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On Some Generalized Inequalities for Two-Dimensional φ-Convex Functions

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Abstract. In this study, we worked at general convexity of functions which is named φ-convexity. Initially, we derived a generalized Ostrowski's type inequality for every positive φ-convex function and obtained a generalized Hadamard's type inequality for every φ-convex functions. Most especially, we identified two-dimensional φ-convex functions and verified Hermite-Hadamard inequality for these functions. Concordantly, we generalized the above mentioned inequalities for every positive two-dimensional φ-convex function and every two-dimensional φ-convex function, respectively.

Key Words and Phrases: φ-Convexity, two-dimensional function, on the coordinates, Ostrowski's type inequality, Hermite-Hadamard's type inequality.

1. Introduction and Preliminaries

The function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is referred convex, if the inequality satisfies for $x, y \in I$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) ; \; \lambda \in [0,1] .$$

It becomes famous the following Hermite-Hadamard inequality for convex functions [1]

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)\, dx \leq \frac{f(a)+f(b)}{2}.$$

Alomari [2] generalized this classical Hermite-Hadamard type inequalities for every convex function and obtained some sharps Ostrowski's type inequalities for every positive convex function on $[a,b]$ as follows, respectively:

$$h \sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) \leq \int_a^b f(t)\, dt \leq \frac{h}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right];$$

$$\int_a^b f(x)\, dx - (b-a) f(t) \leq \frac{h}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right].$$

Recently, Dragomir [3] proved the following Hadamard's type inequality for convex functions on the coordinates as follows:

$$f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\leq\frac{1}{2(b-a)}\int_a^b f\left(x,\frac{c+d}{2}\right)dx+\frac{1}{2(d-c)}\int_c^d f\left(\frac{a+b}{2},y\right)dy$$

$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy$$

$$\leq\frac{1}{4(b-a)}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{1}{4(d-c)}\int_c^d[f(a,y)+f(b,y)]\,dy$$

$$\leq\frac{1}{4}[f(a,c)+f(a,d)+f(b,c)+f(b,d)].$$

Like as Alomari [2], Mwaeze [4] proved some generalizations inequalities of the above inequality on $[a,b]\times[c,d]$, as follows:

$$\frac{d-c}{2n}\sum_{k=1}^{n}\int_a^b f\left(x,\frac{y_{k-1}+y_k}{2}\right)dx+\frac{b-a}{2n}\sum_{k=1}^{n}\int_c^d f\left(\frac{x_{k-1}+x_k}{2},y\right)dy$$

$$\leq\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy$$

$$\leq\frac{d-c}{4n}\int_a^b[f(x,c)+f(x,d)]\,dx+\frac{b-a}{4n}\int_c^d[f(a,y)+f(b,y)]\,dy$$

$$+\frac{d-c}{2n}\sum_{k=1}^{n-1}\int_a^b f(x,y_k)\,dx+\frac{b-a}{2n}\sum_{k=1}^{n-1}\int_c^d f(x_k,y)dy\ ;$$

$$\int_a^b\int_c^d f(x,y)dxdy\leq\frac{b-a}{4n}\left[\begin{array}{c}(n+1)\int_c^d f(a,y)\,dy+(n+1)\int_c^d f(b,y)dy\\+2\sum_{k=1}^{n-1}\int_c^d f(x_k,y)\,dy\end{array}\right]$$

$$\leq\frac{d-c}{4n}\left[\begin{array}{c}(n+1)\int_a^b f(x,c)\,dx+(n+1)\int_a^b f(x,d)dx\\+2\sum_{k=1}^{n-1}\int_a^b f(x,y_k)\,dx\end{array}\right].$$

The Hermite-Hadamard type integral inequality for convex functions has received renewed attention in recent years and the remarkable varieties of refinements and generalizations have been found in [1]-[14]. In this context, the Hermite-Hadamard type inequalities for φ-convex functions were obtained by many researchers in literature (see, [5]-[10]). For example, in 1999, Youness [5] defined E-convexity such that a function $f:\mathbb{R}^n\to\mathbb{R}$ is said to be E-convex on a set $M\subset\mathbb{R}^n$ iff there is a map $E:\mathbb{R}^n\to\mathbb{R}^n$ such that M is an E-convex set and

$$f(\lambda E(t)+(1-\lambda)E(s))\leq\lambda f(E(t))+(1-\lambda)f(E(s))$$

for each $t,\ s \in M$ and $\lambda \in [0,1]$. Also, Sarikaya et al. [6] improved Youness's definition such that a function $f : [a,b] \subset \mathbb{R} \rightarrow \mathbb{R}$ is called φ-convex function on $[a,b]$, if the following inequality holds

$$f\left(\lambda\varphi(t) + (1-\lambda)\varphi(s)\right) \leq \lambda f(\varphi(t)) + (1-\lambda) f(\varphi(s))$$

for all $t,\ s \in [a,b],\ \lambda \in [0,1]$ and $\varphi : [a,b] \rightarrow [a,b],\ a < b$ is a function. Besides, Cristescu [7] presented the following inequality for φ-convex functions:

$$f\left(\frac{\varphi(u)+\varphi(v)}{2}\right) \leq \frac{1}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} f(x)\,dx \leq \frac{f(\varphi(u)) + f(\varphi(v))}{2}.$$

Concordantly, we intend to establish some important generalizations of the last above inequality for φ-convex functions on real number line and on the coordinates, respectively.

2. Main Results

This section contains three sub-sections. In the first sub-section we obtained a sharp Ostrowski's type inequality for every positive φ-convex function and a generalized Hadamard's type inequalities for every φ-convex functions. In the second sub-section we introduced two-dimensional φ-convex functions and derived Hermite-Hadamard inequality for these functions. In the third sub-section we verified the above mentioned inequalities for positive two-dimensional φ-convex functions and two-dimensional φ-convex functions, respectively.

