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A generalized multiplier transform on a univalent integral operator

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Abstract. The focus of this paper is to obtain a generalization of a multiplier transformation for analytic univalent functions in relation to an integral operator of the form
$F_\mu = \int_0^z \prod_{i=1}^k \left(\frac{I_{\alpha,\beta,\gamma} f_i(t)}{t}\right)^{\frac{1}{\mu}} dt, \mu \in \mathbb{C}$ and $|\mu| \leq 1$ and obtain its coefficient estimates using the relationship between starlike and convex functions. Also, we obtain the growth and distortion theorems for the operator.
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1. Introduction and Preliminaries

Let A denote the class of normalized univalent functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$.
For the function f of the form (1) in A, the following results are well known: f is said to be starlike respectively convex if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, |z| < 1$$

and

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, |z| < 1.$$

From the above, it is clear that f is convex if and only if $zf'(z)$ is starlike. Swamy [4], introduced a multiplier differential operator of the form $I^n_{\alpha,\beta}$ defined by:

$$I^n_{\alpha,\beta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{\alpha + k\beta}{\alpha + \beta}\right)^n a_n z^n$$

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which is analytic and univalent in the unit disk. For details see [4].
Definition: Let $s, \beta, \gamma \geq 0, \alpha$ a real number such that $\alpha + \beta + \gamma > 0$. Then for a subclass f of A, of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n^i z^n \tag{2}$$

and $i(1 \leq i \leq k)$ we define the linear operator $I^s_{\alpha,\beta,\gamma} f(z)$ by:

$$I_{\alpha,\beta,\gamma} f(z) = \frac{\alpha f(z) + \beta z f'(z) + \gamma z (z f'(z))'}{\alpha + \beta + \gamma} \tag{3}$$

$$I^2_{\alpha,\beta,\gamma} f(z) = I_{\alpha,\beta,\gamma}(I_{\alpha,\beta,\gamma} f(z)) \tag{4}$$

$$I^s_{\alpha,\beta,\gamma} f(z) = I_{\alpha,\beta,\gamma}(I^{s-1}_{\alpha,\beta,\gamma} f(z)) \tag{5}$$

Thus from (3),(4) and (5), we obtain the representation for $I^s_{\alpha,\beta,\gamma} f(z)$ given by:

$$I^s_{\alpha,\beta,\gamma} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha + n\beta + n^2\gamma}{\alpha + \beta + \gamma} \right)^s a_n^i z^n \tag{6}$$

Remarks: For a function $f(z)$ of the form (1), it follows from (5) that:
$I_{\alpha,0,0} f(z) = f(z)$. The operator $I^s_{\alpha,\beta,\gamma}$ is a generalization of many operators of this kind in the literature.

$$\begin{aligned} I^s_{\alpha,\beta,0} f(z) &= I^n_{\alpha,\beta} f(z), [4] \\ I^s_{\alpha,1,0} f(z) &= I^n_{\alpha} f(z), \alpha > -1, [1,2] \\ I^s_{1,\beta,0} f(z) &= N^n_{\beta}, [4] \end{aligned}$$

Now, let

$$I^s_{\alpha,\beta,\gamma} f_i(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha + n\beta + n^2\gamma}{\alpha + \beta + \gamma} \right)^s a_n^i z^n, \tag{7}$$

appealing to the integral operator of the form $F_\alpha = \int_0^z \prod_{i=1}^k \left(\frac{f_i(s)}{s}\right)^{\frac{1}{\alpha}} ds$ studied in [3], we define $F_\mu(z)$ by:

$$F_\mu(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{I_{\alpha,\beta,\gamma} f_i(t)}{t} \right)^{\frac{1}{\mu}} dt, \mu \in C \text{ and } |\mu| \leq 1, s, \beta, \gamma \geq 0, \tag{8}$$

where

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n \tag{9}$$

and the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1, \zeta_2, \delta)$ to be

