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Schwarz inequality on the boundary for holomorphic functions

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Abstract. In this paper, a boundary version of the Schwarz inequality is investigated. The function $f(z) = c_p z^p + c_{p+1} z^{p+1}$..... holomorphic in the unit disc D which satisfies $|f(z)| < 1$ for $|z| < 1$ is considered. The Carathéodory and Schwarz inequalities at the boundary are obtained by taking into account the zeros and the poles of $f(z)$ function. The sharpness of these inequalities is also proved.

Key Words and Phrases: Schwarz lemma on the boundary, Holomorphic function, Zeros and Poles.
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1. Introduction

Let f be a holomorphic function in the disc $D = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point z in the disc D, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = cz$, $|c| = 1$ ([4], p.329).

The first generalization of Shwarz lemma, which is known as Schwarz-Pick lemma in literature, has been made by Pick and it is shown as below:

The Schwarz-Pick lemma states that, if $f(z)$ is holomorphic in $|z| < 1$ and $|f(z)| < 1$ for $|z| < 1$, then

$$\left|f'(z)\right| \leq \frac{1 - |f(z)|^2}{1 - |z|^2},$$

with equality only for

$$f(z) = \gamma \frac{z + \varsigma}{1 + \overline{\varsigma} z},$$

where $|\varsigma| < 1$, $|\gamma| = 1$ (see [4]).

In 2000, Osserman has given the inequalities which are called the boundary Schwarz lemma. Osserman has first showed that

$$\left|f'(z_0)\right| \geq \frac{2}{1 + |f'(0)|}$$

under the assumption $f(0)=0$ where f is a holomorphic function mapping the unit disc into itself and z_0 is a boundary point to which f extends continuously. Subsequently, using the Möbius transformation, he has generalized the inequality on the case of $f(0)\neq 0$ (see [6]).

In 2004, Dubinin has continued this line and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z)=c_pz^p+c_{p+1}z^{p+1}.....$ with a zero set $\{a_k\}$ (see [2]).

In 2010, Boas has discussed classical versions of the Schwarz lemma at the boundary of the unit disc in the complex plane. The author has given the geometric meaning of the function of $z_0\frac{f'(z_0)}{f(z_0)}$ and also the following has been stated:

If a holomorphic function f (not identically 0) maps a closed disk centered at 0 into itself, fixing 0, and if the modulus $|f|$ attains a maximum on the closed disk at the boundary point z_0, then $z_0\frac{f'(z_0)}{f(z_0)}$ is a real number greater than or equal to 1 (see [1]).

Consider the following product:

$$B(z)=\prod_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}$$

$B(z)$ is called a finite Blaschke product, where $a_1,a_2,...,a_n\in\mathbb{C}$. $B(z)$ is a holomorphic in the unit disc D, and

$$|B(z)|=1 \quad for \quad z\in\partial D,$$

since $\left|\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\right|=1$ when $|z|=1$. Therefore, the Maximum Modul Principle implies

$$|B(z)|<1 \quad for \quad z\in D.$$

Let the function f have zeros $a_1,a_2,...,a_m$.

Similarly, the extremal function is often given by the Blaschke function

$$B_b(z)=\prod_{k=1}^{n}\frac{1-\overline{b_k}z}{z-b_k},$$

which is generally defined for any set $b=(b_1,...b_n)$ of poles.

The following Theorem 1 is a simple example of the application of the maximum principle for holomorphic functions.

Theorem 1. *Let f be a holomorphic function in the unit disc D except at the poles $b_1,...,b_n$. Suppose that none of the limiting values of $|f(z)|$ as z approaches the boundary of the unit disc D exceed 1. Then*

$$|f(z)|\leq\prod_{k=1}^{n}\left|\frac{1-\overline{b_k}z}{z-b_k}\right|,\quad z\in D.$$

Equality at a point z (not a pole) is attained only if f is the function of the form

$$f(z)=c\prod_{k=1}^{n}\frac{1-\overline{b_k}z}{z-b_k},$$

where $|c| = 1$ *and* $|b_k| < 1$, $k = 1, ..., n$ *([3], p.286).*

In this paper, we obtain the new inequalities for the following expressions:

$$\Re\left(z_0 \frac{f'(z_0)}{f(z_0)}\right), \quad \left|\frac{f'(z_0)}{f(z_0)}\right|.$$

Also, we show the sharpness of each inequality by giving the concrete functions which are attaining each equality.

