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Maximal regularity of parameter dependent differential-operator equations on the halfline

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Abstract. This paper focuses on boundary value problems for differential-operator equations in half line. The equations and boundary conditions contain certain small and spectral parameters. The uniform L_p- separability is obtained. Here the explicit formula for the solution is given and behavior of solution is established with small parameter. It used to obtain singular perturbation result for the convolution differential-operator equation.

Key Words and Phrases: Singular perturbation; Boundary value problems; Differential-operator equations; Uniformly R-boundedness; Maximal L_p- regularity; Operator-valued multipliers; Coercive uniformly estimaete.

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1. Introduction

It is well known that the differential equations with parameters play important role in modelling of physical processes. Differential-operator equations (DOEs) with small parameters have also significant applications to the developed theory to problems in mathematical physics. Note that, DOEs are studied e.g., in $[1-3]$, $[5-11]$, [21] and the references therein. Moreover, convolution differential-operator equations have been studied e.g., in $[12-17]$.

The main aim of this paper is to show the uniform separability properties of boundary value problems (BVPs) for the following DOE with parameters

$$-\varepsilon u''(t) + A_\lambda u(t) = f(t), \tag{1.1}$$

where $A_\lambda = A+\lambda I$, A are linear operators in a Banach space E, $a = a(t)$ is a scalar valued function on $(0;\infty)$, ε is a small and λ is a complex parameter, $u(t) = u(\varepsilon, t)$.

We derive the representation of solution involving semigroup generated by operator A which allows to obtain the maximal regularity properties of DOEs and the sharp coercive L_p estimates of solution uniformly with respect to small and spectral parameter.

The treatment of the singular perturbation problem for the abstract integro-differential equations studied e.g in [8] . In contrast to these, here uniform separability properties of the problem (1.1) is derived in $L_p(0,\infty;E)$.

Since the Banach space E and linear operator A are arbitrary, by chousing the space E and operators A we can obtained different results which occur in a wide variety of physical systems.

The present paper is organized as follows. The first section in this paper contains an introduction. Section 2 collects definitions, some notations and basic propertis of vector-valued function spaces, in particular, weighted $L_{p,\gamma}-$spaces and weighted multiplier conditions. In section 3 we consider the corresponding homogeneous problem and prepares for the proof of the main result of this paper. In section 4 by applying this results the uniform L_p-separability of nonhomogeneous problem (1.1) is proved.

2. Notation and conventions

Let E be a Banach space and $L_p(\Omega; E)$ denotes the space of strongly measurable E-valued functions that are defined on the measurable subset $\Omega \subset \mathbb{R}^n$ with the norm

$$\|f\|_{L_p} = \|f\|_{L_p(\Omega;E)} = \{.\left(\int\limits_{\Omega} \|f(x)\|_E^p \, dx\right)^{\frac{1}{p}}, 1 \leq p < \infty, ess \sup_{x\in\Omega} \|f(x)\|_E\,,\ p = \infty.$$

Let $\mathbb{N}, \mathbb{R}$ denote the sets of natural and real numbers, respectively.

Let $\mathbb{C}$ be the set of the complex numbers and

$$S_\varphi = \{\lambda;\ \ \lambda \in \mathbb{C},\ |\arg \lambda| \leq \varphi\} \cup \{0\}\,, 0 \leq \varphi < \pi.$$

Let Ω be a domain in $\mathbb{R}^n$. $C(\Omega, E)$ and $C^{(m)}(\Omega; E)$ will denote the spaces of E -valued bounded, uniformly strongly continuous and m -times continuously differentiable functions on Ω, respectively.

A linear operator A is said to be φ-positive in a Banach space E with bound $M > 0$ if $D(A)$ is dense on E and

$$\|R(-\lambda, A)\|_{B(E)} = \left\|(A + \lambda I)^{-1}\right\|_{B(E)} \leq M\,(1 + |\lambda|)^{-1}\,,$$

for any $\lambda \in S_\varphi$, $0 \leq \varphi < \pi$, where I is the identity operator in E. For convenience, sometimes we write $A + \lambda$ instead of $A + \lambda I$ and denoted by A_λ. It is known [19, §1.15.1] that there exist the fractional powers A^θ of a positive operator A. Let $E\left(A^\theta\right)$ denote the space $D\left(A^\theta\right)$ with the graph norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\|A^\theta u\right\|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty,\ \ -\infty < \theta < \infty.$$

