Journal of Contemporary Applied Mathematics
V. 3, No 1, 2013, September
ISSN 2222-5498

A Class of Strong Limit Theorems for Multivariate Function Sequence of mth-Order Nonhomogeneous Markov Chains on the Generalized Gambling Systems

Kangkang Wang, Xiaofei Ye, Ting Teng, Jingping Jia

Abstract. In this paper, we study the local limit theorems for the sequence of multivariate functions of the mth-order nonhomogeneous Markov chain with respect to the generalized random selection system by constructing a nonnegative super martingale. As corollaries, some strong limit theorems for a multivariate function of an mth-order homogeneous Markov chain, the occurrence frequencies of the state group and sample point sets of an mth-order nonhomogeneous Markov chain are obtained. The result which has been obtained is extended. In the proof, we apply a new technique to the study of the mth-order nonhomogeneous Markov chain.

Key Words and Phrases: strong limit theorem, mth-order nonhomogeneous Markov chain, generalized random selection system, sample point set.
2000 Mathematics Subject Classifications: 60F15

1. Introduction.

Let $\{X_n, n \geq 0\}$ be an arbitrary stochastic sequence defined on the probability space $(\Omega, \mathcal{F}, P)$ which takes values in an infinite alphabet set $S = \{s_0, s_1, s_2, \cdots\}$, with the joint distribution:

$$P(X_0 = x_0, \cdots, X_n = x_n) = p(x_0, \cdots, x_n) > 0, \quad x_i \in S, \quad 0 \leq i \leq n. \tag{1}$$

We have by the definition of the conditional probability

$$p(x_0, \cdots, x_n) = p(x_0) \prod_{k=1}^{n} p_k(x_k | x_0, \cdots, x_{k-1}). \tag{2}$$

Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov chain on the measure P with the m-dimensional initial distribution and mth-order transition probabilities as follows:

$$q_o(i_0, \cdots, i_{m-1}) = P(X_0 = i_0, \cdots, X_{m-1} = i_{m-1}), \quad i_0, \cdots, i_{m-1} \in S. \tag{3}$$

$$p_n(j|i_1,\cdots,i_m)=P_n(X_n=j|X_{n-m}=i_1,\cdots,X_{n-1}=i_m),\quad i_1,\cdots,i_m,j\in S,\ n\geq m. \tag{4}$$

The joint distribution (1) of $\{X_n, n\geq 0\}$ can be written as

$$P(X_0=x_0,\cdots,X_n=x_n)=p(x_0,\cdots,x_n)=q_o(x_0,\cdots,x_{m-1})\prod_{k=m}^{n}P_k(x_k|x_{k-m},\cdots,x_{k-1}),$$
$$x_i\in S,\quad 0\leq i\leq n. \tag{5}$$

Study for strong limit properties of nonhomogeneous Markov chain is always one of central parts of the limit theory of probability theory. Many scholars have studied the subject until now. Liu and and Yang(see[1]) have studied the asymptotic equipartition properties (AEP) and limit properties of function sequence of nonhomogeneous Markov chain. Liu (see[2]) has discussed the strong limit theorems relative to the geometric average of random transition properties of finite nonhomogeneous Markov chain. Liu and Yang (see[3]) have investigated the strong deviation theorems of nonhomogeneous Markov chain relative to arbitrary stochastic sequence and AEP approximation of nonhomogeneous Markov information source. Liu (see[5]) has discussed the strong limit theorems for the harmonic mean of random transition probabilities of nonhomogeneous Markov chain. Liu and Wang (see[6]) have proved the strong limit properties for the state couples of nonhomogeneous Markov chain on the random selection system. Wang (see[7]) have studied the AEP and limit theorems for nonhomogeneous Markov chain on the generalized gambling system.

Many of practical information sources, such as language and image information, are often mth-order Markov chains, and always nonhomogeneous. mth-order nonhomogeneous Markov chain is a natural generalization of the general nonhomogeneous Markov chain. Hence it is of importance to study the limit properties for the mth-order nonhomogeneous Markov chain in the information theory and the probability theory. Yang and Liu (see[9]) have proved the limit theorem for averages of the functions of $m+1$ variables of mth-order nonhomogeneous Markov chain and the AEP for mth-order nonhomogeneous Markov information source. Wang (see[10]) have discussed some Shannon-McMillan theorems for mth-order nonhomogeneous Markov information source.

The conception of random selection derives from gambling. We consider a sequence of Bernoulli trial, and suppose that at each trial the bettor has the free choice of whether or not to bet. A theorem on gambling system asserts that under any non-anticipative system the successive bets form a sequence of Bernoulli trial with unchanged probability for success. The importance of this statement was recognized by von Mises, who introduced the impossibility of a successful gambling system as a fundamental axiom (see [11], [12]). This topic was discussed still further by Kolmogrov (see[13]) and Liu and Wang (see [14] and [15]).