2.1. Generalized Inequalities for φ-Convex Functions on the Real Line

Initially, we need to obtain our main results the following propositions. For this reason, we used for their proof the similar methods of the papers in [12]-[14]. Let $\varphi : [u,v] \rightarrow [u,v]$, $a < b$ a continuous function and $H_\varphi : [0,1] \rightarrow \mathbb{R}$ be defined by

$$H_\varphi(\lambda) := \frac{1}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} f\left(\lambda t + (1-\lambda)\frac{\varphi(u)+\varphi(v)}{2}\right) dt,$$

where $f : [u,v] \rightarrow \mathbb{R}$ is *a* φ-convex function $[u,v] \subset \mathbb{R}$, $\lambda \in [0,1]$;

Proposition 2.1. *Let* $f : [u,v] \rightarrow \mathbb{R}$ *be as above. Then* **i)** $\mathrm{H}_\varphi(\lambda)$ *is* φ*-convex on* $[0,1]$, **ii)** *we get*

$$\inf_{\lambda \in [0,1]} H_\varphi(\lambda) = H_\varphi(0) = f\left(\frac{\varphi(u)+\varphi(v)}{2}\right);$$

$$\sup_{\lambda \in [0,1]} H_\varphi(\lambda) = H_\varphi(1) = \frac{1}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} f(t)\,dt,$$

iii) $H_\varphi(\lambda)$ *is monotonic nondecreasing on* $[0,1]$.

Proof. **i)** Let $\gamma,\delta\geq 0$ with $\gamma+\delta=1$ and $\lambda_1,\lambda_2\in[0,1]$. Then

$$H_{\varphi}(\gamma\lambda_1+\delta\lambda_2)=\frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)}f\begin{pmatrix}\gamma\left(\lambda_1 t+(1-\lambda_1)\frac{\varphi(u)+\varphi(v)}{2}\right)\\+\delta\left(\lambda_2 t+(1-\lambda_2)\frac{\varphi(u)+\varphi(v)}{2}\right)\end{pmatrix}dt$$

$$\leq\gamma\frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)}f\left(\lambda_1 t+(1-\lambda_1)\frac{\varphi(u)+\varphi(v)}{2}\right)dt$$

$$+\delta\frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)}f\left(\lambda_2 t+(1-\lambda_2)\frac{\varphi(u)+\varphi(v)}{2}\right)dt=\gamma H_{\varphi}(\lambda_1)+\delta H_{\varphi}(\lambda_2)$$

which shows that $H_{\varphi}(\lambda)$ is φ-convex on $[0,1]$.
ii) By Jensen's inequality

$$H_{\varphi}(\lambda)\geq f\left(\frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)}\left[\lambda t+(1-\lambda)\frac{\varphi(u)+\varphi(v)}{2}\right]dt\right)=f\left(\frac{\varphi(u)+\varphi(v)}{2}\right).$$

Using φ-convexity of f

$$H_{\varphi}(\lambda)\leq\frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)}\left[\lambda f(t)+(1-\lambda)f\left(\frac{\varphi(u)+\varphi(v)}{2}\right)\right]dt$$

$$=\lambda\frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)}f(t)\,dt+(1-\lambda)f\left(\frac{\varphi(u)+\varphi(v)}{2}\right).$$

Combining the last inequalities

$$f\left(\frac{\varphi(u)+\varphi(v)}{2}\right)\leq H_{\varphi}(\lambda)$$

$$\leq\lambda\frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)}f(t)\,dt+(1-\lambda)f\left(\frac{\varphi(u)+\varphi(v)}{2}\right)\leq\frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)}f(t)\,dt$$

for all $\lambda\in[0,1]$.
iii) Let $\lambda_1,\lambda_2\in(0,1)$ with $\lambda_1<\lambda_2$. Then

$$\frac{H_{\varphi}(\lambda_2)-H_{\varphi}(\lambda_1)}{\lambda_2-\lambda_1}\geq H'_{\varphi+}(\lambda_1)$$

$$=\frac{1}{\varphi(v)-\varphi(u)}\int_{\varphi(u)}^{\varphi(v)}f'_{+}\left(\lambda_1 t+(1-\lambda_1)\frac{\varphi(u)+\varphi(v)}{2}\right)\left(t-\frac{\varphi(u)+\varphi(v)}{2}\right)dt.$$

Since f is φ-convex on $[u,v]\subset\mathbb{R}$, we deduce that

$$f\left(\frac{\varphi(u)+\varphi(v)}{2}\right)-f\left(\lambda_1 t+(1-\lambda_1)\frac{\varphi(u)+\varphi(v)}{2}\right)$$

$$\geq \lambda_1 \int_{\varphi(u)}^{\varphi(v)} f'_+ \left(\lambda_1 t + (1-\lambda_1)\frac{\varphi(u)+\varphi(v)}{2}\right)\left(\frac{\varphi(u)+\varphi(v)}{2} - t\right) dt.$$

For every $t \in [u, v]$. So,

$$\frac{1}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} f'_+ \left(\lambda_1 t + (1-\lambda_1)\frac{\varphi(u)+\varphi(v)}{2}\right)\left(t - \frac{\varphi(u)+\varphi(v)}{2}\right) dt$$

$$\geq \frac{1}{\lambda_1}\left[\frac{1}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} f\left(\lambda_1 t + (1-\lambda_1)\frac{\varphi(u)+\varphi(v)}{2}\right) dt - f\left(\frac{\varphi(u)+\varphi(v)}{2}\right)\right]$$

$$\geq \frac{1}{\lambda_1}\left[H_\varphi(\lambda_1) - f\left(\frac{\varphi(u)+\varphi(v)}{2}\right)\right] \geq 0.$$

Consequently, $H_\varphi(\lambda_2) - H_\varphi(\lambda_1) \geq 0$ for $1 \geq \lambda_2 > \lambda_1 \geq 0$. That completes the proof.