Let $S(\mathbb{R}^n; E)$ denote the Schwartz class, i.e. the space of E-valued rapidly decreasing smooth functions on $\mathbb{R}^n$ and F denote the Fourier transformation. If the map $u \to \Lambda u = F^{-1}\Psi(\xi)\,Fu$, $u \in S(\mathbb{R}^n; E_1)$ is well defined and extends to a bounded linear operator

$$\Lambda:\ L_p(\mathbb{R}^n; E_1) \to\ L_p(\mathbb{R}^n; E_2)$$

then a function $\Psi \in C(\mathbb{R}^n; B(E_1, E_2))$ is called a Fourier multiplier from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$. The set of all multipliers from $L_p(\mathbb{R}^n; E_1)$ to $L_p(\mathbb{R}^n; E_2)$ will be denoted by $M_p^p(E_1, E_2)$.

Let $\Phi_h = \{\Psi_h \in M_p^p(E_1, E_2),\ h \in Q\}$. We say that W_h is a collection of uniformly bounded multipliers (UBM) if there exists a positive constant M independent on $h \in Q$ such that

$$\left\|F^{-1}\Psi_h F u\right\|_{L_p(\mathbb{R}^n; E_2)} \leq M \left\|u\right\|_{L_p(\mathbb{R}^n; E_1)},$$

for all $h \in Q$ and $u \in S(\mathbb{R}^n; E_1)$,where Q denote a set of some parameters.

A family of operators $T \subset B(E_1, E_2)$ is called R-bounded (see e.g. [4] , [19], [20]) if there is a constant $C > 0$ such that for all $T_1, T_2, ..., T_m \in T$ and $u_{1,} u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$ the inequality

$$\int\limits_{\Omega} \left\| \sum_{j=1}^{m} r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int\limits_{\Omega} \left\| \sum_{j=1}^{m} r_j(y) u_j \right\|_{E_1} dy,$$

is valid, where $\{r_j\}$ is a sequence of independent symmetric $\{-1, 1\}$-valued random variables on Ω. The smallest C for which the above estimate holds is called a R-bound of the collection T and denoted by $R\{T\}$.

Let family of operators $T_h \subset B(E_1, E_2)$ depending on the parameter $h \in Q$. Here T_h is called uniformly R-bounded with respect to h if there is a constant C independent of $h \in Q$, such that

$$\sup_{h \in Q} R\{T_h\} \leq C.$$

A Banach space E is said to be a space satisfying a multiplier condition if, for any $\Psi \in C^{(1)}(\mathbb{R}; B(E))$, the R-boundedness of the set

$$\left\{|\xi|^j D^j \Psi(\xi) : \xi \in \mathbb{R} \backslash \{0\},\ j = 0, 1\right\}, \tag{2.1}$$

implies that Ψ is a Fourier multiplier, i.e. $\Psi \in M_p^p(E)$ for any $p \in (1, \infty)$.

The φ-positive operator A is said to be R-positive in a Banach space E if the set

$$\left\{\xi (A + \xi)^{-1} :\ \xi \in S_\varphi\right\},\ 0 \leq \varphi < \pi,$$

is R-bounded.

The operator $A(t)$ is said to be uniformly φ-positive in E with respect to t with bound $M > 0$, if $D(A(t))$ is independent on t, dense in E, and

$$\left\|(A(t) + \lambda)^{-1}\right\| \leq \frac{M}{1 + |\lambda|},$$

for all $\lambda \in S_\varphi, 0 \leq \varphi < \pi$,where M does not depend on t and λ.

A positive operator $A(t)$ is said to be a uniformly R-positive in a Banach space E if there exists a $\varphi \in [0, \pi)$ such that the set

$$\left\{\xi\,(A+\xi I)^{-1};\quad \xi \in S_\varphi\right\}$$

is uniformly R-bounded.

Let E_0 and E be two Banach spaces and E_0 is continuously and densely embedded into E. Let m is a positive integer. $W_p^m\,(0,\infty;E_0,E)$ denotes the collection of E-valued functions $u \in L_p\,(0,\infty;E_0)$ that have the generalized derivatives $u^{(m)} \in L_p\,(0,\infty;E)$ with the norm

$$\|u\|_{W_p^m} = \|u\|_{W_p^m(0,\infty;E_0,E)} = \|u\|_{L_p(0,\infty;E_0)} + \left\|u^{(m)}\right\|_{L_p(0,\infty;E)} < \infty.$$

For $E_0 = E$ it denotes by $W_p^m\,(\Omega;E)$.

