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On ergodic distribution a cyclical inventory-queuing model with delay

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Abstract. In this study, a inventory-queuing model with $X(t)$ delay is investigated. Based on shaper form of the renewal theorem under some assumptions, an asymptotic expansion for the ergodic distribution of the stochastic process $W_a(t) = X(t)/a$ is obtained, as $a \to \infty$ and exact expression of the limit distribution is derived.

Key Words and Phrases: ergodic distribution; renewal function; G/G/1 queuing system; inventory model with delay.

1. Introduction

It is known that inventory control theory is closely related to queuing theory. In recent years, there have been many studies devoted on queuing-inventory systems. Such systems are closely related to the inventory control model (s,S). Some interesting studies exist in literature in this area. For instance, in paper Janssen et.al.(1998) an approximation method to compute the reorder point s in inventory model with a service level restriction was introduced, when demand is modeled as a compound Bernoulli process. In paper Sobel and Zhang (2001) a periodic review inventory system with demand arriving simultaneously from a deterministic source and a random source was considered and shown that the (s,S) policy is optimal. In paper Nair and Jacob (2013) considered a multi server queuing model with the special feature that the servers act as an inventory with standard (s,S) policy and positive lead time. The demands are considered as arrivals and the time for serving the inventory is considered as the service time. With each replenishment, the level of the inventory is restocked to S ($< \infty$). The lead time follows an exponential distribution. We assume that during stock out, new arrivals do not join the queue and customers being in the system, wait until their service is completed.

In paper Nair and Jacob (2015) considered a multiserver Markovian queuing system where each server provides service only to one customer. Arrival of customers is according to a Poisson process and whenever a customer leaves the system after getting service, that server is also removed from the system. Here the servers are considered as a standard (s,S) production inventory. The condition for checking ergodicity and the steady state solutions are obtained using matrix analytic method.

In paper Ko (2016) is consider an (s,S) continuous inventory model with perishable items, impatient customers, and random lead times. Two characteristic behaviors of impatient customers are balking and reneging. Balking is when a customer departs the system if the item they desire is unavailable. Reneging occurs when a waiting customer leaves the system if their demand is not met within a set period of time. In study Melikov et. al. (2017) Markov models of the queuing-inventory systems with variable order size are investigated. Two classes of models, with instantaneous and non-zero service times of customers are considered.

Unlike of the above mention studies, in the present paper we will be consider generalization of classical control model type of (s,S) with delay.

Let $\{\xi_n\}$,$\{\eta_n\}$ and $\{\theta_n\}$ - are independent sequences of random variables defined on probability space $(\Omega, \Im, P)$, such that variables in each sequence independent and identically distributed. Suppose that ξ_n, η_n, θ_n and ζ_n take only positive values and these distribution functions be denoted by

$$\Phi(t) = P\{\xi_1 \le t\},\ t > 0, F(x) = P\{\eta_1 \le x\}, x > 0, H(u) = P\{\theta_1 \le u\}, u > 0.$$

Let introduce also, sequence of random variables $\{\zeta_n\}$, $n \ge 1$which describes the discrete interference of chance and form an ergodic Markov chain with stationary distribution $\pi_a(z)$, $z \ge 0$, depends on parameter a > 0, i.e., $\pi_a(z) \equiv P\{\zeta \le z\} = \lim_{n\to\infty} P\{\zeta_n \le z\}$.

In particular, we assume that $\zeta_n = a\tilde{\zeta}_n$, $a > 0$, $n = 1, 2, \ldots$. In this case, $\pi_a(z) = \pi_1(z/a)$, where $\pi_1(z) \equiv P\{\tilde{\zeta} \le z\} = \lim_{n\to\infty} P\{\tilde{\zeta}_n \le z\}$ is a distribution function of the variable $\tilde{\zeta}$.

Define independent renewal sequence $\{T_n\}$ and $\{Y_n\}$ as follows using the initial sequences of the random variables $\{\xi_n\}$ and $\{\eta_n\}$ as:

$$T_n = \sum_{i=1}^{n} \xi_i,\ \ Y_n = \sum_{i=1}^{n} \eta_i,\ n = 1, 2, ...;\ T_0 = Y_0 = 0.$$

We also introduce following integer-valued random variables, which represent the number of jumps of the process X(t), before it passages the level 0:

$$N_0 = 0,$$

$$N_1 \equiv N(z) = \min\{k \ge 1 : z - Y_k < 0\};$$

$$N_n \equiv N_n(\zeta_{n-1}) = \min\{k \ge N_{n-1} + 1 : \zeta_{n-1} - (Y_k - Y_{N_n}) < 0\}, n = 2, 3, ...$$

Let also introduce following random variables:

$$\tau_0 = 0, \tau_1 = T_{N_1}, \gamma_1 = \tau_1 + \theta_1,$$

$$\tau_2 = \gamma_1 + T_{N_2} - T_{N_1}, \gamma_2 = \tau_2 + \theta_2,$$

$$...$$

$$\tau_n = \gamma_{n-1} + T_{N_n} - T_{N_{n-1}},\ \gamma_n = \tau_n + \theta_n, n = 1, 2, ...$$

Define also the counting process $\nu\,(\mathrm{t})$ as:

$$\nu\,(\mathrm{t}) = \max\left\{n \geq 0\ :\ T_n \leq t\right\}.$$

Thus the following stochastic process can be constructed using these notations:

$X(t) = \max\{0\,,\ \zeta_n - Y_{\nu(t)} + Y_{N_n}\}$ as $\gamma_n \leq t < \gamma_{n+1}, \quad n = 0, 1, 2, ...$

where $\zeta_0 = z$.