Proposition 2.2. *Suppose that $f : [u, v] \to \mathbb{R}$ is φ-convex and the mapping $G_\varphi : [0, 1] \to \mathbb{R}$ is defined by*

$$G_\varphi(\lambda) := \frac{1}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} \left[f\left(\frac{1+\lambda}{2}\varphi(u) + \frac{1-\lambda}{2}s\right) + f\left(\frac{1+\lambda}{2}\varphi(v) + \frac{1-\lambda}{2}s\right)\right] ds.$$

*Then **i)** $G_\varphi(\lambda)$ is φ-convex and monotonic nondecreasing on $[0, 1]$, **ii)** we get*

$$\inf_{\lambda\in[0,1]} G_\varphi(\lambda) = G_\varphi(0) = \frac{1}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} f(t)\, dt;$$

$$\sup_{\lambda\in[0,1]} G_\varphi(\lambda) = G_\varphi(1) = \frac{f(\varphi(u)) + f(\varphi(v))}{2}.$$

Proof. It is obviously proved like as Proposition 2.1.

Remark 2.1. *Let $f : [u, v] \to \mathbb{R}$ be φ-convex function on $[u, v]$ and the mappings $H_{\varphi,i}$, $F_{\varphi,i} : [0, 1] \to \mathbb{R}$ be defined by*

$$H_{\varphi,i}(\lambda) := \frac{n}{\varphi(v)-\varphi(u)} \int_{\varphi(t_{i-1})}^{\varphi(t_i)} f\left(\lambda s + (1-\lambda)\frac{\varphi(t_{i-1}) - \varphi(t_i)}{2}\right) ds;$$

$$G_{\varphi,i}(\lambda) := \frac{n}{\varphi(v)-\varphi(u)} \int_{\varphi(t_{i-1})}^{\varphi(t_i)} \left[f\left(\frac{1+\lambda}{2}\varphi(t_{i-1}) + \frac{1-\lambda}{2}s\right) + f\left(\frac{1+\lambda}{2}\varphi(t_i) + \frac{1-\lambda}{2}s\right)\right] ds,$$

for all $s \in [t_{i-1}, t_i]$.

*Applying Propositions 2.1 and 2.2, then we obtain for $f : [t_{i-1}, t_i] \to \mathbb{R}$, $i = 1, \ldots, n$ **i)** $H_{\varphi,i}$ and $G_{\varphi,i}$ are φ-convex for all $\lambda \in [0, 1]$ and $s \in [t_{i-1}, t_i]$; **ii)** $G_{\varphi,i}$ and $F_{\varphi,i}$ are monotonic nondecreasing for all $\lambda \in [0, 1]$ and $s \in [t_{i-1}, t_i]$; **iii)** we get*

$$H_{\varphi,i}(0) = f\left(\frac{\varphi(t_{i-1}) + \varphi(t_i)}{2}\right);\ H_{\varphi,i}(1) = \frac{n}{\varphi(v)-\varphi(u)} \int_{\varphi(t_{i-1})}^{\varphi(t_i)} f(s)\, ds;$$

$$G_{\varphi,i}(0) = \frac{n}{\varphi(v)-\varphi(u)} \int_{\varphi(t_{i-1})}^{\varphi(t_i)} f(s)\,ds;\ G_{\varphi,i}(1) = \frac{f(\varphi(t_{i-1})) + f(\varphi(t_i))}{2}.$$

Proposition 2.3. *Let $f : [u,v] \to \mathbb{R}$ be a φ-convex function and define the mappings $H_\varphi,\ F_\varphi : [0,1] \to \mathbb{R}$ given by*

$$H_\varphi = \sum_{i=1}^{n} H_{\varphi,i}(\lambda);\ G_\varphi = \sum_{i=1}^{n} G_{\varphi,i}(\lambda)$$

*where $H_{\varphi,i}$ and $G_{\varphi,i}$ are defined in Remark 2.1. Then **i)** H_φ and G_φ are φ-convex for all $\lambda \in [0,1]$ and $s \in [u,v]$; **ii)** H_φ and G_φ are monotonic nondecreasing for all $\lambda \in [0,1]$ and $s \in [u,v]$; **iii)** we get*

$$\underset{\lambda\in[0,1]}{inf}\ H_\varphi(\lambda)\ = H_\varphi(0) = \sum_{i=1}^{n} f\left(\frac{\varphi(t_{i-1}) + \varphi(t_i)}{2}\right);$$

$$\underset{\lambda\in[0,1]}{sup}\ H_\varphi(\lambda)\ = H_\varphi(1) = \frac{n}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} f(s)\,ds;$$

$$\underset{\lambda\in[0,1]}{inf}\ G_\varphi(\lambda)\ = G_\varphi(0) = \frac{n}{\varphi(v)-\varphi(u)} \int_{\varphi(u)}^{\varphi(v)} f(s)\,ds;$$

$$\underset{\lambda\in[0,1]}{sup}\ G_\varphi(\lambda)\ = G_\varphi(1) = \frac{1}{2}\left[f(u) + f(v) + 2\sum_{i=1}^{n-1} f(\varphi(t_i))\right].$$

Proof. Taking the sum over from i to n by $H_{\varphi,i}(0);\ H_{\varphi,i}(1)$ and $G_{\varphi,i}(0); G_{\varphi,i}(1)$, this proof is completed.

Let us give firstly the generalized Ostrowki's type inequality for every positive φ-convex functions on the interval $[u,v] \subset \mathbb{R}$.