$$\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1, \zeta_2, \delta) = \left\{ I^s_{\alpha,\beta,\gamma} f_i(z) \in A : \left| \frac{H(z) + \frac{1}{\mu} - 1}{\zeta_1 \left(H(z) + \frac{1}{\mu} \right) + \zeta_2} \right| < \delta \right\}, \tag{10}$$

where $H(z) = \frac{zF''_\mu(z)}{F'_\mu(z)}$
Furthermore, let
$I^s_{\alpha,\beta,\gamma} f_i(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+n\beta+n^2\gamma}{\alpha+\beta+\gamma}\right)^s a^i_n z^n$ and $I^s_{\alpha,\beta,\gamma} g_i(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha+n\beta+n^2\gamma}{\alpha+\beta+\gamma}\right)^s b^i_n z^n$,
we define the convolution of f_i and g_i, $I^s_{\alpha,\beta,\gamma} f(z) * I^s_{\alpha,\beta,\gamma} g_i(z) = (I^s_{\alpha,\beta,\gamma} f * I^s_{\alpha,\beta,\gamma} g)(z)$ by:

$$(I^s_{\alpha,\beta,\gamma} f_i * I^s_{\alpha,\beta,\gamma} g_i)(z) = z + \sum_{n=2}^{\infty} \left(\frac{\alpha + n\beta + n^2\gamma}{\alpha + \beta + \gamma}\right)^s a^i_n b^i_n z^n \tag{11}$$

In this paper, we study some coefficients of the generalized operator
$I^n_{\alpha,\beta,\gamma} : A \to A$
defined in (6) in relation to the integral operator in (8) which gives rise to the subclass of the univalent analytic functions in (10).
Lemma [3]: f_i is in the class $\Gamma_\alpha(\zeta_1, \zeta_2, \gamma)$ if and only if $\sum_{i=1}^{k} \sum_{n=2}^{\infty} n\big[(1 + \gamma\zeta_1) + \alpha(\gamma\zeta_2 - 1)\big]|a^i_n| \le \gamma|\zeta_1 + \alpha\zeta_2| - |1 - \alpha|, 0 \le \zeta_1, \zeta_2 \le 1$
We now state and prove the results in this paper.

2. Main Results

Theorem 1: Let F_μ be as defined in (8). Then $I^s_{\alpha,\beta,\gamma} f_i(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1, \zeta_2, \delta)$ if
$\sum_{i=1}^{k} \sum_{n=2}^{\infty} n \left(\frac{\alpha+n\beta+n^2\gamma}{\alpha+\beta+\gamma}\right)^s \big[(1 + \delta\zeta_1) + \mu(\delta\zeta_2 - 1)\big]|a^i_n| \le \delta|\zeta_1 + \mu\zeta_2| - |1 - \mu|$,
$0 \le \zeta_1, \zeta_2 \le 1, 0 < \delta < 1, \beta, \gamma \ge 0, \mu \in \mathbb{C}$ and $|\mu| \le 1$
Proof: Given that
$F_\mu(z) = \int_0^z \prod_{i=1}^{k} \left(\frac{I_{\alpha,\beta,\gamma} f_i(t)}{t}\right)^{\frac{1}{\mu}} dt$ and $I^s_{\alpha,\beta,\gamma} f_i(z) = z + \sum_{n=2}^{\infty} (\frac{\alpha+n\beta+n^2\gamma}{\alpha+\beta+\gamma})^s a^i_n z^n$,
with simple calculation, we have

$$\left|\frac{H(z) + \frac{1}{\mu} - 1}{\zeta_1\Big(H(z) + \frac{1}{\mu}\Big) + \zeta_2}\right| = \left|\frac{\dfrac{\sum_{i=1}^{k}\Big(z+\sum_{n=2}^{\infty} nx^s a^i_n z^n\Big) - \sum_{i=1}^{k}\mu\Big(z+\sum_{n=2}^{\infty} x^s a^i_n z^n\Big)}{\sum_{i=1}^{k}\mu\Big(z+\sum_{n=2}^{\infty} x^s a^i_n z^n\Big)}}{\dfrac{\sum_{i=1}^{k}\zeta_1\Big(z+\sum_{n=2}^{\infty} nx^s a^i_n z^n\Big) + \zeta_2\sum_{i=1}^{k}\mu\Big(z+\sum_{n=2}^{\infty} x^s a^i_n z^n\Big)}{\sum_{i=1}^{k}\mu\Big(z+\sum_{n=2}^{\infty} x^s a^i_n z^n\Big)}}\right|$$