Theorem 2. *Let* $f(z) = c_p z^p + ...$, $c_p \neq 0$, $p \geq 1$ *be a holomorphic function in the unit disc* D *except at the poles* $b_1, ..., b_n$ *and* $a_1, ..., a_m$ *zeros of* $f(z)$ *in the unit disc* D *that are different from zero. Suppose that none of the limiting values of* $|f(z)|$ *as* z *approaches the boundary of the unit disc* D *exceed* 1*. Assume that* f *and* f' *are defined at the boundary* $z_0 \in \partial D$*. Then we have the inequality*

$$\Re\left(z_0 \frac{f'(z_0)}{f(z_0)}\right) \geq p + \sum_{\lambda=1}^{m} \frac{1 - |a_\lambda|^2}{|z_0 - a_\lambda|^2} - \sum_{k=1}^{n} \frac{1 - |b_k|^2}{|z_0 - a_k|^2} + \frac{\prod_{\lambda=1}^{m} |a_\lambda| + |c_p| \prod_{k=1}^{n} |b_k|}{\prod_{\lambda=1}^{m} |a_\lambda| - |c_p| \prod_{k=1}^{n} |b_k|}. \tag{1}$$

The inequality (1) *is sharp, with equality at a point* z *(which is not a pole) for the function*

$$f(z) = z^p \frac{z - \alpha}{1 - \alpha z} \prod_{\lambda=1}^{m} \frac{z - a_\lambda}{1 - \overline{a_\lambda} z} \prod_{k=1}^{n} \frac{1 - \overline{b_k} z}{z - b_k}, \ 0 < \alpha < 1,$$

where $|b_k| < 1$, $k = 1, ..., n$, $|a_\lambda| < 1$, $\lambda = 1, ..., m$, $a_1, a_2, ..., a_m$ *and* $b_1, b_2, ..., b_n$ *are positive real numbers.*

Proof. Consider the function

$$\varphi(z) = \frac{f(z) \prod_{k=1}^{n} \frac{z - b_k}{1 - \overline{b_k} z}}{z^p \prod_{\lambda=1}^{m} \frac{z - a_\lambda}{1 - \overline{a_\lambda} z}} = d_0 + d_p z + ..., \tag{2}$$

where $|d_0| = \frac{|c_p| \prod_{k=1}^{n} |b_k|}{\prod_{\lambda=1}^{m} |a_\lambda|}$. $\varphi(z)$ is a holomorphic function in the unit disc D, $|\varphi(z)| \leq 1$ for $z \in D$ (see, Theorem1).

Since $\varphi(z)$ function satisfies the conditions of the Schwarz-Pick lemma, we obtain

$$|\varphi'(z)| \leq \frac{1 - |\varphi(z)|^2}{1 - |z|^2}$$

and

$$\left|\Re\left(z \frac{\varphi'(z)}{\varphi(z)}\right)\right| \leq \left|z \frac{\varphi'(z)}{\varphi(z)}\right| \leq \frac{|z|}{1 - |z|^2} \left(\frac{1}{|\varphi(z)|} - |\varphi(z)|\right). \tag{3}$$

From (2) and Schwarz lemma, we take

$$|\varphi(z)| \geq \frac{|d_0| - |z|}{1 - |d_0|\,|z|}.$$

Therefore, from (3)

$$\left|\Re\left(z\frac{\varphi'(z)}{\varphi(z)}\right)\right| \leq \frac{|z|\left(1-|d_0|^2\right)}{(|d_0|-|z|)\,(1-|d_0|\,|z|)}. \tag{4}$$

Moreover, it can be seen that

$$\begin{aligned}\Re\left(z\frac{f'(z)}{f(z)}\right) &= p + \Re\left(\sum_{\lambda=1}^{m}\frac{\left(1-|a_\lambda|^2\right)z}{(1-\overline{a_\lambda}z)\,(z-a_\lambda)}\right)\\ &\quad -\Re\left(\sum_{k=1}^{n}\frac{\left(1-|b_k|^2\right)z}{(1-\overline{b_k}z)\,(z-b_k)}\right) + \Re\left(z\frac{\psi'(z)}{\psi(z)}\right)\end{aligned}$$

and

$$\begin{aligned}\Re\left(z\frac{f'(z)}{f(z)}\right) &\geq p + \Re\left(\sum_{\lambda=1}^{m}\frac{\left(1-|a_\lambda|^2\right)z}{(1-\overline{a_\lambda}z)\,(z-a_\lambda)}\right)\\ &\quad -\Re\left(\sum_{k=1}^{n}\frac{\left(1-|b_k|^2\right)z}{(1-\overline{b_k}z)\,(z-b_k)}\right) - \Re\left(z\frac{\psi'(z)}{\psi(z)}\right).\end{aligned}$$