Theorem 2.1. *Let $f : [u,v] \to \mathbb{R}_+$ be a positive φ-convex function and $f \in L[u,v]$. Then the following inequality holds*

$$\int_{\varphi(u)}^{\varphi(v)} f(t)\,dt - [\varphi(v) - \varphi(u)]\, f(\varphi(s))$$

$$\leq \frac{\varphi(v)-\varphi(u)}{2n}\left[f(\varphi(u)) + f(\varphi(v)) + 2\sum_{k=1}^{n-1} f(\varphi(t_k))\right] \quad (1)$$

foralls $\in [u,v], \varphi(t_k) = \varphi(u) + k\frac{\varphi(v)-\varphi(u)}{n}, k = 0,1,2,\ \ldots, n; n \in N.$

Proof. Since f is φ-convex on $[u,v]$, then f is φ-convex on each subinterval $[t_{k-1}, t_k]$. So, for $s \in [t_{k-1}, t_k]$, $k = 1,2,\ \ldots, n$

$$f(\lambda\varphi(t_{k-1}) + (1-\lambda)\varphi(s)) \leq \lambda f(\varphi(t_{k-1})) + (1-\lambda) f(\varphi(s)),\quad k = 1,2,\ \ldots, n \quad (2)$$

Integrating (2) with respect to λ on $[0,1]$

$$\int_0^1 f\left(\lambda\varphi\left(t_{k-1}\right)+(1-\lambda)\varphi\left(s\right)\right)d\lambda \leq \frac{f\left(\varphi\left(t_{k-1}\right)\right)+f\left(\varphi\left(s\right)\right)}{2}. \tag{3}$$

Changing of variable $t=\lambda\varphi\left(t_{k-1}\right)+(1-\lambda)\varphi\left(s\right)$ in (3)

$$\int_{\varphi(t_{k-1})}^{\varphi(s)} f\left(t\right)dt \leq \frac{\varphi\left(s\right)-\varphi\left(t_{k-1}\right)}{2}\left[f\left(\varphi\left(t_{k-1}\right)\right)+f\left(\varphi\left(s\right)\right)\right]. \tag{4}$$

Using the above method

$$\int_{\varphi(s)}^{\varphi(t_k)} f\left(t\right)dt \leq \frac{\varphi\left(t_k\right)-\varphi\left(s\right)}{2}\left[f\left(\varphi\left(s\right)\right)+f\left(\varphi\left(t_k\right)\right)\right]. \tag{5}$$

Adding the inequalities (4) and (5)

$$\int_{\varphi(t_{k-1})}^{\varphi(s)} f\left(t\right)dt+\int_{\varphi(s)}^{\varphi(t_k)} f\left(t\right)dt=\int_{\varphi(t_{k-1})}^{\varphi(t_k)} f\left(t\right)dt$$

$$\leq \frac{\varphi\left(s\right)-\varphi\left(t_{k-1}\right)}{2}\left[f\left(\varphi\left(t_{k-1}\right)\right)+f\left(\varphi\left(s\right)\right)\right]+\frac{\varphi\left(t_k\right)-\varphi\left(s\right)}{2}\left[f\left(\varphi\left(s\right)\right)+f\left(\varphi\left(t_k\right)\right)\right]$$

$$\leq \frac{\varphi\left(t_k\right)-\varphi\left(t_{k-1}\right)}{2}\left[f\left(\varphi\left(t_{k-1}\right)\right)+f\left(\varphi\left(t_k\right)\right)\right]+\frac{\varphi\left(v\right)-\varphi\left(u\right)}{n}f\left(\varphi\left(s\right)\right) \tag{6}$$

Taking the sum over k from 1 to n on (6)

$$\sum_{k=1}^{n}\int_{\varphi(t_{k-1})}^{\varphi(t_k)} f\left(t\right)dt=\int_{\varphi(u)}^{\varphi(v)} f\left(t\right)dt$$

$$\leq \sum_{k=1}^{n}\frac{\varphi\left(t_k\right)-\varphi\left(t_{k-1}\right)}{2}\left[f\left(\varphi\left(t_{k-1}\right)\right)+f\left(\varphi\left(t_k\right)\right)\right]+\sum_{k=1}^{n}\frac{\varphi\left(v\right)-\varphi\left(u\right)}{n}f\left(\varphi\left(s\right)\right).$$

$$\leq \frac{1}{2}\max_k\left\{\varphi\left(t_k\right)-\varphi\left(t_{k-1}\right)\right\}\sum_{k=1}^{n}\left[f\left(\varphi\left(t_{k-1}\right)\right)+f\left(\varphi\left(t_k\right)\right)\right]+\left(\varphi\left(v\right)-\varphi\left(u\right)\right)f\left(\varphi\left(s\right)\right)$$

$$=\frac{\varphi\left(v\right)-\varphi\left(u\right)}{2n}\left[\begin{array}{c} f\left(\varphi\left(t_0\right)\right)+f\left(\varphi\left(t_1\right)\right)+f\left(\varphi\left(t_{n-1}\right)\right)+f\left(\varphi\left(t_n\right)\right) \\ +\sum_{k=2}^{n-1}\left[f\left(\varphi\left(t_{k-1}\right)\right)+f\left(\varphi\left(t_k\right)\right)\right]\end{array}\right]+\left(\varphi\left(v\right)-\varphi\left(u\right)\right)f\left(\varphi\left(s\right)\right)$$

$$=\frac{\varphi\left(v\right)-\varphi\left(u\right)}{2n}\left[f\left(\varphi\left(u\right)\right)+f\left(\varphi\left(v\right)\right)+2\sum_{k=1}^{n-1}f\left(\varphi\left(t_k\right)\right)\right]+\left(\varphi\left(v\right)-\varphi\left(u\right)\right)f\left(\varphi\left(s\right)\right). \tag{7}$$

This completes the proof.

Let us obtain the generalized Hadamard's type inequality for every φ-convex functions.

