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On extremality of a class of algebraic varieties

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Abstract. In this paper one studies extremality of a class of algebraic varieties given by all multivariate monomials of degree not exceeding a natural number. One of conjectures of V. G. Sprindzuk states that such varieties are extremal. We prove this conjecture using theorem on convergence exponent for the special integral of Terry's problem.

Key Words and Phrases: Algebraic variety, Terry's problem, extremality, Diophantine Approximation

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1. Introduction

Let's consider a set of all polynomials with integral coefficients of a degree not exceeding a natural number n:

$$\Pi = \left\{ f(x) = \sum_{i=1}^{n} a_i x^i | a_i \in R \right\}.$$

A number

$$h(f) = \max(|a_0|,\ |a_1|, ..., |a_n|)$$

is called to be the height of the polynomial $f(x)$. Let a transcendental number α be given (consequently, α is not a root of none of polynomials of Π). Let $h > 0$ be a real number. For a given $h > 0$, we shall consider such polynomials the heights of which does not exceed h (it is clear that the number of such polynomials is finite). Denote by $\omega_n(\alpha)$ the supremum γ_0 of such positive numbers $\gamma > 0$ for which the relation

$$|f(\alpha)| < h^{-\gamma};\ h = h(f) \tag{1}$$

is satisfied for infinitely many polynomials from Π, when $h \to \infty$; it means that for any given $\varepsilon > 0$ there exists an unbounded and increasing sequence of positive numbers $h_1, h_1, ...$ such that for all h_m we have

$$\gamma = \gamma_0 + \varepsilon.$$

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This number was firstly studied by Mahler K. in [10, 11]. The conjecture $\gamma_0 = n$ sets up a content of the problem B' of the work [13, p. 433] of Sprindzuk V. G. The equality $\gamma_0 = n$ is equivalent to the extremality of the variety defined by monomials of considered polynomial, by the Khintchin's transference principle.

Nowadays the conjecture of Sprindzuk V. G. is proven (more general result was established by Kleinbock D. Y. and Margulis G. A. [9]). In this paper we prove the Sprindzuk's conjecture for real algebraic varieties by a new method, using estimations above for the convergence exponent of the special integral of Terry's problem. We prove the theorem below.

Theorem 1.1. *Let n be a natural number, $P(x_1, x_2, ..., x_k)$ a polynomial of degree not exceeding n, i. e.*

$$P(x_1, x_2, ..., x_k) = \sum_{\substack{0 \le i_1, \cdots, 0 \le i_k \\ i_1 + i_2 + \cdots + i_k \le n}} a_{i_1 i_2 \cdots i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}.$$

Define $v(\omega_1, \omega_2, ..., \omega_k)$ as a supremum of such $v > 0$ for which there exist an infinite set of polynomials with integral coefficients satisfying inequality

$$|P(\omega_1, \omega_2, ..., \omega_k)| < h^{-v},$$

$$h = \max_{(i_1, i_2, \cdots, i_k)} |a_{i_1 i_2 \cdots i_k}|.$$

Then, for almost all $\bar{\omega} = (\omega_1, \omega_2, ..., \omega_k)$ can one state that

$$v(\omega_1, \omega_2, ..., \omega_k) = C_{n+k}^k - 1?$$

2. Auxiliary statements

For establishing our results it is necessarily to pass to equivalent variant of the problem expressing the question as a problem on extremality of varieties. To realize the last we interpret relations above in the different form. Let we are given with some system of real numbers $\beta_1, \beta_2, ..., \beta_n$. Consider a system of inequalities:

$$\|b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n\| \le b^{-u}, \tag{2}$$

where $\|x\|$ denotes the distance from x to the nearest integral number, $b = \max(|b_1|, ..., |b_n|)$, moreover, $b > 0$, $u > 0$. Denote by $u(\beta_1, \beta_2, ..., \beta_n)$ the supremum of such $u > 0$ for which (2) is satisfied for infinitely many integral vectors $(b_1, ..., b_n)$.

We, simultaneously, consider following system of inequalities:

$$\max\left(\|q\beta_1\|, \|q\beta_2\|, ..., \|q\beta_n\|\right) \le q^{-v}. \tag{3}$$

Now we denote be $v(\beta_1, \beta_2, ..., \beta_n)$ the supremum of that $v > 0$ for which the condition (3) satisfied for infinitely many integral numbers $q > 0$. There is a principle in the theory of Diophantine Approximations, called as Khintchin's transference principle (see [4, 12]), which states that the equalities $u(\beta_1, \beta_2, ..., \beta_n) = n$ and $v(\beta_1, \beta_2, ..., \beta_n) = 1/n$ are equivalent. Again, in accordance with the Dirichlet's principle, we have

$$v(\beta_1, \beta_2, ..., \beta_n) \geq 1/n.$$

Lemma 2.1. *Let A_q, $(q = 1, ..., n)$ be a sequence of measurable sets in R^n and*

$$\sum_{q=1}^{\infty} mesA_q < \infty.$$

Then the measure of a set E of such points $\bar{x} \in R^n$ which fall into infinite number of sets A_q equals to zero.