Let ε is a positive parameter. We define in the space $W_{p,\varepsilon}^m\,(0,\infty;E_0,E)$ the following parametrized norm

$$\|u\|_{W_{p,\varepsilon}^m(0,\infty;E_0,E)} = \|u\|_{L_p(0,\infty;E_0)} + \left\|\varepsilon u^{(m)}\right\|_{L_p(0,\infty;E)}.$$

3. DOEs with small parameter

Consider the following BVP for elliptic differential-operator equation with small parameters

$$\{.Lu = -\varepsilon u''(t) + A_\lambda u(t) = f(t), \quad t \in (0;\infty)\,, L_1 u = \varepsilon^{\frac{p+1}{2p}} \alpha u'\,(0) + \varepsilon^{\frac{1}{2p}} \beta u(0) = f_0 \quad (3.1)$$

where $u\,(t) = u\,(\varepsilon,t)$ is a solution of (3.1), A are linear operator in a Banach space E, $A_\lambda = A + \lambda I$, $a = a(t)$ is a scalar valued function on $\mathbb{R}_+ = (0;\infty)\,,\ f_0 \in E_p = (E(A),E)_{\theta,p}\,,$ here $(E(A),E)_{\theta,p}$ denotes the real interpolation space between $E(A)$ and E, $p \in (1,\infty)$, $\theta = \frac{1+p}{2p}$, α, β are complex numbers, ε is a small positive, and λ is a complex parameters, i.e., $\varepsilon \in (0,1)\,.$

For investigation of main problem first all of consider the corresponding homogeneous problem

$$\{.-\varepsilon u''(t) + A_\lambda u(t) = 0 L_1 u = f_0. \quad (3.2)$$

Condition 3.1. Assume the following conditions are satisfied:

1) E is a Banach space satisfying the uniformly multiplier condition for $p \in (1,\infty)\,;$

2) A is a $R-$positive operator in E for $0 \leq \varphi < \pi$.

Theorem 3.1. Assume Condition 3.1 holds and $-\beta\alpha^{-1} \in S_{\varphi_1}$, $0 \leq \varphi_1 + \varphi < \pi$. Then the problem (3.2) for $\forall f_0 \in E_p$ has a unique solution $u(t) \in W_p^2(\mathbb{R}_+; E(A), E)$ and the coercive estimate

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{\frac{i}{2}} \left\| u^{(i)}(t) \right\|_{L_p(\mathbb{R}_+;E)} + \|Au(t)\|_{L_p(\mathbb{R}_+;E)} \leq \tag{3.3}$$

$$C\left[|\lambda|^{1-\theta} \|f_0\|_E + \|f_0\|_{E_p} \right]$$

holds uniformly with respect to ε and $\lambda \in S_\varphi$ with sufficiently large $|\lambda|$.

Proof. We consider the BVP equivalent to (3.2)

$$\{.-u''(t) + \varepsilon^{-1} A_\lambda u(t) = 0 L_1 u = f_0. \tag{3.4}$$

By definition of positive operator, $\varepsilon^{-1}A_\lambda$ is $R-$positive uniformly with respect to ε. Then we have the estimate $\left\| \left(\varepsilon^{-1}A_\lambda + \mu\right)^{-1} \right\| \leq C|\mu|^{-1}$, where $|\arg\lambda| \leq \varphi$, $|\arg\mu| \leq \varphi_1$, $\varphi_1 + \varphi < \pi$ and C independent of ε, depending on φ only. Then in view of [5] and [21] an arbitrary solution of equation (3.4) belonging to the space $W_p^2(\mathbb{R}_+; E(A), E)$, has the form

$$u(t) = e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} g,$$

where $g \in (E(A), E)_{\frac{1}{2p},p}$.