By (4) and taking into account, f and f' are defined at the boundary point $z_0 \in \partial D$. Thus, we get the following inequality:

$$\begin{aligned}\Re\left(z_0\frac{f'(z_0)}{f(z_0)}\right) &\geq p + \Re\left(\sum_{\lambda=1}^{m}\frac{\left(1-|a_\lambda|^2\right)z_0}{(1-\overline{a_\lambda}z_0)\,(z_0-a_\lambda)}\right)\\ &\quad -\Re\left(\sum_{k=1}^{n}\frac{\left(1-|b_k|^2\right)z_0}{\left(1-\overline{b_k}z_0\right)(z_0-b_k)}\right) - \frac{|z_0|\left(1-|d_0|^2\right)}{(|d_0|-|z_0|)\,(1-|d_0|\,|z_0|)}\end{aligned}$$

which implies that

$$\Re\left(z_0\frac{f'(z_0)}{f(z_0)}\right) \geq \sum_{\lambda=1}^{m}\frac{1-|a_\lambda|^2}{|z_0-a_\lambda|^2} - \sum_{k=1}^{n}\frac{1-|b_k|^2}{|z_0-a_k|^2} + p + \frac{1+|d_0|}{1-|d_0|}.$$

The equality in (1) is obtained for function

$$f(z) = z^p\frac{z-\alpha}{1-\alpha z}\prod_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\prod_{k=1}^{n}\frac{1-\overline{b_k}z}{z-b_k}.$$

as show simple calculations.

In the following theorem, we formulated boundary "Carathéodory inequality" (see, [5]) as long the Schwarz lemma at the boundary (see, [6]). Besides, we has following the result, which can be offered as the boundary refinement of the Carathéodory inequality.

Theorem 3. *Let $f(z)=c_pz^p+.....,\ c_p\neq 0,\ p\geq 1$ be a holomorphic function in the unit disc D except at the poles $b_1,...,b_n$ and $a_1,...,a_m$ zeros of $f(z)$ in the unit disc D that are different from zero. Suppose that none of the limiting values of $\Re f(z)$ as z approaches the boundary of the unit disc D exceed A. Assume that f and f' are defined at the boundary $z_0\in\partial D$. Then we have the inequality*

$$\left|\frac{f'(z_0)}{f(z_0)}\right|\geq\frac{1}{2}\sum_{\lambda=1}^{m}\frac{1-|a_\lambda|^2}{|z_0-a_\lambda|^2}-\frac{1}{2}\sum_{k=1}^{n}\frac{1-|b_k|^2}{|z_0-a_k|^2}+\frac{1}{2}\left(p+\frac{2A\prod\limits_{\lambda=1}^{m}|a_\lambda|+|c_p|\prod\limits_{k=1}^{n}|b_k|}{2A\prod\limits_{\lambda=1}^{m}|a_\lambda|-|c_p|\prod\limits_{k=1}^{n}|b_k|}\right). \quad (5)$$

The inequality (5) *is sharp, with equality at a point z (which is not a pole) for the function*

$$f(z)=2A\frac{z^p\frac{z-\upsilon}{1-\upsilon z}\prod\limits_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\prod\limits_{k=1}^{n}\frac{1-\overline{b_k}z}{z-b_k}}{1+z^p\frac{z-\upsilon}{1-\upsilon z}\prod\limits_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}\prod\limits_{k=1}^{n}\frac{1-\overline{b_k}z}{z-b_k}},\ 0<\upsilon<1$$

where $|b_k|<1,\ k=1,...,n,\ |a_\lambda|<1,\ \lambda=1,...,m,\ a_1,a_2,...,a_m$ and $b_1,b_2,...,b_n$ are positive real numbers.

Proof.

$$\omega(z)=\frac{f(z)}{z^p\left(2A-f(z)\right)\prod\limits_{\lambda=1}^{m}\frac{z-a_\lambda}{1-\overline{a_\lambda}z}}\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}=b_0+b_pz+.... \quad (6)$$

is a holomorphic function in the unit disc D, $|\omega(z)|\leq 1$ for $z\in D$ (see, Theorem1), where $|b_0|=\frac{|c_p|\prod\limits_{k=1}^{n}|b_k|}{2A\prod\limits_{\lambda=1}^{m}|a_\lambda|}$.