Theorem 2.2. *Let $f:[u,v]\to\mathbb{R}$ be a φ-convex function and $f\in L[u,v]$. Then the double inequality holds*

$$\frac{\varphi\left(v\right)-\varphi\left(u\right)}{n}\sum_{k=1}^{n}f\left(\frac{\varphi\left(t_{k-1}\right)+\varphi\left(t_k\right)}{2}\right)\leq\int_{\varphi(u)}^{\varphi(v)} f\left(t\right)dt$$

$$\leq \frac{\varphi(v)-\varphi(u)}{2n}\left[f(\varphi(u))+2\sum_{k=1}^{n-1}f(\varphi(t_k))+f(\varphi(v))\right] \tag{8}$$

where $\varphi(t_k)=\varphi(u)+k\frac{\varphi(v)-\varphi(u)}{n}$, $k=0,1,2,\ \ldots,\ n;\ n\in N$.

Proof. As similar method of Theorem 2.1, using φ-convexity of f on each sub-interval $[t_{k-1},t_k]\subseteq[u,v]$, $k=1,2,\ \ldots,n$ we obtain the right side of the inequality (7). Accordingly

$$f\left(\frac{\varphi(t_{k-1})+\varphi(t_k)}{2}\right)=f\left(\frac{\lambda\varphi(t_k)+(1-\lambda)\varphi(t_{k-1})}{2}+\frac{(1-\lambda)\varphi(t_k)+\lambda\varphi(t_{k-1})}{2}\right)$$

$$\leq\frac{1}{2}\left[f(\lambda\varphi(t_k)+(1-\lambda)\varphi(t_{k-1}))+f((1-\lambda)\varphi(t_k)+\lambda\varphi(t_{k-1}))\right]. \tag{9}$$

Following method of procedure between the inequalities (3) and (7) to the inequality (9), we get the left side of the inequality (7).

Remark 2.2. *In Theorem 2.2 for $n=1$, then we refer to Hermite-Hadamard inequality on the real number line.*

2.2. Two-dimensional φ-Convex Functions and Related Inequality

Let $\Delta:=[u_1,u_2]\times[v_1,v_2]\subseteq\mathbb{R}^2$, $u_1<u_2$, $v_1<v_2$, $\varphi:\Delta\to\Delta$ be a continuous function. Definition of a two-dimensional φ-convex function (or φ-convex functions on Δ) is follows:

Definition 2.1. *A function $F{:}\Delta\to\mathbb{R}$ is called φ-convex on Δ, if the following inequality holds*

$$F(\lambda\varphi(t_1)+(1-\lambda)\varphi(t_2),\lambda\varphi(s_1)+(1-\lambda)\varphi(s_2))$$
$$\leq\lambda F(\varphi(t_1),\varphi(s_1))+(1-\lambda)F(\varphi(t_2),\varphi(s_2))$$

for all (t_1,s_1), $(t_2,s_2)\in\Delta$ and $\lambda\in[0,1]$. If the above inequality is reversed then F is said to be φ-concave on Δ.

Definition 2.2. *A function $F:\Delta\to\mathbb{R}$ is called φ-convex on the coordinates, if the following partial mappings are defined φ-convex $F_s:[u_1,u_2]\ \to\mathbb{R}$, $F_s(u):=F(u,s)$ and $F_t:[v_1,v_2]\ \to\mathbb{R}$, $F_t(v):=F(t,v)$ for all $t\in[u_1,u_2]$ and $s\in[v_1,v_2]$.*

Lemma 2.1. *Every φ-convex function on Δ is φ-convex on the coordinates, but converse is not true.*

Proof. Let $F:\Delta\to\mathbb{R}$ be a φ-convex function on Δ. Consider $F_t:\ [v_1,v_2]\to\mathbb{R}$ defined by $F_t(v):=F(t,\ v)$. Now for $s_1,s_2\in[v_1,v_2]$ and $\lambda\in[0,1]$

$$F_t(\lambda\varphi(s_1)+(1-\lambda)\varphi(s_2))$$
$$=F(t,\ \lambda\varphi(s_1)+(1-\lambda)\varphi(s_2))=F(\lambda t+(1-\lambda)t,\ \lambda\varphi(s_1)+(1-\lambda)\varphi(s_2))$$
$$\leq\lambda F(t,\varphi(s_1))+\ (1-\lambda)F(t,\varphi(s_2))=\lambda F_t(\varphi(s_1))+(1-\lambda)F_t(\varphi(s_2)).$$

Then, F_t is φ-convex on $[u_1,u_2]$. Similarly, it can be show that F_s is φ-convex on $[v_1,v_2]$. Thus, F is a φ-convex function on the coordinates. Let's give a counter example for converse:

Example 2.1. *Let us consider a function* $F_0 : [0,1]^2 \to [0,\infty)$ *defined as* $F_0(t,s) := ts$ *and* $\varphi : [0,1]^2 \to [0,1]^2$, $\varphi(t,s) := (t,s)$. *Then ?? is not* φ*-convex on* $[0,1]^2$.
Proof. Let us assume that F is φ-convex on $[0,1]^2$. But for $(t,0)$, $(0,s) \in [0,1]^2$

$$F_0(\lambda\varphi(t,0) + (1-\lambda)\varphi(0,s)) = F_0(\lambda t, (1-\lambda)s) = \lambda(1-\lambda)ts;$$

$$\lambda F_0(\varphi(t,0)) + (1-\lambda)F_0(\varphi(0,s)) = \lambda F_0(t,0) + (1-\lambda)F_0(0,s) = 0.$$

