y, x) = y - x$ and consequently invexity reduces to convexity. Thus, it is true that every convex set is also an invex set with respect to $\mu(y, x) = y - x$, but the converse is not necessarily true.

Definition 2.6. *[17] Let $I \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : I \times I \longrightarrow \mathbb{R}^n$. Then the function (not necessarily differentiable) $f : I \to \mathbb{R}$ is said to be preinvex with respect to η if*

$$f(x + \lambda\eta(y, x)) \leq (1 - \lambda)f(x) + \lambda f(y) \tag{1}$$

for each $x, y \in I$ and $\lambda \in [0, 1]$. Any convex function is preinvex with respect to $\eta(y, x) = y - x$, but the converse is not necessarily true.

Definition 2.7. *[17] For a differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be invex if there exists a vector function $\eta : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that*

$$f(x) - f(u) \geq [\nabla f(u)]^T \eta(x, u) \tag{2}$$

for all $x, u \in \mathbb{R}^n$.

An invex function may not be preinvex, $f(x) = \exp(x)$ is a counterexample, it is invex with respect to $\eta(x, u) = -1$, but not preinvex with respect to same η. Mohan and Neogy [24] proved that an invex function is also preinvex under following Condition C.

Condition C. Let $\eta : I \times I \longrightarrow \mathbb{R}^n$. It is told that the function η satisfies Condition C if,

(C1) $\eta(x, x + \lambda\eta(y, x)) = -\lambda\eta(y, x)$

(C2) $\eta(y, x + \lambda\eta(y, x)) = (1 - \lambda)\eta(y, x)$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Additionally, note that from condition C, we have

$$\eta(x + \lambda_2\eta(y, x), x + \lambda_1\eta(y, x)) = (\lambda_2 - \lambda_1)\eta(y, x) \tag{3}$$

for all $x, y \in I$ and $\lambda_1, \lambda_2 \in [0, 1]$.

Definition 2.8. *[18] Let $I \subseteq \mathbb{R}^n$ be an invex set with respect to η. Then the function (not necessarily differentiable) $f : I \to (0, \infty)$ is said to be strongly preinvex with modulus $c > 0$ if*

$$f(x + \lambda\eta(y, x)) \leq (1 - \lambda)f(x) + \lambda f(y) - c\lambda(1 - \lambda)\eta^2(y, x) \tag{4}$$

for each $x, y \in I$ and $\lambda \in [0, 1]$. Any convex function is strongly preinvex with respect to $\eta(y, x) = y - x$, but the converse is not necessarily true.

The classical Hermite-Hadamard inequality for strongly preinvex functions is as follows [18]:

$$f\left(\frac{2u+\eta(v,u)}{2}\right)+\frac{c}{12}\eta^2(v,u)\leq\frac{1}{\eta(v,u)}\int_u^{u+\eta(v,u)} f(x)\,dx\leq\frac{f(u)+f(v)}{2}-\frac{c}{6}\eta^2(v,u)$$

where $f:[u,u+\eta(v,u)]\subset I\to\mathbb{R}$ is a strongly preinvex function on the interval I and $u,v\in I$ with $u<v$ and $u+\eta(v,u)\leq v$ respect with η.

3. Main Results

In this paper, we prove using Sugeno integral another refinement of the Hermite-Hadamard type inequality for strongly preinvex functions. We assume that (X,Σ,μ) is a fuzzy measure space. To simplify the calculation of the fuzzy integral, for a given $f,g\in M^+$ and $A\in\Sigma$, we write

$$\Gamma=\{\alpha\mid\alpha\geq 0,\ \mu(A\cap F_\alpha)>\mu(A\cap F_\beta)\ \text{for any}\ \beta>\alpha\}$$

It is easy to see that

$$(s)\int_A f d\mu=\bigvee_{\alpha\in\Gamma}\left[\alpha\wedge\mu(A\cap F_\alpha)\right].$$

J.Caballero et all [7] proved with the help of certain examples that the classical Hermite-Hadamard inequalities for fuzzy integrals.