The function $\omega(z)$ satisfies the hypothesis of the Schwarz-Pick lemma. Thus, we obtain

$$\left|\omega'(z)\right|\leq\frac{1-|\omega(z)|^2}{1-|z|^2}$$

and

$$\left|\Re\left(z\frac{\omega'(z)}{\omega(z)}\right)\right|\leq\left|z\frac{\omega'(z)}{\omega(z)}\right|\leq\frac{|z|}{1-|z|^2}\left(\frac{1}{|\omega(z)|}-|\omega(z)|\right). \quad (7)$$

From (6) and Schwarz lemma, we take

$$|\omega(z)|\geq\frac{|b_0|-|z|}{1-|b_0|\,|z|}.$$

Therefore, from (7)

$$\left|\Re\left(z\frac{\omega'(z)}{\omega(z)}\right)\right| \leq \frac{|z|\left(1-|b_0|^2\right)}{(|b_0|-|z|)\left(1-|b_0|\,|z|\right)}. \tag{8}$$

In addition, it can be seen that

$$\begin{aligned}\Re\left(z\frac{f'(z)}{f(z)}\right)+\Re\left(z\frac{f'(z)}{2A-f(z)}\right) &= p+\Re\left(\sum_{\lambda=1}^{m}\frac{\left(1-|a_\lambda|^2\right)z}{(1-\overline{a_\lambda}z)(z-a_\lambda)}\right)\\ &\quad -\Re\left(\sum_{k=1}^{n}\frac{\left(1-|b_k|^2\right)z}{(1-\overline{b_k}z)(z-b_k)}\right)+\Re\left(z\frac{\omega'(z)}{\omega(z)}\right)\end{aligned}$$

and

$$\begin{aligned}\left|z\frac{f'(z)}{f(z)}\right|+\left|z\frac{f'(z)}{2A-f(z)}\right| &\geq p+\Re\left(\sum_{\lambda=1}^{m}\frac{\left(1-|a_\lambda|^2\right)z}{(1-\overline{a_\lambda}z)(z-a_\lambda)}\right)\\ &\quad -\Re\left(\sum_{k=1}^{n}\frac{\left(1-|b_k|^2\right)z}{(1-\overline{b_k}z)(z-b_k)}\right)+\Re\left(z\frac{\omega'(z)}{\omega(z)}\right).\end{aligned}$$

Since $|2A-f(z)|>|f(z)|$, we take

$$\begin{aligned}\left|z\frac{f'(z)}{f(z)}\right|+\left|z\frac{f'(z)}{f(z)}\right| &\geq p+\Re\left(\sum_{\lambda=1}^{m}\frac{\left(1-|a_\lambda|^2\right)z}{(1-\overline{a_\lambda}z)(z-a_\lambda)}\right)\\ &\quad -\Re\left(\sum_{k=1}^{n}\frac{\left(1-|b_k|^2\right)z}{(1-\overline{b_k}z)(z-b_k)}\right)-\Re\left(z\frac{\omega'(z)}{\omega(z)}\right).\end{aligned}$$

By (8) and taking into account, f and f' are defined at the boundary point $z_0\in\partial D$. Thus, we get the following inequality:

$$\begin{aligned}2\left|z_0\frac{f'(z_0)}{f(z_0)}\right| &\geq p+\Re\left(\sum_{\lambda=1}^{m}\frac{\left(1-|a_\lambda|^2\right)z_0}{(1-\overline{a_\lambda}z_0)(z_0-a_\lambda)}\right)\\ &\quad -\Re\left(\sum_{k=1}^{n}\frac{\left(1-|b_k|^2\right)z_0}{(1-\overline{b_k}z_0)(z_0-b_k)}\right)-\frac{|z_0|\left(1-|b_0|^2\right)}{(|b_0|-|z_0|)\left(1-|b_0|\,|z_0|\right)}\end{aligned}$$

which implies that

$$\left|\frac{f'(z_0)}{f(z_0)}\right| \geq \frac{1}{2}\sum_{\lambda=1}^{m}\frac{1-|a_\lambda|^2}{|z_0-a_\lambda|^2}-\frac{1}{2}\sum_{k=1}^{n}\frac{1-|b_k|^2}{|z_0-a_k|^2}+\frac{1}{2}\left(p+\frac{1+|b_0|}{1-|b_0|}\right).$$

We obtain the inequality (5) with an obvious equality case.

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