Taking into account boundary conditions we obtain that the representation of the solution of the problem (3.4)

$$u(t) = \varepsilon^{-\frac{1}{2p}} e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} \left[-\alpha A_\lambda^{\frac{1}{2}} + \beta \right]^{-1} f_0. \tag{3.5}$$

Then in view of positivity of A we obtain from (3.5)

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{\frac{i}{2}} \left\| u^{(i)}(t) \right\|_{L_p(\mathbb{R}_+;E)} + \|Au(t)\|_{L_p(\mathbb{R}_+;E)} \leq$$

$$\leq \left\| \varepsilon^{-\frac{1}{2p}} |\lambda| e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} \left(-\alpha A_\lambda^{\frac{1}{2}} + \beta \right)^{-1} f_0 \right\|_{L_p(\mathbb{R}_+;E)} +$$

$$\left\| \varepsilon^{-\frac{1}{2p}} |\lambda|^{\frac{1}{2}} \left(-A_\lambda^{\frac{1}{2}} \right) e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} \left[-\alpha A_\lambda^{\frac{1}{2}} + \beta \right]^{-1} f_0 \right\|_{L_p(\mathbb{R}_+;E)} +$$

$$\left\| \varepsilon^{-\frac{1}{2p}} A_\lambda e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} \left[-\alpha A_\lambda^{\frac{1}{2}} + \beta \right]^{-1} f_0 \right\|_{L_p(\mathbb{R}_+;E)} +$$

$$\left\| \varepsilon^{-\frac{1}{2p}} A e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} \left[-\alpha A_\lambda^{\frac{1}{2}} + \beta \right]^{-1} f_0 \right\|_{L_p(\mathbb{R}_+;E)} \leq$$

$$\leq \varepsilon^{-\frac{1}{2p}} \left\{ C_1 \left\| \bar{A}_\lambda^{\frac{1}{2}} e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} f_0 \right\|_{L_p(\mathbb{R}_+;E)} + C_2 \left\| A^{\frac{1}{2}} e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} f_0 \right\|_{L_p(\mathbb{R}_+;E)} \right\} \leq$$

$$\varepsilon^{-\frac{1}{2p}} \left\{ C_1 \left(\int_0^\infty \left\| A_\lambda^{\frac{1}{2}} e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} f_0 \right\|_E^p dt \right)^{\frac{1}{p}} + C_2 \left(\int_0^\infty \left\| A^{\frac{1}{2}} e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{\frac{1}{2}}} f_0 \right\|_E^p dt \right)^{\frac{1}{p}} \right\}.$$

By changing of variable $t\varepsilon^{-\frac{1}{2}} = z$ redenoting and in view of Theorem 2.1 (see, [5]) we obtain

$$\int_0^\infty \left\| A_\lambda^{\frac{1}{2}} e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{-\frac{1}{2}}} f_0 \right\|_E^p dt \leq C \left[\ |\lambda|^{1-\theta} \ \| \ f_0\|_E^p + \| \ f_0\|_{E_p}^p \right]. \tag{3.6}$$

On the other hand, by using of the positivity of operator A we have

$$\left\| A A_\lambda^{-\frac{1}{2}} \right\| = \left\| (A_\lambda - \lambda)\, A_\lambda^{-\frac{1}{2}} \right\| = \left\| A_\lambda^{\frac{1}{2}} - \lambda A_\lambda^{-\frac{1}{2}} \right\| \leq \left(1 + \|\lambda A_\lambda^{-1}\|\right) \left\| A_\lambda^{\frac{1}{2}} \right\|.$$

By using the similar technique as in (3.6) we have

$$\int_0^\infty \left\| A^{\frac{1}{2}} e^{-t\varepsilon^{-\frac{1}{2}} A_\lambda^{-\frac{1}{2}}} f_0 \right\|_E^p dt \leq C \left[\ |\lambda|^{1-\theta} \ \| \ f_0\|_E^p + \| \ f_0\|_{E_p}^p \right]. \tag{3.7}$$

The from (3.6) and (3.7) we obtain (3.3).

Note that the solution of the problem (3.2) dependes on the parameter ε, i.e., $u = u(x, \varepsilon)$.