We first explain the conception of the generalized random selection system, which is the crucial part of the gambling system.

Definition 1. *Let* $\{f_n(x_0,\cdots,x_n), n\geq 0\}$ *be a set of real-valued functions defined on* $S^{n+1}(n=0,1,2,\cdots)$*, which will be called the generalized random selection functions if they take values in an interval of* $[0,M]$*. Then let*

$$Y_0 = y(y \textit{ is an arbitrary real number}),$$

$$Y_{n+1} = f_n(X_0\cdots,X_n), \quad n\geq 0. \tag{6}$$

where $\{Y_n, n\geq 0\}$ *is called the generalized gambling system or the generalized random selection system. The traditional random selection system*[14] *takes values in the set of* $\{0,1\}$.

In order to explain the real meaning of the notion of the random selection, we consider the traditional gambling model. Let $\{X_n, n\geq 0\}$ be an mth-order nonhomogeneous Markov chain, and $g(x_0,\cdots,x_m)$ be a real-valued function defined on S^{m+1}. Interpret X_n as the result of the nth trial, the type of which may change at each step. Let $\mu_n = Y_n g(X_{n-m},\cdots,X_n)$ denote the gain of the bettor at the nth trial, where Y_n represents the bet size, $g(X_{n-m},\cdots,X_n)$ is determined by the gambling rules, and $\{Y_n, n\geq 0\}$ is called a gambling system or a random selection system. The bettor's strategy is to determine $\{Y_n, n\geq 0\}$ by the results of the last m trials. Let the entrance fee that the bettor pays at the nth trial be b_n. Also suppose that b_n depends on $X_{n-1},\cdots,X_{n-m}$ as $n\geq m$, and $b_0,\cdots,b_{m-1}$ are constants. Thus $\sum_{k=m}^n Y_k g(X_{k-m},\cdots,X_k)$ represents the total gain, $\sum_{k=m}^n b_k$ the accumulated entrance fees, and $\sum_{k=m}^n [Y_k g(X_{k-m},\cdots,X_k)-b_k]$ the accumulated net gain. Motivated by the classical definition of "fairness" of game of chance (see Kolmogorov 1982), we introduce the following definition:

Definition 2. *The game is said to be fair, if for almost all* $\omega\in\{\omega:\sum_{k=m}^\infty Y_k=\infty\}$*, the accumulated net gain in the first* n *trial is to be of smaller order of magnitude than the accumulated stake* $\sum_{k=m}^n Y_k$ *as* n *tends to infinity, that is*

$$\lim_{n\to\infty}\frac{1}{\sum_{k=m}^n Y_k}\sum_{k=m}^n [Y_k g(X_{k-m},\cdots,X_k)-b_k]=0 \qquad a.s. \quad on \ \ \{\omega:\sum\nolimits_{k=m}^\infty Y_k=\infty\}.$$

In this paper, our aim is to generalize the study for mth-order nonhomogeneous Markov chains to the case of the generalized random selection system. We establish a class of strong limit theorems for a sequence of multivariate functions of the mth-order nonhomogeneous Markov chain on the generalized random selection system by constructing a nonnegative super-martingale. We apply a new type of techniques to study of the strong limit properties for the mth-order nonhomogeneous Markov chain. As corollaries, a class of strong limit theorems for the occurrence frequencies of states groups and some limit properties for sample point sets of the mth-order Markov chain on the generalized random selection system are obtained.

We denote $X_m^n = \{X_m, \cdots, X_n\}$. x_m^n the realization of X_m^n.

2. Lemma and its proof.

Lemma 1. *Let $\alpha \geq 0$, $0 \leq \beta \leq 1$, then for all $\lambda > 0$, we have*

$$\lambda^{\alpha\beta} \leq 1 + (\lambda^\alpha - 1)\beta. \tag{7}$$

Proof . Let $f(\lambda) = \lambda^{\alpha\beta} - 1 - (\lambda^\alpha - 1)\beta$, in the case $0 < \lambda \leq 1$, $f'(\lambda) = \alpha\beta(\lambda^{\alpha\beta-1} - \lambda^{\alpha-1}) \geq 0$, we get $f(\lambda) \leq f(1) = 0$. In the case $\lambda > 1$, $f'(\lambda) = \alpha\beta(\lambda^{\alpha\beta-1} - \lambda^{\alpha-1}) < 0$, we attain $f(\lambda) \leq f(1) = 0$. In conclusion, we always achieve $\lambda^{\alpha\beta} - 1 - (\lambda^\alpha - 1)\beta \leq 0$ for all $\lambda > 0$. The lemma is valid.