Thus

$$F_0(\lambda\varphi(t,0) + (1-\lambda)\varphi(0,s)) > \lambda F_0(\varphi(t,0)) + (1-\lambda)F_0(\varphi(0,s))$$

for all $\lambda \in [0,1]$, that is F is not φ-convex on $[0,1]^2$.
Let us derive Hermite-Hadamard inequality for φ-convex functions on the coordinates.
Theorem 2.3. *Let* $F : \Delta \to \mathbb{R}$ *be a two-dimensional* φ*-convex function on* Δ. *Then the following inequality holds for* $\varphi(u_i) < \varphi(v_i)$, $i = 1,2$ *as follows:*

$$\begin{aligned} F\left(\frac{\varphi(u_1)+\varphi(u_2)}{2}, \frac{\varphi(v_1)+\varphi(v_2)}{2}\right) &\le \frac{1}{2(\varphi(u_2)-\varphi(u_1))}\int_{\varphi(u_1)}^{\varphi(u_2)} F\left(t, \frac{\varphi(v_1)+\varphi(v_2)}{2}\right)dt \\ &+\frac{1}{2(\varphi(v_2)-\varphi(v_1\ ))}\int_{\varphi(v_1)}^{\varphi(v_2)} F\left(\frac{\varphi(u_1)+\varphi(u_2)}{2}, s\right)ds \\ &\le \frac{1}{(\varphi(u_2)-\varphi(u_1))(\varphi(v_2)-\varphi(v_1))}\int_{\varphi(u_1)}^{\varphi(u_1)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)\,dtds \\ &\le \frac{1}{4(\varphi(u_2)-\varphi(u_1))}\int_{\varphi(u_1)}^{\varphi(u_2)} [F(t,\varphi(v_1)) + F(t,\varphi(v_2))]\,dt \\ &+\frac{1}{4(\varphi(v_2)-\varphi(v_1))}\int_{\varphi(v_1)}^{\varphi(v_2)} [F(\varphi(u_1),s) + F(\varphi(u_2),s)]\,ds \\ &\le \frac{1}{4}[F(\varphi(u_1),\varphi(v_1)) + F(\varphi(u_1),\varphi(v_2)) + F(\varphi(u_2),\varphi(v_1)) + F(\varphi(u_2),\varphi(v_2))]. \end{aligned} \tag{10}$$

Proof. Using φ-convexity of the partial mapping $F_t(v)$ on $[\varphi(v_1), \varphi(v_2)]$

$$F_t\left(\frac{\varphi(v_1)+\varphi(v_2)}{2}\right) \le \frac{1}{\varphi(v_2)-\varphi(v_1)}\int_{\varphi(v_1)}^{\varphi(v_2)} F_t(s)\,ds \le \frac{F_t(\varphi(v_1)) + F_t(\varphi(v_2))}{2}.$$

Because of $F_t(v) := F(t,v)$

$$F\left(t, \frac{\varphi(v_1)+\varphi(v_2)}{2}\right) \le \frac{1}{\varphi(v_2)-\varphi(v_1)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)\,ds \le \frac{F(t,\varphi(v_1)) + F(t,\varphi(v_2))}{2}. \tag{11}$$

Integrating (11) with respect to t on $[\varphi(u_1), \varphi(u_2)]$

$$\frac{1}{\varphi(u_2)-\varphi(u_1)}\int_{\varphi(u_1)}^{\varphi(u_2)} F\left(t, \frac{\varphi(v_1)+\varphi(v_2)}{2}\right) dt$$

$$\leq \frac{1}{(\varphi(u_2)-\varphi(u_1))(\varphi(v_2)-\varphi(v_1))}\int_{\varphi(u_1)}^{\varphi(u_2)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)\,dsdt$$

$$\leq \frac{1}{2(\varphi(u_2)-\varphi(u_1))}\int_{\varphi(u_1)}^{\varphi(u_2)} [F(t,\varphi(v_1))+F(t,\varphi(v_2))]dt. \tag{12}$$

Similarly for $F_s(u)$

$$\frac{1}{\varphi(v_2)-\varphi(v_1)}\int_{\varphi(v_1)}^{\varphi(v_2)} F\left(\frac{\varphi(u_1)+\varphi(u_2)}{2}, s\right) ds$$

$$\leq \frac{1}{(\varphi(u_2)-\varphi(u_1))(\varphi(v_2)-\varphi(v_1))}\int_{\varphi(u_1)}^{\varphi(u_2)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)\,dsdt$$

$$\leq \frac{1}{2(\varphi(v_2)-\varphi(v_1))}\int_{\varphi(v_1)}^{\varphi(v_2)} [F(\varphi(u_1),s)+F(\varphi(u_2),s)]ds. \tag{13}$$

Using (12) and (13), we get the second and third inequality of (10). Additionally by the left side of (11) for $F_t(v)$ and $F_s(u)$, respectively

$$F\left(\frac{\varphi(u_1)+\varphi(u_2)}{2}, \frac{\varphi(v_1)+\varphi(v_2)}{2}\right) \leq \frac{1}{\varphi(u_2)-\varphi(u_1)}\int_{\varphi(u_1)}^{\varphi(u_2)} F\left(t, \frac{\varphi(v_1)+\varphi(v_2)}{2}\right) dt;$$

$$F\left(\frac{\varphi(u_1)+\varphi(u_2)}{2}, \frac{\varphi(v_1)+\varphi(v_2)}{2}\right) \leq \frac{1}{\varphi(v_2)-\varphi(v_1)}\int_{\varphi(v_1)}^{\varphi(v_2)} F\left(\frac{\varphi(u_1)+\varphi(u_2)}{2}, s\right) ds.$$

Adding the above inequalities, we get the first inequality of (10). Also, by the right side of (11) for $F_t(v)$ and $F_s(u)$, respectively

$$\frac{1}{\varphi(u_2)-\varphi(u_1)}\int_{\varphi(u_1)}^{\varphi(u_2)} F(t,\varphi(v_1))\,dt \leq \frac{F(\varphi(u_1),\varphi(v_1))+F(\varphi(u_2),\varphi(v_1))}{2};$$

$$\frac{1}{\varphi(u_2)-\varphi(u_1)}\int_{\varphi(u_1)}^{\varphi(u_2)} F(t,\varphi(v_2))\,dt \leq \frac{F(\varphi(u_1),\varphi(v_2))+F(\varphi(u_2),\varphi(v_2))}{2};$$

$$\frac{1}{\varphi(v_2)-\varphi(v_1)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_1),s)ds \leq \frac{F(\varphi(u_1),\varphi(v_1))+F(\varphi(u_1),\varphi(v_2))}{2};$$

$$\frac{1}{\varphi(v_2)-\varphi(v_1)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_2),s)\,ds \leq \frac{F(\varphi(u_2),\varphi(v_1))+F(\varphi(u_2),\varphi(v_2))}{2}.$$

Accordingly, we obtain the last inequality of (10) adding the above four inequalities.