where $x = \left(\frac{\alpha+n\beta+n^2\gamma}{\alpha+\beta+\gamma}\right), H(z) = \frac{zF_\mu(z)''}{F_\mu(z)'}$

and $\frac{zF''_\mu(z)}{F'_\mu(z)} = \sum_{i=1}^{k} \frac{1}{\mu}\left(\frac{zI^s_{\alpha,\beta,\gamma} f'_i(z)}{I^s_{\alpha,\beta,\gamma} f_i(z)} - 1\right)$ and for $z \to 1^-$ we have
$\left|\frac{H(z)+\frac{1}{\mu}-1}{\zeta_1\Big(H(z)+\frac{1}{\mu}\Big)+\zeta_2}\right| \le \frac{|1-\mu|+\sum_{i=1}^{k}\sum_{n=2}^{\infty} x^s|n-\mu||a^i_n|}{|\zeta_1+\mu\zeta_2|-\sum_{i=1}^{k}\sum_{n=2}^{\infty} x^s|n\zeta_1+\mu\zeta_2||a^i_n|}$

which is bounded by δ only if
$|1-\mu| + \sum_{i=1}^{k}\sum_{n=2}^{\infty} x^s |n-\mu| \left|a_n^i\right| \le \delta \left(|\zeta_1 + \mu\zeta_2| - \sum_{i=1}^{k}\sum_{n=2}^{\infty} x^s n(\zeta_1+\mu\zeta_2) \left|a_n^i\right| \right)$
fixing the value of x and restructuring, we have the result.

Remark: The above theorem shows the relationship between the convexity of the integral operator in (8) and the starlikeness of the differential operator in (7).

Corollary 1: Let F_μ be as defined in (8). and $I^s_{\alpha,\beta,\gamma} f_i(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$, also let f_i belong to the class $\Gamma_\alpha(\zeta_1,\zeta_2,\gamma)$ as given in lemma 1. Then $f_i \subset I^s_{\alpha,\beta\gamma} f_i(z)$, $0 \le \zeta_1,\zeta_2 \le 1$,
$0 < \delta < 1, \beta,\gamma \ge 0, \mu \in \mathbb{C}$ and $|\mu| \le 1$.
Proof: Let F_μ be as defined in (8). and $I^s_{\alpha,\beta\gamma} f_i(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. From theorem 1, we have:

$$\begin{aligned}\sum_{i=1}^{k}\sum_{n=2}^{\infty}\left[n(1+\delta\zeta_1)+\mu(\delta\zeta_2-1)\right]|a_n^i| &\le \frac{(\alpha+\beta+\gamma)^s\left[\delta|\zeta_1+\mu\zeta_2|-|1-\mu|\right]}{(\alpha+n\beta+n^2\gamma)^s}\\ &\le \delta|\zeta_1+\mu\zeta_2|-|1-\mu|\end{aligned}$$

which Proves the result.

Corollary 2: Let $I^s_{\alpha,\beta\gamma} f_i(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. Then $|a_n^i| \le \frac{(\alpha+\beta+\gamma)^s\left[(\delta|\zeta_1+\mu\zeta_2|)-|1-\mu|\right]}{\left[n(1+\delta\zeta_1)+\mu(\delta\zeta_2-1)\right](\alpha+n\beta+n^2\gamma)^s}$,
$0 \le \zeta_1,\zeta_2 \le 1, 0 < \delta < 1, \beta,\gamma \ge 0, \mu \in \mathbb{C}$ and $|\mu| \le 1$.