Lemma 2. *Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov chain with the m dimensional initial distribution (3) and the mth-order transition probabilities (4), $\{Y_n, n \geq 0\}$ be the generalized random selection system defined by (6). Let $\{g_n(x_0, \cdots, x_m),\ n \geq 1\}$ be a multivariate real functional sequence defined on S^{m+1} which takes values in the interval $[0, 1]$. Denote $\mathcal{F}_n = \sigma(X_0, \cdots, X_n)$,*

$$U_n(\lambda, \omega) = \frac{\lambda^{\sum_{k=m}^{n} Y_k g_k(X_{k-m}^k)}}{\prod_{k=m}^{n} [1 + (\lambda^{Y_k} - 1)E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})]}, \tag{8}$$

where $\lambda > 0$. Then $\{U_n(\lambda, \omega), \mathcal{F}_n, n \geq m\}$ is a nonnegative superior martingale.

Proof . Obviously, when $n \geq m$, we have

$$U_n(\lambda, \omega) = U_{n-1}(\lambda, \omega) \frac{\lambda^{Y_n g_n(X_{n-m}^n)}}{[1 + (\lambda^{Y_n} - 1)E_P(g_n(X_{n-m}^n)|X_{n-m}^{n-1})]}. \tag{9}$$

By (8) we can write

$$\begin{aligned}
& E_P[U_n(\lambda, \omega)|\mathcal{F}_{n-1}] \\
= \;& U_{n-1}(\lambda, \omega) E_P\left[\frac{\lambda^{Y_n g_n(X_{n-m}^n)}}{1 + (\lambda^{Y_n} - 1)E_P(g_n(X_{n-m}^n)|X_{n-m}^{n-1})}|X_{n-m}^{n-1}\right] \\
= \;& U_{n-1}(\lambda, \omega) E_P\left[\frac{\lambda^{Y_n g_n(X_{n-m}^n)}}{1 + (\lambda^{Y_n} - 1)E_P(g_n(X_{n-m}^n)|X_{n-m}^{n-1})}|X_{n-m}^{n-1}\right] \\
= \;& U_{n-1}(\lambda, \omega) \sum_{x_n \in S} \frac{\lambda^{Y_n g_n(X_{n-m}^{n-1}, x_n)}}{1 + (\lambda^{Y_n} - 1)E_P(g_n(X_{n-m}^n)|X_{n-m}^{n-1})} P_n(x_n|X_{n-m}^{n-1})
\end{aligned}$$

$$
\begin{aligned}
&= U_{n-1}(\lambda,\omega)\frac{E_P(\lambda^{Y_n g_n(X_{n-m}^n)}|X_{n-m}^{n-1})}{1+(\lambda^{Y_n}-1)E_P(g_n(X_{n-m}^n)|X_{n-m}^{n-1})}\\
&= U_{n-1}(\lambda,\omega)\frac{E_P(\lambda^{Y_n g_n(X_{n-m}^n)}|X_{n-m}^{n-1})}{E_P(1+(\lambda^{Y_n}-1)g_n(X_{n-m}^n)|X_{n-m}^{n-1})}. \qquad (10)
\end{aligned}
$$

By virtue of Lemma 1, for $Y_n \geq 0$ and $0 \leq g_n(X_{n-m}^{n-1}) \leq 1$, we attain

$$
\frac{\lambda^{Y_n g_n(X_{n-m}^n)}}{1+(\lambda^{Y_n}-1)g_n(X_{n-m}^n)} \leq 1. \qquad (11)
$$

Therefore, by (10), (11) we obtain

$$
E_P[U_n(\lambda,\omega)|\mathcal{F}_{n-1}] \leq U_{n-1}(\lambda,\omega), \qquad P-a.s.
$$

It implies that $\{U_n(\lambda,\omega),\mathcal{F}_n, n>1\}$ is a nonnegative super martingale under the mth-order Markov probability measure P.

3. Main results .

Theorem 1. *Let $\{X_n, n\geq 0\}$ be an mth-order nonhomogeneous Markov chain with the m dimensional initial distribution (3) and the mth-order transition probabilities (4), $\{Y_n, n\geq 0\}$ be a generalized random selection system defined by (6). Let $\{g_n(x_0,\cdots,x_m),\ n\geq 1\}$ be a multivariate real functional sequence defined on S^{m+1} which takes values in the interval $[0,1]$, $\{a_n, n\geq 0\}$ be a nonnegative stochastic sequence. Denote*

$$
D(\omega)=\{\omega:\lim_n a_n=\infty,\ \limsup_n \frac{1}{a_n}\sum_{k=m}^{n} E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}]=G<\infty\}. \qquad (12)
$$