Firstly, let us show with the following examples whether the classical Hermite-Hadamard type inequality for strongly preinvex functions is valid in the fuzzy context or not:

Example 3.1. *Consider $X=[0,\eta(1,0)]$ and let μ be the Lebesgue measure on X. If we take the function $f(x)=x^2$, then $f(x)$ is a strongly preinvex function. In fact*

$$f(\lambda\eta(1,0))\leq(1-\lambda)f(0)+\lambda f(\eta(1,0))-c\lambda(1-\lambda)\eta^2(1,0)$$

thus $0<c\leq 1$, for all $\lambda\in[0,1]$. To calculate the Sugeno integral related to this function, let's consider the distribution function F associated to f to $[0,\eta(1,0)]$ by Remark 1, this is

$$\begin{aligned}(s)\int_0^{\eta(1,0)} x^2 d\mu &= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([0,\eta(1,0)]\cap\{x^2\geq\alpha\}\right)\right)\\ &= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([0,\eta(1,0)]\cap\{x\geq\sqrt{\alpha}\}\right)\right)\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left([\sqrt{\alpha},\eta(1,0)]\right)\right)\\
&= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\left(\eta(1,0)-\sqrt{\alpha}\right)\right).
\end{aligned}
$$

In this expression, $(\eta(1,0)-\sqrt{\alpha})$ *may be negative, but it is a decreasing continuous function of* α *when* $\alpha\geq 0$*. Hence, the supremum will be attained at the point which is one of the solutions of the equation* $\eta(1,0)-\sqrt{\alpha}=\alpha$*, that is, at* $0<c\leq 0,3820$*. So we have by Remark 1, we get*

$$
0\leq (s)\int_{0}^{\eta(1,0)} f d\mu \leq 0,3820.
$$

On the other hand

$$
\frac{f(0)+f(\eta(1,0))}{2}-\frac{c}{6}\eta^2(1,0)=\eta^2(1,0)\left(\frac{1}{2}-\frac{c}{6}\right)
$$

and $0\leq\eta(1,0)\leq 1$*. Then* $c\leq -6,168$*.It is a contradiction so that* $0<c\leq 1$*.Therefore it is occurred this proving that the right part of Hermite-Hadamard inequality is not satisfied in the fuzzy context.*

Example 3.2. *Consider* $X=(0,\eta(1,0)]$ *and let* μ *be the Lebesgue measure on* X*. If we take the function* $f(x)=\frac{1}{x}$*, then* $f(x)$ *is a strongly preinvex function. To calculate the Sugeno integral related to this function, let's consider the distribution function* F *associated to* f *to* $(0,\eta(1,0)]$ *by Remark 1, this is*

$$
\begin{aligned}
(s)\int_{0}^{\eta(1,0)}\frac{1}{x}d\mu &= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left((0,\eta(1,0)]\cap\left\{\frac{1}{x}\geq\alpha\right\}\right)\right)\\
&= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left((0,\eta(1,0)]\cap\left\{x\geq\frac{1}{\alpha}\right\}\right)\right)\\
&= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\mu\left((0,\frac{1}{\alpha}]\right)\right)\\
&= \bigvee_{\alpha\geq 0}\left(\alpha\wedge\frac{1}{\alpha}\right)
\end{aligned}
$$

and we solve the equation $F(\alpha)=\alpha$*. It is easily proved that the solutions of the last equation is* $\alpha=1$ *and Remark 1, we get*

$$
\frac{1}{\eta(1,0)}(s)\int_{0}^{\eta(1,0)} f d\mu=\frac{1}{\eta(1,0)}.
$$

On the other hand from the Hermite Hadamard inequality, this below equation has been become:

$$f\left(\frac{\eta(1,0)}{2}\right)+\frac{c}{12}\eta^2(1,0)\leq\frac{1}{\eta(1,0)}$$

and $0\leq\eta(1,0)\leq 1$. *Then* $c\leq -96$.*It is a contradiction so that* $0<c\leq 1$.*Therefore it is occurred this proving that the left part of Hermite-Hadamard inequality is not satisfied in the fuzzy context.*