4. Separability properties of parameter dependent nonhomogeneous differential-operator equation

Now we are ready to present our main result. Consider the following nonhomogeneous problem

$$\{.-\varepsilon u''(t) + A_\lambda u(t) = f(t) \qquad L_1 u = f_0 \tag{4.1}$$

Theorem 4.1. Assume that the all conditions of Theorem 3.1 are satisfied. Then for $f(t) \in L_p(\mathbb{R}_+;E)$, $\forall f_0 \in E_p$ and λ, with sufficiently large $|\lambda|$, the problem (4.1) has a unique solution $u \in W_p^2(\mathbb{R}_+;E(A),E)$ and the following coercive uniformly estimate holds

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\left\|u^{(i)}(t)\right\|_{L_p(\mathbb{R}_+;E)}+\|Au(t)\|_{L_p(\mathbb{R}_+;E)}\leq \tag{4.2}$$

$$\leq C\left[\|f\|_{L_p(\mathbb{R}_+;E)}+|\lambda|^{1-\theta}\|f_0\|_E+\|f_0\|_{E_p}\right].$$

Proof. Let us define

$$\overline{f(t)}=\{.f(t),\quad t\in[0,\infty)0,\qquad t\notin[0,\infty).$$

We now show that the problem (4.1) has a solution $u\in W_p^2(\mathbb{R}_+;E(A),E)$ for all $f(t)\in L_p(\mathbb{R}_+;E)$, $f_0\in E_p$ and $u(t)=u_1(t)+u_2(t)$, where $u_1(t)$ is the restriction on $[0,\infty)$ of the solution $\overline{u}_1(t)$ to the equation

$$-\varepsilon\overline{u}_1''(t)+A_\lambda\overline{u}_1(t)=\overline{f(t)},\qquad t\in(-\infty;+\infty) \tag{4.3}$$

and $u_2(t)$ is a solution of the problem

$$\{.-\varepsilon u_2''(t)+A_\lambda u_2(t)=0\qquad L_1u_2=f_0-L_1u_1$$

It is not hard to see that the solution of (4.3) is given by the formula

$$\overline{u}_1(t)=F^{-1}L_0^{-1}(\lambda,\varepsilon,\xi)F\overline{f},$$

where $L_0(\lambda,\varepsilon,\xi)=A+\varepsilon\xi^2+\lambda I$ and F denotes the Fourier transformation .

It follows from the expression above that

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\left\|\overline{u}_1^{(i)}(t)\right\|_{L_p(\mathbb{R};E)}+\|A\overline{u}_1(t)\|_{L_p(\mathbb{R};E)}=$$

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\left\|F^{-1}\xi^iL_0^{-1}(\lambda,\varepsilon,\xi)F\overline{f}\right\|_{L_p(\mathbb{R};E)}+\left\|F^{-1}AL_0^{-1}(\lambda,\varepsilon,\xi)F\overline{f}\right\|_{L_p(\mathbb{R};E)}. \tag{4.4}$$

Let us show that operator-functions $AL_0^{-1}(\lambda,\varepsilon,\xi)$ and $\sum\limits_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^iL_0^{-1}(\lambda,\varepsilon,\xi)$ are Fourier multipliers in $L_p(\mathbb{R};E)$.

Taking into account positivity of A and [5, Lemma 2.3] we have

$$\left\|L_0^{-1}(\lambda,\varepsilon,\xi)\right\|_{B(E)}=\left\|\left(A+\varepsilon\xi^2+\lambda\right)^{-1}\right\|_{B(E)}\leq C\left(1+\varepsilon\xi^2+|\lambda|\right)^{-1},$$

$$\left\|AL_0^{-1}(\lambda,\varepsilon,\xi)\right\|_{B(E)}=\left\|A\left(A+\varepsilon\xi^2+\lambda\right)^{-1}\right\|_{B(E)}\leq C. \tag{4.5}$$

It is known that

$$\xi\frac{d}{d\xi}\left[A\left(A+\varepsilon\xi^{2}+\lambda\right)^{-1}\right]=-2\varepsilon\xi^{2}\left(A+\varepsilon\xi^{2}+\lambda\right)^{-1}A\left(A+\varepsilon\xi^{2}+\lambda\right)^{-1}$$

The $R-$positivity of the operator A imply that the sets

$$\left\{-2\varepsilon\xi^{2}\left(A+\varepsilon\xi^{2}+\lambda\right)^{-1};\ \xi\in\mathbb{R}\backslash\{0\}\right\},\left\{A\left(\ A+\varepsilon\xi^{2}+\lambda\right)^{-1};\ \xi\in\mathbb{R}\backslash\{0\}\right\}$$

are $R-$bounded and it is $R-$bounds are independent on ε and λ.