Corollary 3: Let $I^s_{\alpha,\beta\gamma} f_i(z)$ belongs to the class $\Gamma_{1;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. Then $|a_n^i| \le \frac{(\alpha+\beta+\gamma)^s\delta|\zeta_1+\zeta_2|}{\left[n(1+\delta\zeta_1)+(\delta\zeta_2-1)\right](\alpha+n\beta+n^2\gamma)^s}$,
$0 \le \zeta_1,\zeta_2 \le 1, 0 < \delta < 1, \alpha > 0$.

Corollary 4: Let $I^s_{\alpha,\beta\gamma} f_i(z)$ belongs to the class $\Gamma_{1;1,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. Then $|a_n^i| \le \frac{(1+\beta+\gamma)^s\delta|\zeta_1+\zeta_2|}{\left[n(1+\delta\zeta_1)+(\delta\zeta_2-1)\right](1+n\beta+n^2\gamma)^s}$,
$0 \le \zeta_1,\zeta_2 \le 1, 0 < \delta < 1, \alpha > 0$.

Corollary 5: Let $I^s_{\alpha,\beta\gamma} f_i(z)$ belongs to the class $\Gamma_{\mu;\alpha,0,0}(\zeta_1,\zeta_2,\delta)$. Then $|a_n^i| \le \frac{\delta|\zeta_1+\mu\zeta_2|-|1-\mu|}{\left[n(1+\delta\zeta_1)+\mu(\delta\zeta_2-1)\right]}$,
$0 \le \zeta_1,\zeta_2 \le 1, 0 < \delta < 1, \beta,\gamma \ge 0, \mu \in \mathbb{C}$ and $|\mu| \le 1$.

Corollary 6: Let $I^s_{\alpha,\beta\gamma} f_i(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(1,1,\delta)$. Then

$|a_n^i| \leq \frac{(\alpha+\beta+\gamma)^s\delta|1+\mu|-|1-\mu|}{\left[n(1+\delta)+\mu(\delta-1)\right](\alpha+n\beta+n^2\gamma)^s}$,
$0 \leq \zeta_1, \zeta_2 \leq 1, 0 < \delta < 1, \beta, \gamma \geq 0, \mu \in C$ and $|\mu| \leq 1$.

Corollary 7: Let $I^s_{\alpha,\beta\gamma}f_i(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(0, 1, \delta)$. Then $|a_n^i| \leq \frac{(\alpha+\beta+\gamma)^s(\delta\mu-|1-\mu|)}{n\left[1+\mu(\delta-1)\right](\alpha+n\beta+n^2\gamma)^s}$,
$0 \leq \zeta_1, \zeta_2 \leq 1, 0 < \delta < 1, \beta, \gamma \geq 0, \mu \in \mathbb{C}$ and $|\mu| \leq 1$.

Theorem 2: Let F_μ be as defined in (8) and $I^s_{\alpha,\beta\gamma}f_i(z)$, $I^s_{\alpha,\beta\gamma}g_i(z)$ belong to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1, \zeta_2, \delta)$. Then $(I^s_{\alpha,\beta\gamma}f_i * I^s_{\alpha,\beta\gamma}g_i(z))(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1, \zeta_2, \delta)$ if
$\lambda \leq \frac{|1-\mu|^2-\delta|\zeta_1+\mu\zeta_2||1-\mu|-|1-\mu|\sum_{i=1}^{k}\sum_{n=2}^{\infty}x^s(\delta|n\zeta_1+\mu\zeta_2|+|n-\mu|)}{\delta|\zeta_1+\mu\zeta_2|^2-|\zeta_1+\mu\zeta_2||1-\mu|-\sum_{i=1}^{k}\sum_{n=2}^{\infty}x^s(\delta|n\zeta_1+\mu\zeta_2|+|n-\mu|)|\zeta_1+\mu\zeta_2|}$.
Proof: Let $I^s_{\alpha,\beta\gamma}f_i(z)$, $I^s_{\alpha,\beta\gamma}g_i(z)$ belong to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1, \zeta_2, \delta)$,
then from theorem 1 we have,