One of the realization of the considered process $X(t)$ has the following form:

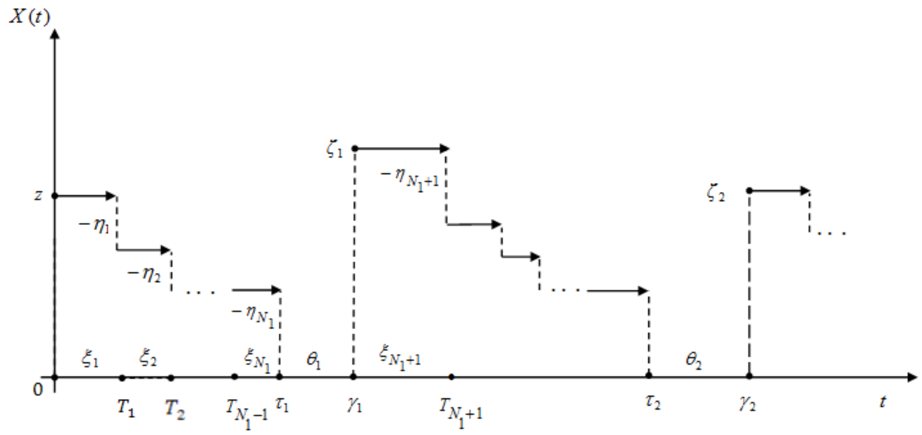

Figure 1: A trajectory of the process $X(t)$.

In literature process X(t) is called control model of type (s,S), in case, if $\pi_1(z)$ has a special distribution in between control level s and maximum stock level S and if $P\left\{\theta_n = 0\right\} = 1$ (see, for example Prabhu (1980)). Process X(t) is closely related to the queuing system G/G/1, since random variables ξ_n and η_n has the arbitrary distribution. Random variables θ_n, $n = 1, 2, ...$ describe delay times and γ_n, $n = 1, 2, ...$ are the moments of beginning of new cycle.

2. Asymptotic expansion for the ergodic distribution of the process $W_a(t)$

In this section we will investigate ergodic distribution of the process $X(t)$.

Let the initial sequences of the random variables $\{\xi_n\}$, $\{\eta_n\}$, $\{\theta_n\}$and $\{\zeta_n\}$, $n \geq 1$ be satisfied the following supplementary conditions:
1) random variables ξ_n, η_n and θ_n are independent;

$$2) E\xi_1 < \infty;$$

$$3) m_2 = E\eta_1^2 < \infty;$$

$$4) E\zeta_1 < \infty;$$

$$5) E\theta_1 < \infty.$$

6) η_1 is non-arithmetic random variable.

Under conditions 1) – 6) in Aliyev (2015) ergodicity of the process $X(t)$ is proved and explicit form of the ergodic distribution function $Q_X(x) \equiv \lim_{t\to\infty} P\{X(t) \leq x\}$is derived:

$$Q_X(x) = 1 - \frac{\int_x^\infty U(z)\, d\pi_a(z)}{EU(\zeta) + K}, x > 0 \tag{2.1}$$

Let us introduce the process $W_a(t) = X(t)/a$. From (2.1) is not difficult to obtain ergodic distribution $(Q_{W_a}(x))$ of the process $W_a(t)$:

$$Q_{W_a}(x) \equiv \lim_{t\to\infty} P\{W_a(t) \leq x\} = 1 - \frac{\int_{ax}^\infty U(z - ax)\, d\pi_a(z)}{EU(\zeta) + K}. \tag{2.2}$$

where
$K = E\theta_1/E\xi_1-$ delay coefficient,

$$EU(\zeta_1) = \int_0^\infty U(z) d\pi_a(z),$$

$U(z) = \sum_{n=0}^\infty F^{*(n)}(z)$ – is a renewal function generated by the sequence $\{\eta_n\}$, $n \geq 1$, $F^{*(n)}(z)$ is denoted n-fold convolution of the distribution function $F(z)$ with itself and define as $F^{*(n)}(z) = \int_0^z F^{*(n-1)}(z-y) dF(y)$, $n \geq 1$.

In this section we will study the asymptotic behavior of the ergodic distribution $(Q_{W_a}(x))$ of the process $W_a(t) = X(t)/a$, as $a \to \infty$. Note that, analogical problem consider in Aliyev (2015), when Cramer condition is hold, which equivalent exponential decreasing of tail of distribution of the random variable η_1:

$\int_0^\infty e^{\mu y} dF(y) = E(e^{\mu\eta_1}) < \infty$ for some $\mu > 0$.