Moreover, by using of additional and product properties and Kahane's contraction principle for family of $R-$bounded (see, [4]) operators we have the $R-$boundedness of collection $\xi\frac{d}{d\xi}\left[A\left(A+\varepsilon\xi^{2}+\lambda\right)^{-1}\right]$. This means that

$$\sup_{\lambda,\varepsilon}R\left\{\xi\frac{d}{d\xi}\left[A\left(A+\varepsilon\xi^{2}+\lambda\right)^{-1}\right];\ \xi\in\mathbb{R}\backslash\{0\}\right\}\le C.$$

Now, we prove the uniformly $R-$boundedness of the family of operator-functions

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_{0}^{-1}\left(\lambda,\varepsilon,\xi\right).$$

It is clear to see that

$$\left\|\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_{0}^{-1}\left(\lambda,\varepsilon,\xi\right)\right\|_{B(E)}\le C\left|\lambda\right|\sum_{i=0}^{2}\left[\varepsilon^{\frac{1}{2}}\left|\xi\right|\left|\lambda\right|^{-\frac{1}{2}}\right]^{i}\left\|L_{0}^{-1}\left(\lambda,\varepsilon,\xi\right)\right\|_{B(E)}.$$

Then setting $y=\varepsilon^{\frac{1}{2}}\left|\xi\right|\left|\lambda\right|^{-\frac{1}{2}}$ in the following well known inequality

$$y^{i}\le C\left(1+y^{k}\right),\ \text{for } y\ge 0,\ i\le k,$$

we obtain

$$\left\|\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_{0}^{-1}\left(\lambda,\varepsilon,\xi\right)\right\|_{B(E)}\le C\left(|\lambda|+\varepsilon\xi^{2}\right)\left(1+\varepsilon\xi^{2}+|\lambda|\right)^{-1}\le C. \tag{4.6}$$

Due to $R-$positivity A, we get that the set

$$\left\{\left(|\lambda|+\varepsilon\xi^{2}\right)L_{0}^{-1}\left(\lambda,\varepsilon,\xi\right);\ \xi\in\mathbb{R}\backslash\{0\}\ \right\}$$

is uniformly $R-$bounded. Then by Kahane's contraction principle we have the $R-$boundedness of set

$$\left\{\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_{0}^{-1}\left(\lambda,\varepsilon,\xi\right);\ \xi\in\mathbb{R}\backslash\{0\}\ \right\}.$$

Since,

$$\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_0^{-1}(\lambda,\varepsilon,\xi)=\left[\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}\right]\left(|\lambda|+\varepsilon\xi^{2}\right)^{-1}\left(|\lambda|+\varepsilon\xi^{2}\right)L_0^{-1}(\lambda,\varepsilon,\xi).$$

Moreover, it is clear to see that

$$\xi\frac{d}{d\xi}\left[\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_0^{-1}\right]=\xi\left[(|\lambda|+|\lambda|^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\xi+\varepsilon\xi^{2})L_0^{-1}\right]'_{\xi}=$$

$$=\left(|\lambda|^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\xi+2\varepsilon\xi^{2}\right)L_0^{-1}-2(|\lambda|+|\lambda|^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\xi+\varepsilon\xi^{2})L_0^{-1}(\varepsilon\xi^{2})L_0^{-1}=$$

$$=\left(|\lambda|^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\xi+2\varepsilon\xi^{2}\right)L_0^{-1}-2\,|\lambda|\,L_0^{-1}(\varepsilon\xi^{2})L_0^{-1}-2(|\lambda|^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}\xi+\varepsilon\xi^{2})L_0^{-1}(\varepsilon\xi^{2})L_0^{-1},$$

where

$$L_0^{-1}=L_0^{-1}(\lambda,\varepsilon,\xi).$$

Taking into account well known inequality, as we mentioned before, we have $|\lambda|^{\frac{1}{2}}\varepsilon^{\frac{1}{2}}|\xi|=|\lambda|\,|\lambda|^{-\frac{1}{2}}\varepsilon^{\frac{1}{2}}|\xi|\le C\,|\lambda|\,(1+\varepsilon\,|\lambda|^{-1}|\xi|^{2})=C(|\lambda|+\varepsilon\,|\xi|^{2})$.