$$\frac{\sum_{i=1}^{k}\sum_{n=2}^{\infty}x^s(\delta|n\zeta_1+\mu\zeta_2|+|n-\mu|)|a_n^i|}{\delta|\zeta_1+\mu\zeta_2|-|1-\mu|} \leq 1$$

and

$$\frac{\sum_{i=1}^{k}\sum_{n=2}^{\infty}x^s(\delta|n\zeta_1+\mu\zeta_2|+|n-\mu|)|b_n^i|}{\delta|\zeta_1+\mu\zeta_2|-|1-\mu|} \leq 1$$

respectively. We need to find the smallest λ such that

$$\frac{\sum_{i=1}^{k}\sum_{n=2}^{\infty}x^s(\lambda|n\zeta_1+\mu\zeta_2|+|n-\mu|)|a_n^i b_n^i|}{\lambda|\zeta_1+\mu\zeta_2|-|1-\mu|} \leq 1 \tag{12}$$

and by Cauchy Schwartz inequality, we have

$$\frac{\sum_{i=1}^{k}\sum_{n=2}^{\infty}x^s(\delta|n\zeta_1+\mu\zeta_2|+|n-\mu|)\sqrt{|a_n^i b_n^i|}}{\delta|\zeta_1+\mu\zeta_2|-|1-\mu|} \leq 1. \tag{13}$$

Thus, it suffices to show that:

$$\frac{\sum_{i=1}^{k}\sum_{n=2}^{\infty}x^s(\lambda|n\zeta_1+\mu\zeta_2|+|n-\mu|)|a_n^i b_n^i|}{\lambda|\zeta_1+\mu\zeta_2|-|1-\mu|} \leq \frac{\sum_{i=1}^{k}\sum_{n=2}^{\infty}x^s(\delta|n\zeta_1+\mu\zeta_2|+|n-\mu|)\sqrt{|a_n^i b_n^i|}}{\delta|\zeta_1+\mu\zeta_2|-|1-\mu|}$$

from where we have:

$$\sqrt{|a_n^i b_n^i|} \leq \frac{\lambda|\zeta_1+\mu\zeta_2|-|1-\mu|}{\lambda|\zeta_1+\mu\zeta_2|+|1-\mu|}. \tag{14}$$

But from (13), we have:

$$\sqrt{|a_n^i b_n^i|} \leq \frac{\delta|\zeta_1+\mu\zeta_2|-|1-\mu|}{\sum_{n=2}^{\infty}nx^s(\delta|\zeta_1+\mu\zeta_2|+|1-\mu|)}. \tag{15}$$

Combining (14) and (15), gives:

$$\frac{\delta|\zeta_1+\mu\zeta_2|-|1-\mu|}{\sum_{n=2}^{\infty} nx^s(\delta|\zeta_1+\mu\zeta_2|+|1-\mu|)} \leq \frac{\lambda|\zeta_1+\mu\zeta_2|-|1-\mu|}{\lambda|\zeta_1+\mu\zeta_2|+|1-\mu|}.$$

Thus, we have:

$$\lambda \leq \frac{|1-\mu|^2-\delta|\zeta_1+\mu\zeta_2||1-\mu|-|1-\mu|\sum_{n=2}^{\infty} x^s(\delta|n\zeta_1+\mu\zeta_2|+|n-\mu|)}{\delta|\zeta_1+\mu\zeta_2|^2-|\zeta_1+\mu\zeta_2||1-\mu|-\sum_{n=2}^{\infty} x^s(\delta|n\zeta_1+\mu\zeta_2|+|n-\mu|)|\zeta_1+\mu\zeta_2|} \qquad (16)$$

which proves the result.