Then

$$
\lim_{n\to\infty}\frac{1}{a_n(\omega)}\sum_{k=m}^{n} Y_k[g_k(X_{k-m}^k)-E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})]=0. \qquad P-a.s. \quad \omega\in D(\omega) \qquad (13)
$$

Proof. Let $U_n(\lambda,\omega)$ be defined as (8), we know by lemma $\{U_n(\lambda,\omega),\mathcal{F}_n,n>1\}$ is a nonnegative super martingale. By Doob's martingale convergence theorem, we have

$$
\lim_n U_n(\lambda,\omega)=U(\lambda,\omega)<\infty. \qquad P-a.s. \qquad (14)
$$

By the first equation of (12) and (14) we attain

$$
\limsup_n \frac{1}{a_n}\ln U_n(\lambda,\omega)\leq 0, \qquad P-a.s. \quad \omega\in D \qquad (15)
$$

By use of (8), we can rewrite (15) as

$$\limsup_{n} \frac{1}{a_n} \left\{ \sum_{k=m}^{n} Y_k g_k(X_{k-m}^k) \ln \lambda - \sum_{k=m}^{n} \ln[1 + (\lambda^{Y_k} - 1) E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})] \right\} \le 0.$$

$$P - a.s. \quad \omega \in D \tag{16}$$

In the case of $\lambda > 1$, dividing both sides of (16) by $\ln \lambda$, we get

$$\limsup_{n} \frac{1}{a_n} \left\{ \sum_{k=m}^{n} Y_k g_k(X_{k-m}^k) - \sum_{k=m}^{n} \frac{\ln[1 + (\lambda^{Y_k} - 1) E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{\ln \lambda} \right\} \le 0.$$

$$P - a.s. \quad \omega \in D \tag{17}$$

By virtue of (12), (17) and the inequalities $1 - 1/x \le \ln x \le x - 1 (x > 0)$, $\lambda^x - 1 - x \ln \lambda \le (x \ln \lambda)^2 e^{|x \ln \lambda|}$, noticing that $|Y_n| \le M$, according to the property of the limit superior

$$\limsup_{n \to \infty} (a_n - b_n) \le 0 \Rightarrow \limsup_{n \to \infty} (a_n - c_n) \le \limsup_{n \to \infty} (b_n - c_n),$$

we can write

$$\begin{aligned}
& \limsup_{n} \frac{1}{a_n} \sum_{k=m}^{n} Y_k [g_k(X_{k-m}^k) - E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})] \\
\le & \limsup_{n} \frac{1}{a_n} \sum_{k=m}^{n} \{ \frac{\ln[1 + (\lambda^{Y_k} - 1) E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{\ln \lambda} - Y_k E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1}) \} \\
\le & \limsup_{n} \frac{1}{a_n} \sum_{k=m}^{n} \frac{\lambda^{Y_k} - 1 - Y_k \ln \lambda}{\ln \lambda} E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1}) \\
\le & \ln \lambda \cdot \limsup_{n} \frac{1}{a_n} \sum_{k=m}^{n} Y_k^2 e^{|Y_k| \ln \lambda} \cdot E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1}) \\
\le & M \ln \lambda \cdot \limsup_{n} \frac{1}{a_n} \sum_{k=m}^{n} Y_k e^{M \ln \lambda} \cdot E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1}) \\
\le & \lambda^M M \ln \lambda \cdot \limsup_{n} \frac{1}{a_n} \sum_{k=m}^{n} Y_k \cdot E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1}) \\
\le & \lambda^M M (\lambda - 1) G < \infty.
\end{aligned}$$

$$P - a.s. \quad \omega \in D. \tag{18}$$

Since $\lambda^M M(\lambda - 1) G \to 0$ as $\lambda \to 1 + 0$, we can reach by (18)

$$\limsup_{n} \frac{1}{a_n} \sum_{k=m}^{n} Y_k [g_k(X_{k-m}^k) - E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})] \le 0. \qquad P - a.s. \quad \omega \in D. \tag{19}$$

Taking $0<\lambda<1$, dividing two sides of (16) by $\ln\lambda$, we obtain

$$\liminf_{n} \frac{1}{a_n}\left\{\sum_{k=m}^{n} Y_k g_k(X_{k-m}^k) - \sum_{k=m}^{n} \frac{\ln[1+(\lambda^{Y_k}-1)E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})]}{\ln\lambda}\right\} \geq 0.$$

$$p-a.s. \qquad \omega\in D \tag{20}$$

Taking into account the inequalities $1-1/x\leq \ln x\leq x-1 (x>0)$, $\lambda^x-1-x\ln\lambda\leq (x\ln\lambda)^2 e^{|x\ln\lambda|}$ and the property of limit inferior, by (12), (20) we can deduce