In case if Cramer condition holds, all moments of random variable η_1 exist and finite.

In present paper, instead of the Cramer condition, we assume that satisfied condition $m_2 = E\eta_1^2 < \infty$.

We formulate this result in the form of the following theorem.

Theorem 2.1. *Let the conditions 1)-6) be satisfied. Then as $a \to \infty$ for the each $x > 0$ the asymptotic expansion for the ergodic distribution function $(Q_{W_a}(x))$ of the process $W_a(t) = X(t)/a$ is as follows:*

$$Q_{W_a}(x) = R(x) + ((m_{21} + Km_1)(1 - R(x)) - m_{21}(1 - \pi_1(x)))\frac{1}{e_1 a} + o\left(\frac{1}{a}\right), \qquad (2.3)$$

where

$$R(x) = \frac{1}{e_1}\int_0^x (1 - \pi_1(z))\,dz;$$

$$m_k = E\eta_1^k, k = 1, 2, m_{21} = \frac{m_2}{2m_1}, e_1 = E\tilde{\zeta}_1.$$

Proof. For the investigate asymptotic behavior of $Q_{W_a}(x)$ *we obtain asymptotic expansions for the* $(EU(\zeta_1) + K)^{-1}$ and $\int_{ax}^{\infty} U(z - ax)d\pi_a(z)$ in Eq.(2.2).

By using Proposition 4.3 from Aliyev et. al. (2016), we have:

$$EU(\zeta) + K = \frac{e_1 a}{m_1} + \frac{m_2}{2m_1^2} + K + o(1). \qquad (2.4)$$

From Eq.(2.4) can be obtained the following expansion, as $a \to \infty$:

$$(EU(\zeta) + K)^{-1} = \frac{m_1}{e_1 a}\left(1 - (m_{21} + m_1 K)\frac{1}{e_1 a} + o\left(\frac{1}{a}\right)\right). \qquad (2.5)$$

Now consider $\int_{ax}^{\infty} U(z - ax)d\pi_a(z)$ from formula (2.2).
Since $m_2 = E\eta_1^2 < +\infty$, then using shaper form of the renewal theorem (see, Feller (1971), p.415) as $z \to \infty$ we can write:

$$U(z) = \frac{z}{m_1} + \frac{m_2}{2m_1^2} + g(z), \qquad (2.6)$$

where $\lim_{z\to\infty} g(z) = 0$.

Therefore, using (2.6) it is not difficult to see that:

$$\int_{ax}^{\infty} U(z - ax)d\pi_a(z) = \frac{1}{m_1}\int_{ax}^{\infty} z d\pi_a(z) - \frac{1}{m_1} ax \int_{ax}^{\infty} d\pi_a(z)$$

$$+\frac{m_2}{2m_1^2}\int_{ax}^{\infty} d\pi_a(z) + \int_{ax}^{\infty} g(z - ax)d\pi_a(z) =$$

$$= \frac{1}{m_1}\left(\int_0^{\infty} z d\pi_a(z) - \int_0^{ax} z d\pi_a(z) - ax(1 - \pi_a(ax))\right) + \frac{m_2}{2m_1^2}(1 - \pi_a(ax)) + J(a, x) =$$

$$= \frac{1}{m_1}\left(ae_1 - \int_0^{ax}(1 - \pi_a(z))dz\right) + \frac{m_2}{2m_1^2}(1 - \pi_a(ax)) + J(a, x), \qquad (2.7)$$

where $J(a, x) = \int_{ax}^{\infty} g(z - ax)d\pi_a(z)$.
It is easy to see that when $a \to \infty$ for arbitrary $x > 0$:

$$J(a, x) = o(1). \qquad (2.8)$$

Hence, from (2.7) and (2.8) we have, as $a \to \infty$:

$$\int_{ax}^{\infty} U(z - ax)d\pi_a(z) = \frac{e_1 a}{m_1}\left(1 - R(x) + m_{21}(1 - \pi_1(x))\frac{1}{e_1 a} + o\left(\frac{1}{a}\right)\right), \qquad (2.9)$$

where $R(x) = \frac{1}{e_1}\int_0^x (1-\pi_1(z))\,dz$.

Substituting (2.5) and (2.9) into (2.2) after the first the corresponding calculations as $a \to \infty$ we obtain (2.3). This completes the proof of Theorem 2.1.

From Theorem 2.1, for each $x > 0$, we have:

$$\lim_{a\to\infty} Q_{W_a}(x) = R(x) \equiv \frac{1}{e_1}\int_0^x (1-\pi_1(z))\,dz. \tag{2.10}$$

Example 2.1. Let us $\pi_1(z) = 1 - e^{-z},\ z > 0$. In this case limit distribution $R(x)$ from (2.10) has the following form:

$$\lim_{a\to\infty} Q_{W_a}(x) = 1 - e^{-x},\ x > 0.$$

Example 2.2. Let us $\pi_1(z) = 1 - (1+z)e^{-z},\ z > 0$- second-order Erlang distribution with parameter 1. In this case limit distribution $R(x)$ from (2.10) has the following form:

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