Corollary 8: Let F_μ be as defined in (8) and $I^s_{\alpha,\beta\gamma}f_i(z)$, $I^s_{\alpha,\beta\gamma}g_i(z)$ belong to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$ and $(I^s_{\alpha,\beta\gamma}f_i * I^s_{\alpha,\beta\gamma}g_i(z))(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. Then $|a^i_n||b^i_n| \leq \frac{q^s(\lambda|\zeta_1+\mu\zeta_2|-|1-\mu|)}{np^s(\lambda|\zeta_1+\mu\zeta_2|+|1-\mu|)}, 0 \leq, \zeta_1, \zeta_2 \leq 1, 0 < \delta < 1, \beta, \gamma \geq 0, \mu \in C$ and $|\mu| \leq 1$ where $p = \alpha + n\beta + n^2\gamma$ and $q = \alpha + \beta + \gamma$ and λ is as defined in (16).

Corollary 9: Let F_μ be as defined in (8) and $I^s_{\alpha,\beta\gamma}f_i(z)$, $I^s_{\alpha,\beta\gamma}g_i(z)$ belong to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$ and $(I^s_{\alpha,\beta\gamma}f_i * I^s_{\alpha,\beta\gamma}g_i(z))(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. Then $|a^i_n| \leq \frac{q^s(\lambda|\zeta_1+\mu\zeta_2|-|1-\mu|)}{np^s(\lambda|\zeta_1+\mu\zeta_2|+|1-\mu|)|b^i_n|}, 0 \leq, \zeta_1, \zeta_2 \leq 1, 0 < \delta < 1, \beta, \gamma \geq 0, \mu \in C$ and $|\mu| \leq 1$

Corollary 10: Let F_μ be as defined in (8) and $I^s_{\alpha,\beta\gamma}f_i(z)$, $I^s_{\alpha,\beta\gamma}g_i(z)$ belong to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$ and $(I^s_{\alpha,\beta\gamma}f_i * I^s_{\alpha,\beta\gamma}g_i(z))(z)$ belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. Then $|b^i_n| \leq \frac{q^s(\lambda|\zeta_1+\mu\zeta_2|-|1-\mu|)}{np^s(\lambda|\zeta_1+\mu\zeta_2|+|1-\mu|)|a^i_n|}, 0 \leq, \zeta_1, \zeta_2 \leq 1, 0 < \delta < 1, \beta, \gamma \geq 0, \mu \in C$ and $|\mu| \leq 1$.

In what follows, we show that the convex combination of the differential operator belongs to the the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$.

Theorem 3: Let $I^s_{\alpha,\beta\gamma}f_i(z) \in \Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$,
and $I^s_{\alpha,\beta\gamma}g_i(z) \in \Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$.
Then $G_i(z)$ given by

$$\begin{aligned} G_i(z) &= (1-\lambda)I^s_{\alpha,\beta\gamma}f_i(z) + \lambda I^s_{\alpha,\beta\gamma}g_i(z) \\ &= z + \sum_{n=2}^{\infty} x^s C^i_n z^* n \end{aligned}$$

belongs to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$ where $C^i_n = (1-\lambda)a^i_n + \lambda b^i_n, 0 \leq \lambda \leq 1$

Proof: Let $I^s_{\alpha,\beta,\gamma}f_i(z)$ and $I^s_{\alpha,\beta,\gamma}g_i(z)$ belong to the class $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. Then we have

$$(1-\lambda)(\sum_{i=1}^{k}\sum_{n=2}^{\infty} x^s\big[n(1+\delta\zeta_1)+\mu(\delta\zeta_2-1)\big]|a^i_n|) \leq (1-\lambda)(\delta|\zeta_1+\mu\zeta_2|-|1-\mu|) \qquad (17)$$

and respectively

$$\lambda(\sum_{i=1}^{k}\sum_{n=2}^{\infty} x^s\big[n(1+\delta\zeta_1)+\mu(\delta\zeta_2-1)\big]|b^i_n|) \leq \lambda(\delta|\zeta_1+\mu\zeta_2|-|1-\mu|) \qquad (18)$$

adding (17) and (18) gives

$$
\begin{aligned}
(\sum_{i=1}^{k}\sum_{n=2}^{\infty} x^s\big[n(1+\delta\zeta_1)+\mu(\delta\zeta_2-1)\big])((1-\lambda)|a_n^i|+\lambda|b_n^i|) &\leq (1-\lambda)(\delta|\zeta_1+\mu\zeta_2|-|1-\mu|)\\
&+ \lambda(\delta|\zeta_1+\mu\zeta_2|-|1-\mu|)\\
&= (\delta|\zeta_1+\mu\zeta_2|-|1-\mu|)
\end{aligned}
$$

Which prove the result.