$$\begin{aligned}
&\liminf_{n} \frac{1}{a_n}\sum_{k=m}^{n} Y_k[g_k(X_{k-m}^k) - E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})] \\
\geq\ & \liminf_{n} \frac{1}{a_n}\sum_{k=m}^{n} \frac{\lambda^{Y_k}-1-Y_k\ln\lambda}{\ln\lambda} E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1}) \\
\geq\ & \ln\lambda\cdot\limsup_{n} \frac{1}{a_n}\sum_{k=m}^{n} Y_k^2 e^{-|Y_k|\ln\lambda}\cdot E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1}) \\
\geq\ & \frac{\lambda-1}{\lambda}\cdot\limsup_{n} \frac{1}{a_n}\sum_{k=m}^{n} Y_k^2 e^{-M\ln\lambda}\cdot E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1}) \\
\geq\ & \lambda^{-M-1}M(\lambda-1)\cdot\limsup_{n} \frac{1}{a_n}\sum_{k=m}^{n} Y_k\cdot E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1}) \\
=\ & \lambda^{-M-1}M(\lambda-1)G.
\end{aligned}$$

$$P-a.s. \quad \omega\in D. \tag{21}$$

Since $\lambda^{-M-1}M(\lambda-1)G\to 0$ as $\lambda\to 1-0$, we achieve by (21)

$$\liminf_{n} \frac{1}{a_n}\sum_{k=m}^{n} Y_k[g_k(X_{k-m}^k) - E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})] \geq 0. \qquad P-a.s. \quad \omega\in D. \tag{22}$$

Hence (13) follows from (19) and (22).

Corollary 1[9]. *Let $\{X_n, n\geq 0\}$ be an mth-order nonhomogeneous Markov chain with the m dimensional initial distribution (3) and the mth-order transition probabilities (4), $\{g_n(x_0,\cdots,x_m),\ n\geq 1\}$ be a multivariate real functional sequence defined on S^{m+1}. Denote*

$$B(\omega)=\{\omega:\limsup_{n} \frac{1}{n}\sum_{k=m}^{n} E_P[g_k(X_{k-m}^k)|X_{k-m}^{k-1}]=G<\infty\} \tag{23}$$

Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=m}^{n} [g_k(X_{k-m}^k) - E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})]=0. \qquad P-a.s. \quad \omega\in B(\omega) \tag{24}$$

Proof. Letting $a_n = n,\ \ n \geq 1$, $Y_n \equiv 1,\ n \geq 1$, we easily see that $B(\omega) = D(\omega)$. (23), (24) follow from (12), (13) immediately.

Corollary 2. *Let $\{X_n, n \geq 0\}$ be an mth-order nonhomogeneous Markov chain with the m dimensional initial distribution (3) and the mth-order transition probabilities (4), $\{Y_n, n \geq 0\}$ and $\{g_n(x_0, \cdots, x_m),\ n \geq 1\}$ be defined by Theorem 1. Denote*

$$L(\omega) = \{\omega : \lim_n \sum_{k=m}^{n} Y_k = \infty\}. \tag{25}$$

Then

$$\lim_n \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} Y_k[g_k(X_{k-m}^k) - E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})] = 0.$$

$$P - a.s. \quad \omega \in L(\omega) \tag{26}$$

Proof. Letting $a_n = \sum\limits_{k=m}^{n} Y_k,\ n \geq 1$, $G = 1$, noticing that $g_n(x_0, \cdots, x_m) \in [0, 1],\ n \geq 1$, we obtain

$$\limsup_n \frac{1}{a_n} \sum_{k=m}^{n} E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}] \leq \limsup_n \frac{1}{\sum_{k=m}^{n} Y_k} \sum_{k=m}^{n} Y_k = 1 < \infty.$$

Therefore, $L(\omega) = D(\omega)$. (26) follows from (13) respectively.

Corollary 3. *Under the assumption of Theorem 1, we suppose that $\{Y_n, n \geq 0\}$ is the generalized random selection system taking values in the interval $[0, M]$. Denote*

$$D_2(\omega) = \{\omega : \lim_n \sum_{k=m}^{n} E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}] = \infty\}. \tag{27}$$

Then

$$\lim_{n\to\infty} \frac{\sum_{k=m}^{n} Y_k g_k(X_{k-m}^k)}{\sum_{k=m}^{n} E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}]} = 1. \qquad P - a.s. \quad \omega \in D_2(\omega) \tag{28}$$

Proof. Letting $a_n = \sum\limits_{k=m}^{n} E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}]$, $G = 1$, noticing that $Y_n \in [0, M]$, we easily see

$$\limsup_n \frac{1}{a_n} \sum_{k=m}^{n} E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}]$$

$$= \limsup_n \frac{1}{\sum_{k=m}^n E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}]} \sum_{k=m}^n E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}] = 1.$$

$$\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=m}^n Y_k[g(X_{k-m}^k) - E_P(g(X_{k-m}^k)|X_{k-m}^{k-1})] = \lim_{n\to\infty} \frac{\sum_{k=m}^n Y_k g_k(X_{k-m}^k)}{\sum_{k=m}^n E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}]} - 1 = 0.$$

Hence $D_2(\omega) = D(\omega)$. (28) follows (13) immediately.