2.3. Generalized Inequalities for Two-dimensional φ-Convex Functions

Let $\Delta := [u_1, u_2] \times [v_1, v_2] \subseteq \mathbb{R}^2$, $u_1 < u_2$, $v_1 < v_2$ and $\varphi : \Delta \to \Delta$ be a continuous function. Let us prove the generalized Ostrowki's type inequality for related functions.

Theorem 2.4. *Let $F : \Delta \to \mathbb{R}_+$ be two-dimensional positive φ-convex function on Δ. Then the following inequality holds*

$$\int_{\varphi(u_1)}^{\varphi(u_2)} \int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)\,dtds$$

$$\leq \frac{\varphi(u_2)-\varphi(u_1)}{4n}\left[\begin{array}{c}(n+1)\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_1),s)\,ds+(n+1)\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_2),s)\,ds\\ +2\sum_{k=1}^{n-1}\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(t_k),s)\,ds\end{array}\right]$$

$$+\frac{\varphi(v_2)-\varphi(v_1)}{4n}\left[\begin{array}{c}(n+1)\int_{\varphi(u_1)}^{\varphi(u_2)} F(t,\varphi(v_1))\,dt+(n+1)\int_{\varphi(u_1)}^{\varphi(u_2)} F(t,\varphi(v_2))\,dt\\ +2\sum_{k=1}^{n-1}\int_{\varphi(u_1)}^{\varphi(u_2)} F(t,\varphi(s_k))\,dt\end{array}\right],$$

where $(t_k) = \varphi(u_1) + k\frac{\varphi(u_2)-\varphi(u_1)}{n}$, $\varphi(s_k) = \varphi(v_1) + k\frac{\varphi(v_2)-\varphi(v_1)}{n}$, $k = 0, 1, 2, \ldots, n$, $n \in N$.

Proof. Using Theorem 2.1 for the partial mapping $F_s : [u_1, u_2] \to \mathbb{R}_+$ at $t = \varphi(u_2)$

$$\int_{\varphi(u_1)}^{\varphi(u_2)} F_s(t)\,dt - [\varphi(u_2)-\varphi(u_1)]\,F_s(\varphi(u_2))$$

$$\leq \frac{\varphi(u_2)-\varphi(u_1)}{2n}\left[F_s(\varphi(u_1)) + F_s(\varphi(u_2)) + 2\sum_{k=1}^{n-1} F_s(\varphi(t_k))\right].$$

Namely

$$\int_{\varphi(u_1)}^{\varphi(u_2)} F(t,s)\,dt - [\varphi(u_2)-\varphi(u_1)]\,F(\varphi(u_2),s)$$

$$\leq \frac{\varphi(u_2)-\varphi(u_1)}{2n}\left[F(\varphi(u_1),s) + F(\varphi(u_2),s) + 2\sum_{k=1}^{n-1} F(\varphi(t_k),s)\right]. \tag{14}$$

Integrating (14) on $[\varphi(v_1), \varphi(v_2)]$

$$\int_{\varphi(u_1)}^{\varphi(u_2)} \int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)\,dtds$$

$$\leq \frac{\varphi(u_2)-\varphi(u_1)}{2n}\Bigg[\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_1),s)\,ds + (1+2n)\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_2),s)\,ds$$

$$+2\sum_{k=1}^{n-1}\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(t_k),s)ds\Bigg]. \tag{15}$$

Similarly for F_s at $t=\varphi(u_1)$

$$\int_{\varphi(u_1)}^{\varphi(u_2)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)\,dtds$$

$$\leq \frac{\varphi(u_2)-\varphi(u_1)}{2n}\left[\begin{array}{c}(1+2n)\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_1),s)\,ds+\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_2),s)\,ds\\ +2\sum_{k=1}^{n-1}\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(t_k),s))\,ds\end{array}\right]. \quad (16)$$

Adding (15) and (16)

$$\int_{\varphi(u_1)}^{\varphi(u_2)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)dtds$$

$$\leq \frac{\varphi(u_2)-\varphi(u_1)}{2n}\left[\begin{array}{c}(n+1)\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_1),s)\,ds+(n+1)\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(u_2),s)\,ds\\ +2\sum_{k=1}^{n-1}\int_{\varphi(v_1)}^{\varphi(v_2)} F(\varphi(t_k),s))\,ds\end{array}\right] \quad (17)$$

Accordingly by using $F_t:[v_1,v_2]\to\mathbb{R}_+$ at $s=\varphi(v_2)$ and $s=\varphi(v_1)$, respectively

$$\int_{\varphi(u_1)}^{\varphi(u_2)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)\,dtds$$

$$\leq \frac{\varphi(v_2)-\varphi(v_1)}{2n}\left[\begin{array}{c}(n+1)\int_{\varphi(u_1)}^{\varphi(u_2)} F(t,\varphi(v_1))\,dt+(n+1)\int_{\varphi(u_1)}^{\varphi(u_2)} F(t,\varphi(v_2))\,dt\\ +2\sum_{k=1}^{n-1}\int_{\varphi(u_1)}^{\varphi(u_2)} F(t,\varphi(t_k))\,dt\end{array}\right]. \quad (18)$$