That's why; we will establish an upper bound on the Sugeno integral of strongly preinvex functions. In this context, a specific theorem is firstly given to illustrate this result on the discrete interval $[0,\eta(1,0)]$:

Theorem 3.3. *Let* $f:[0,\eta(1,0)]\rightarrow(0,\infty)$ *be a strongly preinvex function such that* $f(0)\neq f(\eta(1,0))$ *and* μ *the Lebesque measure on* $\mathbb{R}$. *Then*

$$(s)\int_0^{\eta(1,0)} f d\mu\leq\bigvee_{\alpha\in\Gamma}(\alpha\wedge\mu([0,\eta(1,0)]\cap\{g\geq\alpha\}))\leq\min\{\alpha,\eta(1,0)\},$$

where

$$\begin{aligned}\Gamma &= [f(0),f(\eta(1,0))), \qquad f(\eta(1,0))>f(0)\\ \Gamma &= [f(\eta(1,0)),f(0)), \qquad f(\eta(1,0))<f(0)\\ g(x) &= \left(1-\frac{x}{\eta(1,0)}\right)f(0)+\frac{x}{\eta(1,0)}f(\eta(1,0))-c\frac{x}{\eta(1,0)}\left(1-\frac{x}{\eta(1,0)}\right)\eta^2(1,0);\end{aligned}$$

i) If $f(\eta(1,0))>f(0)$, *then* α *is root of the following equation*

$$\eta(1,0)-\alpha+\frac{A-\sqrt{A^2-4c\eta^2(1,0)(f(0)-\alpha)}}{2c\eta(1,0)}=0,$$

ii) If $f(0)>f(\eta(1,0))$, *then* α *is root of the following equation*

$$\eta(1,0)-\alpha+\frac{A+\sqrt{A^2-4c\eta^2(1,0)(f(0)-\alpha)}}{2c\eta(1,0)}=0,$$

where $A=f(\eta(1,0))-f(0)-c\eta^2(1,0)$.

Proof. Using the strongly preinvexity of f , we have

$$f(x) = f\left(x.0+\frac{x}{\eta(1,0)}\eta(1,0)\right)$$

$$
\begin{aligned}
&\le \left(1-\frac{x}{\eta(1,0)}\right) f(0)+\frac{xf(\eta(1,0))}{\eta(1,0)}-\frac{cx\eta^2(1,0)}{\eta(1,0)}\left(1-\frac{x}{\eta(1,0)}\right)\\
&= g(x)
\end{aligned}
$$

for $x \in [0, \eta(1,0)]$. Hence, by (3) of Proposition 1, we get

$$
(s)\int_0^{\eta(1,0)} f d\mu \le (s)\int_0^{\eta(1,0)} g d\mu.
$$

In order to calculate the integral in the right-hand part of the last inequality, we consider the distribution function G given by

$$
G(\alpha) = \mu\left([0, \eta(1,0)] \cap \{g \ge \alpha\}\right).
$$

i) If $f(\eta(1,0)) > f(0)$, then

$$
G(\alpha) = \mu\left(\begin{array}{c}[0, \eta(1,0)] \cap \\ \left\{\left(1-\frac{x}{\eta(1,0)}\right) f(0)+\frac{xf(\eta(1,0))}{\eta(1,0)}-\frac{cx\eta^2(1,0)}{\eta(1,0)}\left(1-\frac{x}{\eta(1,0)}\right) \ge \alpha\right\}\end{array}\right)
$$

Hence, we obtain a root of the above equation as follows:

$$
\eta(1,0)-\alpha+\frac{A-\sqrt{A^2-4c\eta^2(1,0)(f(0)-\alpha)}}{2c\eta(1,0)}=0,
$$

where $A = f(\eta(1,0)) - f(0) - c\eta^2(1,0)$.Thus $\Gamma = [f(0), f(\eta(1,0)))$, and we only need to consider $\alpha \in [f(0), f(\eta(1,0)))$.