Theorem 2. *Let $\{X_n, n \geq 0\}$ be an mth-order homogeneous Markov chain, $g(x_0, \cdots, x_m)$ be a multivariate real function defined on S^{m+1}. If*

$$\sum_{i_1,\cdots,i_m\in S} \sum_{j\in S} g(i_1^m, j)P(j|i_1^m) < \infty, \tag{29}$$

then

$$\lim_n \frac{1}{n} \sum_{k=m}^n [g(X_{k-m}^k) - E_P(g(X_{k-m}^k)|X_{k-m}^{k-1})] = 0.$$

$$P - a.s. \quad \omega \in J(\omega) \tag{30}$$

Proof. Letting $a_n = n,\ Y_n \equiv 1,\ g_n(x_0, \cdots, x_m) = g(x_0, \cdots, x_m),\ P_n(y|x_0, \cdots, x_{m-1}) = P(y|x_0, \cdots, x_{m-1}),\ \ n \geq m$ in Theorem 1, it follows from (12) that

$$\begin{aligned}
& \limsup_n \frac{1}{a_n} \sum_{k=m}^n E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1}] \\
= & \limsup_n \frac{1}{n} \sum_{k=m}^n E_P[g(X_{k-m}^k)|X_{k-m}^{k-1}] \\
= & \limsup_n \frac{1}{n} \sum_{k=m}^n \sum_{x_k\in S} g(X_{k-m}^{k-1}, x_k)P(x_k|X_{k-m}^{k-1}) \\
\leq & \limsup_n \frac{1}{n} \sum_{k=m}^n \sum_{i_1,\cdots,i_m\in S} \sum_{j\in S} \delta_{i_1}(X_{k-m}) \cdots \delta_{i_m}(X_{k-1}) g(i_1^m, j)P(j|i_1^m) \\
\leq & \limsup_n \frac{1}{n} \sum_{k=m}^n \sum_{i_1,\cdots,i_m\in S} \sum_{j\in S} g(i_1^m, j)P(j|i_1^m) \\
\leq & \sum_{i_1,\cdots,i_m\in S} \sum_{j\in S} g(i_1^m, j)P(j|i_1^m) \limsup_n \frac{n-m+1}{n} \\
= & \sum_{i_1,\cdots,i_m\in S} \sum_{j\in S} g(i_1^m, j)P(j|i_1^m) < \infty.
\end{aligned} \tag{31}$$

So from (31) we know $D(\omega) = \Omega$, (30) follows from (13).

Theorem 3. *Under the assumption of Corollary 3, we have*

$$D_2(\omega) = D_3(\omega). \qquad a.s. \tag{32}$$

Where

$$D_3(\omega) = \{\omega : \sum_{k=m}^{\infty} Y_k g_k(X_{k-m}^k) = \infty\}. \tag{33}$$

Proof. By virtue of Corollary 3, we easily know $D_2(\omega) \subset D_3(\omega)$. In the next, we try to prove the opposite conclusion. We denote by $D_2^c(\omega)$ and $D_3^c(\omega)$ the complements of the sample sets $D_2(\omega)$ and $D_3(\omega)$, respectively. Letting $\omega_0 \in D_2^c(\omega)$, $\lambda = e$, we reach by (8)

$$\lim_{n\to\infty} U_n(e,\omega_0) = \lim_{n\to\infty} \frac{e^{\sum_{k=m}^{n} Y_k(\omega_0) g_k(X_{k-m}^k(\omega_0))}}{\prod_{k=m}^{n} [1 + (e^{Y_k(\omega_0)} - 1) E_P(g_k(X_{k-m}^k(\omega_0)) | X_{k-m}^k(\omega_0))]} < \infty. \tag{34}$$

By using the inequality $e^x - 1 \le x e^x$, $0 \le Y_n \le M$, we attain

$$\begin{aligned} 0 &\le \sum_{k=m}^{\infty} (e^{Y_k(\omega_0)} - 1) E_P(g_k(X_{k-m}^k(\omega_0)) | X_{k-m}^k(\omega_0)) \\ &\le \sum_{k=m}^{\infty} Y_k(\omega_0) e^{Y_k(\omega_0)} E_P(g_k(X_{k-m}^k(\omega_0)) | X_{k-m}^k(\omega_0)) \\ &\le \sum_{k=m}^{\infty} Y_k(\omega_0) e^{b} E_P(g_k(X_{k-m}^k(\omega_0)) | X_{k-m}^k(\omega_0)) < \infty. \end{aligned} \tag{35}$$