The desired inequality of this theorem is obtained by adding (17) and (18).
Let us obtain the generalized Hadamard's inequality for related functions:
Theorem 2.5. *Let $F:\Delta\to\mathbb{R}$ be a two-dimensional φ-convex function on Δ. The following inequality holds*

$$\frac{\varphi(v_2)-\varphi(v_1)}{2n}\sum_{k=1}^{n}\int_{\varphi(u_1)}^{\varphi(u_2)} F\left(t,\frac{\varphi(s_{k-1})+\varphi(s_k)}{2}\right)dt$$

$$+\frac{\varphi(u_2)-\varphi(u_1)}{2n}\sum_{k=1}^{n}\int_{\varphi(v_1)}^{\varphi(v_2)} F\left(\frac{\varphi(t_{k-1})+\varphi(t_k)}{2},s\right)ds$$

$$\leq \frac{1}{(\varphi(u_2)-\varphi(u_1))(\varphi(v_2)-\varphi(v_1))}\int_{\varphi(u_1)}^{\varphi(u_2)}\int_{\varphi(v_1)}^{\varphi(v_2)} F(t,s)\,dtds$$

$$\leq \frac{\varphi(v_2)-\varphi(v_1)}{4n}\int_{\varphi(u_1)}^{\varphi(u_2)}\left[F(t,\varphi(v_1))+F(t,\varphi(v_2))\right]dt$$

$$+\frac{\varphi(u_2)-\varphi(u_1)}{4n}\int_{\varphi(v_1)}^{\varphi(v_2)}\left[F(\varphi(u_1),s)+F(\varphi(u_2),s)\right]ds$$

$$+\frac{\varphi(v_2)-\varphi(v_1)}{2n}\sum_{k=1}^{n-1}\int_{\varphi(u_1)}^{\varphi(u_2)}F(t,\varphi(s_k))\,dt+\frac{\varphi(u_2)-\varphi(u_1)}{2n}\sum_{k=1}^{n-1}\int_{\varphi(v_1)}^{\varphi(v_2)}F(\varphi(t_k),s)\,ds,$$

where t_k, s_k are defined as in Theorem 2.4 and $\varphi(t_{k-1})<\varphi(t_k)$, $\varphi(s_{k-1})<\varphi(s_k)$.

Proof. Using Theorem 2.2 for $F_t:[v_1,v_2]\to\mathbb{R}$ and $\varphi(s_{k-1})<\varphi(s_k)$.

$$\frac{\varphi(v_2)-\varphi(v_1)}{n}\sum_{k=1}^{n}F_t\left(\frac{\varphi(s_{k-1})+\ \varphi(s_k)}{2}\right)\le\int_{\varphi(v_1)}^{\varphi(v_2)}F_t(s)\,ds$$

$$\le\frac{\varphi(v_2)-\varphi(v_1)}{2n}\left[F_t(\varphi(v_1))+F_t(\varphi(v_2))+2\sum_{k=1}^{n-1}F_t(\ \varphi(s_k))\right].$$

Thus

$$\frac{\varphi(v_2)-\varphi(v_1)}{n}\sum_{k=1}^{n}F\left(t,\frac{\varphi(s_{k-1})+\ \varphi(s_k)}{2}\right)\le\int_{\varphi(v_1)}^{\varphi(v_2)}F(t,s)\,ds$$

$$\le\frac{\varphi(v_2)-\varphi(v_1)}{2n}\left[F(t,\varphi(v_1))+F(t,\varphi(v_2))+2\sum_{k=1}^{n-1}F(t,\varphi(s_k))\right]. \tag{19}$$

Integrating all sides of (19) on $[\varphi(u_1),\varphi(u_2)]$

$$\frac{\varphi(v_2)-\varphi(v_1)}{n}\sum_{k=1}^{n}\int_{\varphi(u_1)}^{\varphi(u_2)}F\left(t,\frac{\varphi(s_{k-1})+\ \varphi(s_k)}{2}\right)dt\le\int_{\varphi(u_1)}^{\varphi(u_2)}\int_{\varphi(v_1)}^{\varphi(v_2)}F(t,s)\,dtds$$

$$\le\frac{\varphi(v_2)-\varphi(v_1)}{2n}\left[\begin{array}{c}\int_{\varphi(u_1)}^{\varphi(u_2)}F(t,\varphi(v_1))\,dt+\int_{\varphi(u_1)}^{\varphi(u_2)}F(t,\varphi(v_2))\,dt\\+2\sum_{k=1}^{n-1}\int_{\varphi(u_1)}^{\varphi(u_2)}F(t,\varphi(s_k))\,dt\end{array}\right]. \tag{20}$$

Similarly for $F_s:[u_1,u_2]\to\mathbb{R}$ and $\varphi(t_{k-1})<\varphi(t_k)$

$$\frac{\varphi(u_2)-\varphi(u_1)}{n}\sum_{k=1}^{n}\int_{\varphi(v_1)}^{\varphi(v_2)}F\left(\frac{\varphi(t_{k-1})+\varphi(t_k)}{2},s\right)ds\le\int_{\varphi(u_1)}^{\varphi(u_2)}\int_{\varphi(v_1)}^{\varphi(v_2)}F(t,s)\,dtds$$

$$\le\frac{\varphi(u_2)-\varphi(u_1)}{2n}\left[\begin{array}{c}\int_{\varphi(v_1)}^{\varphi(v_2)}F(\varphi(u_1),s)\,ds+\int_{\varphi(v_1)}^{\varphi(v_2)}F(\varphi(u_2),s)\,ds\\+2\sum_{k=1}^{n-1}\int_{\varphi(v_1)}^{\varphi(v_2)}F(\varphi(t_k),s)\,ds\end{array}\right]. \tag{21}$$

Adding (20) and (21), this completes the proof.

Remark 2.3. *In Theorem 2.5 for $n=2$, then we refer to Hermite-Hadamard inequality on the coordinates.*

3. Conclusion

In this paper, we investigated some generalized inequality for φ-convex functions on real number line and on the coordinates. Using other inequalities for kinds of convex functions, one can be verify some important generalizations.

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