By using (4.5) and $R-$positivity of the operator A, in a similar way we obtain the $R-$boundedness of the sets

$$\left\{|\lambda|\,L_0^{-1}:\xi\in\mathbb{R}\backslash\{0\}\right\},\ \left\{\varepsilon\xi^{2}L_0^{-1}:\xi\in\mathbb{R}\backslash\{0\}\right\},\ \left\{\left(|\lambda|+\varepsilon\xi^{2}\right)L_0^{-1}:\xi\in\mathbb{R}\backslash\{0\}\right\}.$$

In view of Kahane's contraction principle, from the additional and product properties of the family of $R-$bounded operators, for all $\xi_i\in R$, $u_i\in E$, $i=\overline{1,n}$ we have

$$\int_{\Omega}\left\|\sum_{i=1}^{n}r_i(y)\xi\frac{d}{d\xi}\left(\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_0^{-1}(\lambda,\varepsilon,\xi)\right)u_i\right\|_{E}dy\le$$

$$C\int_{\Omega}\left\|\sum_{i=1}^{n}\left[\ |\lambda|\,L_0^{-1}(\lambda,\varepsilon,\xi)+\left(|\lambda|+\varepsilon\xi^{2}\right)L_0^{-1}(\lambda,\varepsilon,\xi)+\varepsilon\xi^{2}L_0^{-1}(\lambda,\varepsilon,\xi)\right]r_i(y)u_i\right\|_{E}dy\le$$

$$\le C\int_{\Omega}\left\|\sum_{i=1}^{n}r_i(y)u_i\right\|_{E}dy,$$

where $\{r_i(y)\}$ of is sequence of independent symmetric $\{-1;1\}-$valued random variables. It implies that the uniformly $R-$boundedness the family of operator-functions $\xi\frac{d}{d\xi}\left[\sum\limits_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_0^{-1}(\lambda,\varepsilon,\xi)\right]$, i.e.

$$\sup_{\lambda,\varepsilon} R\left\{\xi\frac{d}{d\xi}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_0^{-1}(\lambda,\varepsilon,\xi):\xi\in\mathbb{R}\backslash\{0\}\right\}\leq C.$$

By virtue of definition of Fourier multiplier, from (4.5), (4.6) it follows that the operator-functions $AL_0^{-1}(\lambda,\varepsilon,\xi)$ and $\sum\limits_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon^{\frac{i}{2}}\xi^{i}L_0^{-1}(\lambda,\varepsilon,\xi)$ are the uniform collection of Fourier multipliers in $L_p(\mathbb{R},E)$. Then from (4.4) we obtain that there exists a unique solution $\overline{u}_1\in W_p^2(\mathbb{R};E(A),E)$ of (4.3) and the uniform estimate holds

$$\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}|\lambda|^{1-\frac{i}{2}}\left\|\overline{u}_1^{(i)}\right\|_{L_p(\mathbb{R};E)}+\|A\overline{u}_1\|_{L_p(\mathbb{R};E)}\leq C\left\|\overline{f}\right\|_{L_p(\mathbb{R};E)}. \tag{4.7}$$

Note that the $u_1(t)$ is the restriction on $[0,\infty)$ of the solution $\overline{u}_1(t)$ the eqution (4.3). From (4.7) implies that $u_1(t)\in W_p^2(0,\infty;E(A),E)$. By using of embedding theorems (see [18] and [19,§1.8] we get $u_1'(0)\in E_p$ and $u_1(0)\in E_p$. Hence $L_1u_1\in E_p$. Then by virtue of Theorem 3.1 we obtain the problem

$$\{.-\varepsilon u_2''(t)+A_\lambda u_2(t)=0 \qquad L_1u_2=f_0-L_1u_1$$

has a unique solution $u_2(t)$ that belongs to the space $W_p^2(0,\infty;E(A),E)$ for $f\in L_p(\mathbb{R}_+;E)$. Consequently, for sufficiently large $|\lambda|$ we have

$$\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}|\lambda|^{1-\frac{i}{2}}\left\|u_2^{(i)}\right\|_{L_p(\mathbb{R}_+;E)}+\|Au_2\|_{L_p(\mathbb{R}_+;E)}\leq C(\|f_0\|_{E_p}+|\lambda|^{1-\theta}\|f_0\|_E+$$

$$\left.+\varepsilon^{\theta}\left\|u_1'\right\|_{C(\mathbb{R}_+;E_p)}+\varepsilon^{\theta}|\lambda|^{1-\theta}\|u_1\|_{C(\mathbb{R}_+;E)}\right).$$