Similarly, ii) is obtained, and thus the proof of the theorem is completed. □

Remark 3.4. *In the case $f(0) = f(\eta(1,0))$ in Theorem 2, the function $g(x)$ is*

$$
g(x) = f(0) - cx\eta(1,0)\left(1-\frac{x}{\eta(1,0)}\right)
$$

and (3) Proposition 1, α is root of the equation

$$
\eta(1,0)-\frac{c\eta(1,0)+\sqrt{c^2\eta^2(1,0)-4c(f(0)-\alpha)}}{2c}=\alpha.
$$

Finally, an upper bound on the Sugeno integral of strongly preinvex functions is obtained as follows:

Theorem 3.5. *Let $f:[u,u+\eta(v,u)]\to(0,\infty)$ be a strongly preinvex function such that $f(u)\neq f(u+\eta(v,u))$ and μ the Lebesque measure on $\mathbb{R}$. Then*

$$(s)\int_{u}^{u+\eta(v,u)} f d\mu \leq \bigvee_{\alpha\in\Gamma}\left(\alpha\wedge\mu\left([u,u+\eta(v,u)]\cap\{g\geq\alpha\}\right)\right)\leq\min\{\alpha,\eta(v,u)\},$$

where

$$\begin{aligned}
\Gamma &= [f(u), f(u+\eta(v,u))), \qquad f(u+\eta(v,u))>f(u);\\
\Gamma &= [f(u+\eta(v,u)), f(u)), \qquad f(u+\eta(v,u))<f(u);\\
g(x) &= \left(1-\frac{x}{\eta(1,0)}\right)f(u)+\frac{xf(u+\eta(v,u))}{\eta(1,0)}-\frac{cx\eta^2(1,0}{\eta(1,0)}\left(1-\frac{x}{\eta(1,0)}\right));
\end{aligned}$$

i) If $f(u+\eta(v,u))>f(u)$, then α is root of the following equation

$$\eta(v,u)-\alpha+\frac{A-\sqrt{A^2-4c\eta^2(v,u)(f(u)-\alpha)}}{2c\eta(v,u)}=0,$$

ii) If $f(u)>f(u+\eta(v,u))$, then α is root of the following equation

$$\eta(v,u)-\alpha+\frac{A+\sqrt{A^2-4c\eta^2(v,u)(f(u)-\alpha)}}{2c\eta(v,u)}=0,$$

where $A=f(u+\eta(v,u))-f(u)-c\eta^2(v,u)$.

Proof. Similarly, the proof of theorem is obtained same as Theorem 2. □

Remark 3.6. *In the case $f(u)=f(\eta(v,u))$ in Theorem 3, the function $g(x)$ is*

$$g(x)=f(u)-cx\eta(v,u)\left(1-\frac{x}{\eta(v,u)}\right)$$

and (3) Proposition 1, α is root of the equation

$$\eta(v,u)-\frac{c\eta(v,u)+\sqrt{c^2\eta^2(v,u)-4c(f(u)-\alpha)}}{2c}=\alpha.$$

4. Conclusion

In this paper, we established an upper bound on Sugeno integral of strongly preinvex functions which is a useful tool to estimate unsolvable integrals of this kind. Thus this study is an important topic for further research.

5. Acknowledgments

The author is very grateful to the referees for helpful comments and thanks to Assist. Prof. Hande G. AKDEMİR for her valuable suggestions.

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Received 23 April 2018
Accepted 20 February 2019