By (35) and the convergence theorem of infinite product we achieve

$$0 < \prod_{k=m}^{\infty} [1 + (e^{Y_k(\omega_0)} - 1) E_P(g_k(X_{k-m}^k(\omega_0)) | X_{k-m}^k(\omega_0))] < \infty. \tag{36}$$

(34) and (36) imply

$$\sum_{k=m}^{\infty} Y_k(\omega_0) g_k(X_{k-m}^k(\omega_0)) < \infty, \tag{37}$$

that is $\omega_0 \in D_3^c(\omega)$. Hence $D_2^c(\omega) \subset D_3^c(\omega)$, it means $D_3(\omega) \subset D_2(\omega)$. This implies (32) is valid.

Theorem 4. *Let $\{X_n, n \ge 0\}$ be an mth-order homogeneous Markov chain, $\{Y_n, n \ge 0\}$ be a generalized random selection system defined by (6). Denote by $\sigma_n(i_1, \cdots, i_m, j)$ the number of $(i_1, \cdots, i_m, j)$ in the sequence of random variables $\{(X_0, \cdots, X_m), (X_1, \cdots, X_{m+1}), \cdots, (X_{n-m}, \cdots, X_n)\}$ which are selected by $\{Y_k,\ k \ge m\}$, $\sigma_{n-1}(i_1, \cdots, i_m)$ the number* *of* $(i_1, \cdots, i_m)$ *in the sequence of random variables* $\{(X_0, \cdots, X_{m-1}), (X_1, \cdots, X_m), \cdots, (X_{n-m}, \cdots, X_{n-1})\}$ *which are selected by* $\{Y_k,\ k \geq m\}$. *That is*

$$\sigma_n(i_1, \cdots, i_m, j) = \sum_{k=m}^{n} Y_k I_{i_1}(X_{k-m}) I_{i_2}(X_{k-m+1}) \cdots I_{i_m}(X_{k-1}) I_j(X_k). \tag{38}$$

$$\sigma_{n-1}(i_1, \cdots, i_m) = \sum_{k=m}^{n} Y_k I_{i_1}(X_{k-m}) I_{i_2}(X_{k-m+1}) \cdots I_{i_m}(X_{k-1}). \tag{39}$$

Then

$$\lim_{n\to\infty} \frac{1}{n}[\sigma_n(i_1, \cdots, i_m, j) - \sigma_{n-1}(i_1, \cdots, i_m) P(j|i_1^m)] = 0. \qquad P-a.s. \tag{40}$$

Proof. At the moment, we assume that $\{X_n, n \geq 0\}$ be an mth-order homogeneous Markov chain, hence we have $P_n(y|x_0, \cdots, x_{m-1}) = P(y|x_0, \cdots, x_{m-1}),\ n \geq m.$ Let $g_n(x_{n-m}^n) = I_{i_1}(x_{n-m}) \cdot I_{i_2}(x_{n-m+1}) \cdots I_{i_m}(x_{n-1}) I_j(x_n),\ a_n = n,\ n \geq m$ in Theorem 1, where

$$I_k(x) = \{1, x = k, 0.x \neq k. \tag{41}$$

By the definition of $g_n(x_{n-m}^n)$ and (38), we obtain

$$\sum_{k=m}^{n} Y_k g_k(X_{k-m}^k) = \sum_{k=m}^{n} Y_k I_{i_1}(X_{k-m}) I_{i_2}(X_{k-m+1}) \cdots I_{i_m}(X_{k-1}) I_j(X_k) = \sigma_n(i_1, \cdots, i_m, j), \tag{42}$$

$$\begin{aligned} & \sum_{k=m}^{n} Y_k E_P[g_k(X_{k-m}^k)|X_{k-m}^{k-1}] \\ = & \sum_{k=m}^{n} Y_k E_P[I_{i_1}(X_{k-m}) I_{i_2}(X_{k-m+1}) \cdots I_{i_m}(X_{k-1}) I_j(X_k)|X_{k-m}^{k-1}] \\ = & \sum_{k=m}^{n} \sum_{x_k \in S} Y_k I_{i_1}(X_{k-m}) I_{i_2}(X_{k-m+1}) \cdots I_{i_m}(X_{k-1}) I_j(x_k) P(x_k|X_{k-m}^{k-1}) \\ = & \sum_{k=m}^{n} Y_k I_{i_1}(X_{k-m}) I_{i_2}(X_{k-m+1}) \cdots I_{i_m}(X_{k-1}) P(j|X_{k-m}^{k-1}) \\ = & \sum_{k=m}^{n} Y_k I_{i_1}(X_{k-m}) I_{i_2}(X_{k-m+1}) \cdots I_{i_m}(X_{k-1}) P(j|i_1^m) \\ = & \sigma_{n-1}(i_1, \cdots, i_m) P(j|i_1^m).. \end{aligned} \tag{43}$$