Moreover, from (4.7) for $|\arg\lambda|\leq\varphi$ we obtain

$$\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}|\lambda|^{1-\frac{i}{2}}\left\|u_1^{(i)}\right\|_{L_p(\mathbb{R}_+;E)}+\|Au_1\|_{L_p(\mathbb{R}_+;E)}\leq C\|f\|_{L_p(\mathbb{R}_+;E)}. \tag{4.8}$$

By using of [18] and by estimate (4.8) we obtain

$$\begin{aligned}\varepsilon^{\theta}\left\|u_1'(0)\right\|_{E_p} &\leq C\|u_1\|_{W^2_{p,\varepsilon}(\mathbb{R}_+;E(A),E)}\leq C\|f\|_{L_p(\mathbb{R}_+;E)},\\ \varepsilon^{\theta}\|u_1(0)\|_{E_p} &\leq C\|u_1\|_{W^2_{p,\varepsilon}(\mathbb{R}_+;E(A),E)}\leq C\|f\|_{L_p(\mathbb{R}_+;E)}.\end{aligned}$$

From [18] it follows

$$\begin{aligned}
|\xi|\,\varepsilon^{\theta}\left\|u'(0)\right\| &\leq C\left(|\xi|^{\frac{1}{p}}\left\|u\right\|_{W^2_{p,\varepsilon}(\mathbb{R}_+;,E)}+|\xi|^{2+\frac{1}{p}}\left\|u\right\|_{L_p(\mathbb{R}_+;E)}\right),\\
|\xi|^2\,\varepsilon^{\theta}\left\|u(0)\right\| &\leq C\left(|\xi|^{\frac{1}{p}}\left\|u\right\|_{W^2_{p,\varepsilon}(\mathbb{R}_+;E)}+|\xi|^{2+\frac{1}{p}}\left\|u\right\|_{L_p(\mathbb{R}_+;E)}\right).
\end{aligned}$$

For $\lambda=\xi^2$ we obtain

$$\begin{aligned}
|\lambda|^{1-\theta}\,\varepsilon^{\theta}\left\|u'(0)\right\| &\leq C\left(\left\|u\right\|_{W^2_{p,\varepsilon}(\mathbb{R}_+;E)}+|\lambda|\left\|u\right\|_{L_p(\mathbb{R}_+;E)}\right), (4.9)\\
|\lambda|^{1-\frac{1}{2p}}\,\varepsilon^{\theta}\left\|u(0)\right\| &\leq C\left(\left\|u\right\|_{W^2_{p,\varepsilon}(\mathbb{R}_+;E)}+|\lambda|\left\|u\right\|_{L_p(\mathbb{R}_+;E)}\right).
\end{aligned}$$

Then (4.8) and (4.9) we have estimate for $u_1'(0)$ and $u_1(0)$, i.e.

$$\begin{aligned}
|\lambda|^{1-\theta}\,\varepsilon^{\theta}\left\|u_1'(0)\right\| &\leq C\left(\left\|u_1\right\|_{W^2_{p,\varepsilon}(\mathbb{R}_+;E)}+|\lambda|\left\|u_1\right\|_{L_p(\mathbb{R}_+;E)}\right)\leq C\left\|f\right\|_{L_p(\mathbb{R}_+;E)}, (4.10)\\
|\lambda|^{1-\frac{1}{2p}}\,\varepsilon^{\theta}\left\|u_1(0)\right\| &\leq C\left(\left\|u_1\right\|_{W^2_{p,\varepsilon}(\mathbb{R}_+;E)}+|\lambda|\left\|u_1\right\|_{L_p(\mathbb{R}_+;E)}\right)\leq C\left\|f\right\|_{L_p(\mathbb{R}_+;E)}.
\end{aligned}$$

From estimates (4.7)-(4.10) we have

$$\sum_{i=0}^{2}\varepsilon^{\frac{i}{2}}\,|\lambda|^{1-\frac{i}{2}}\left\|u_2^{(i)}\right\|_{L_p(\mathbb{R}_+;E)}+\left\|Au_2\right\|_{L_p(\mathbb{R}_+;E)}\leq$$
$$C\left(\left\|f\right\|_{L_p(\mathbb{R}_+;E)}+\left\|f_0\right\|_{E_p}+|\lambda|^{1-\theta}\left\|f_0\right\|_E\right).$$

Finally, using the estimate (4.8) and from the above estimate we have (4.2).

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