Now, we establish two of the fundamental theorems about univalent functions in relation to function in the subclass $\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$; the growth and distortion theorems, which provide bounds on $|I^s_{\alpha,\beta,\gamma}f_i(z)|$ and $|I^s_{\alpha,\beta,\gamma}f_i'(z)|$ respectively. Theorems (4) and (5) below are the growth and distortion theorems respectively.

Theorem 4: Let $I^s_{\alpha,\beta\gamma}f_i(z)\in\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. Then
$r-\frac{(\alpha+\beta+\gamma)(\delta|\zeta_1+\mu\zeta_2|-|1-\mu|)}{\big[n(1+\delta\zeta_1)-\mu(1+\delta\zeta_2)\big](\alpha+n\beta+n^2\gamma)}r^2\leq|I^s_{\alpha,\beta\gamma}f_i(z)|\leq r+\frac{(\alpha+\beta+\gamma)(\delta|\zeta_1+\mu\zeta_2|-|1-\mu|)}{\big[n(1+\delta\zeta_1)-\mu(1+\delta\zeta_2)\big](\alpha+n\beta+n^2\gamma)}r^2.$

Proof: For a function $f\in A$,
$|f(z)|\leq r+\sum_{n=2}^{\infty}|a_n|r^n\leq r+r^2\sum_{n=2}^{\infty}|a_n|.$
Similarly,
$|f(z)|\geq r-r^2\sum_{n=2}^{\infty}|a_n|.$
Fixing the value of a_n for the function in $I^s_{\alpha,\beta,\gamma}f_i(z)$ and rearranging gives the result.

Theorem 5: Let $I^s_{\alpha,\beta\gamma}f_i(z)\in\Gamma_{\mu;\alpha,\beta,\gamma}(\zeta_1,\zeta_2,\delta)$. Then

$$
\begin{aligned}
1-\frac{(\alpha+\beta+\gamma)(\delta|\zeta_1+\mu\zeta_2|-|1-\mu|)}{\big[n(1+\delta\zeta_1)-\mu(1+\delta\zeta_2)\big](\alpha+n\beta+n^2\gamma)}r &\leq |I^s_{\alpha,\beta,\gamma}f_i'(z)|\leq\\
&\leq 1+\frac{(\alpha+\beta+\gamma)(\delta|\zeta_1+\mu\zeta_2|-|1-\mu|)}{\big[n(1+\delta\zeta_1)-\mu(1+\delta\zeta_2)\big](\alpha+n\beta+n^2\gamma)}r.
\end{aligned}
$$

Proof: The proof follows from theorem 4.

References

[1] N. E. Cho and H. M. Srivastava , Argument estimates of certain analytic functions defined by a class of multiplier transformations, Math. Comput. Modeling, 37(1-2) (2003), 39 - 49.

[2] N. E. Cho and T. H. Kim, Multiplier transformations and strongly Close-to-Convex functions, Bull. Korean Math. Soc., 40(3) (2003), 399 - 410.

[3] D.O. Makinde and A.T. Oladipo , Some properties of certain subclasses of univalent integral operators, Scientia Magna 9(1)(2013), 80-88.

[4] S. R. Swamy , Inclusion Properties of Certain Subclasses of Analytic Functions, International Mathematical Forum, Vol. 7, no. 36,(2012) 1751 - 1760.

[5] H. Silverman Proceedings of the American Mathematical Society, Vol 51, Number 1, (1975), 109 - 115

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