By (12) and (43) we have $\lim_{n\to\infty} a_n = \lim_{n\to\infty} n = +\infty$,

$$\limsup_{n} \frac{1}{a_n} \sum_{k=m}^{n} E_P[Y_k g_k(X_{k-m}^k)|X_{k-m}^{k-1})]$$

$$
\begin{aligned}
&= \limsup_{n\to\infty} \frac{\sum_{k=m}^{n} Y_k I_{i_1}(X_{k-m}) I_{i_2}(X_{k-m+1}) \cdots I_{i_m}(X_{k-1}) P(j|i_1^m)}{n} \\
&\le \limsup_{n\to\infty} \frac{\sum_{k=m}^{n} Y_k P(j|i_1^m)}{n} \le \limsup_{n\to\infty} \frac{M(n-m+1)P(j|i_1^m)}{n} \\
&= M \cdot P(j|i_1^m) < \infty. \qquad (44)
\end{aligned}
$$

Thus, at this time $D(\omega) = \Omega$. By (42) and (43) we reach

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{1}{a_n} \sum_{k=m}^{n} Y_k [g_k(X_{k-m}^k) - E_P(g_k(X_{k-m}^k)|X_{k-m}^{k-1})] \\
&= \lim_{n\to\infty} \frac{1}{n} [\sigma_n(i_1, \cdots, i_m, j) - \sigma_{n-1}(i_1, \cdots, i_m) P(j|i_1^m)] = 0. \qquad (45)
\end{aligned}
$$

Therefore, (40) follows from (13) naturally.

4. Acknowledgements.

Authors would like to thank the referee for his valuable suggestions. The work is supported by the Project of Higher Schools' Natural Science Basic Research of Jiangsu Province of China (13KJB110006). Wang Kangkang is the correspondence author.

References

[1] W. Liu and W.G. Yang An extension of Shannon-Mcmillan theorem and some limit properties for nonhomogeneous Markov chains. Stochastic. Process. Appl. 61(1): 129-145 1996

[2] W. Liu Some strong limit theorems relative to the geometric average of random transition probabilities of arbitrary finite nonhomogeneous Markov chains. Stat. Probab. Letts. 21(1): 77-83 1994

[3] W. Liu and W.G. Yang The Markov approximation of the sequences of N-valued random variables and a class of small deviation theorems. Stochastic. Process. Appl. 89(1): 117-130 2000

[4] K.L. Chung A Course in Probability Theory. Academic Press, New York. 1974

[5] W. Liu A strong limit theorem for the harmonic mean of random transition probabilities of finite nonhomogeneous Markov chains[J]. Acta. Math. Scie. 20(1): 81-84 (in Chinese) 2000

[6] Z.Z. Wang, W. Liu A strong limit theorem on random selection forcountable nonhomogeneous Markov chains. Chinese J. Math. 24(2): 187-197 1996

[7] K. K. Wang A Class of strong limit theorems for countable nonhomogeneous Markov chains on the generalized gambling system. Czechoslovak Mathematical Journal. 59 (1), 23-37 2009

[8] J.L. Doob stochastic Process. Wiley, New York. 1953

[9] W.G. Yang, and W. Liu The asymptotic equipartition property for Mth-order nonhomogeneous Markov information source. IEEE Trans.Inform.Theory. 50(7): 3326-3330. 2004

[10] K. K. Wang Some Research on Shannon-McMillan Theorems for mth-order nonhomogeneous Markov information source. Stochastic Analysis and Applications. 27 (6): 1117-1128 2009

[10] P. Billingsley, Probability and Measure. Wiley, New York, 1986.

[10] R.V. Mises, Mathematical Theory of Probability and Statistics. Academic Press. New York, 1964.

[10] A.N. Kolmogorov On the logical foundation of probability theory. Lecture Notes in Mathematics. Springer-Verlag. New York. 1021:1-2 1982.

[10] W. Liu and Z. Wang, An extension of a theorem on gambling systems to arbitrary binary random variables. Statistics and Probability Letters. 28: 51-58 1996

[10] Z.Z. Wang A strong limit theorem on random selection for the N-valued random variables. Pure and Applied Mathematics. 15: 56-61 1999

Wang Kangkang, Xiaofei Ye, Ting Teng, Jingping Jia
School of Mathematics and Physics, Jiangsu University of Science and Technology, Zhenjiang 212003, China
E-mail: wkk.cn@126.com

Received 14 March 2013
Accepted 29 August 2013