This lemma is known as Borel-Kantelly's lemma and plays an important role in the questions concerning extremality of manifolds (see[12]). Following lemma belongs to Kavalevskaya E. I. (see [3, 4, 7, 8]).

Lemma 2.2. *Let $m \leq N$, q be natural numbers, $f_j(\bar{x}), j = 1, ..., N$ be a real measurable functions defined in the cube. Denote by $\mu(q)$ the measure of a set of that $\bar{x} \in \Omega = [0, 1]^r$ for which*

$$\|f_j(\bar{x})\| < q^{-r_j} \, (1 \leq j \leq N).$$

Then,

$$\mu(q) << q^{-r} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \int_{\Omega} e^{2\pi i (c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} d\bar{x} \right|,$$

where $r = r_1 + \cdots + r_N$, and the constant in the symbol $<<$ depends on N only.

The proof of this lemma one can find in [2, 12].

3. Proof of the theorem

In accordance with the said above it suffices for us to show that the variety

$$\Gamma = (x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k});\, 0 \leq i_1, \cdots, 0 \leq i_k;\, 0 < i_1 + \cdots + i_k \leq n$$

is extremal. Let's the monomials of the given polynomial order by some way (for example, lexicographically) and accept notations $f_j(\bar{x}) = q x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$ for $j = 1, ..., N$. Take in the lemma 2 $r_j = 1/N + \delta$ with

$$N = \sum_{\substack{0 \leq i_1, \cdots, 0 \leq i_k \\ 0 < i_1 + \cdots + i_k \leq n}} 1 = C_{n+k}^k - 1$$

expressing the number of monomials (here δ denotes an arbitrary small positive number). From the lemma 2 one derives

$$\mu(q) << q^{-r} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \int_\Omega e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} d\bar{x} \right|.$$

As it is clear from the proof of the lemma 1 the set of points $\bar{x} \in R^n$ which fall into infinite number of sets A_q is possible represent as

$$A = \bigcap_{k=1}^{\infty} \bigcup_{q=k}^{\infty} A_q.$$

Since the sets A_q are measurable then the set A is measurable also, independent of the convergence of the series of this lemma. If now the set A is not of zero measure then $mesA > 0$. In this case the set

$$A^{2h} = A \times \cdots \times A$$

has positive measure in Ω^{2h} for the integral h defined below. So, if we have proven that $mesA^{2h} = 0$ then our assumption $mesA > 0$ is false. We will use this reasoning to prove the theorem 1. Define now the set A_q by the condition $A_q = \max \|qf_j(\bar{x})\| \leq (2h)^{-1} q^{-r_j}$. It is obvious that for any pare of vectors $\bar{x}, \bar{y} \in A_q$ we have

$$\max \|q(g_j(\bar{x}_1, ..., \bar{y}_h))\| =$$

$$= \max \|q(f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h) - f_j(\bar{y}_1) - \cdots - f_j(\bar{y}_h))\| \leq q^{-r_j}, \tag{4}$$

where $(\bar{x}_1, ..., \bar{x}_h, \bar{y}_1, ..., \bar{y}_h) \in \Omega^{2h}$ and

$$g_j(\bar{x}_1, ..., \bar{y}_h) = f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h) - f_j(\bar{y}_1) - \cdots - f_j(\bar{y}_h).$$

Then, the set of points in Ω^{2h} for which (4) is satisfied contains $(A_q)^{2h}$ as a subset. By the lemma 2, for the measure μ_q of the points satisfying conditions (4), one has:

$$(\mu(q))^{2h} \leq \mu_q << q^{-1-N\delta} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} 1 \times$$

$$\times \left| \int_\Omega \cdots \int_\Omega e^{2\pi i(c_1 q g_1(\bar{x}_1,..,\bar{y}_h)+\cdots+c_N q g_N(\bar{x}_1,..,\bar{y}_h))} d\bar{x}_1 \cdots d\bar{y}_h \right| =$$

$$= q^{-1-N\delta} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \left(\int_\Omega e^{2\pi i(c_1 q f_1(\bar{x})+\cdots+c_N q f_N(\bar{x}))} d\bar{x} \right)^h \right|^2.$$

Now we write:

$$\left(\int_\Omega e^{2\pi i(c_1 q f_1(\bar{x})+\cdots+c_N q f_N(\bar{x}))} d\bar{x} \right)^h =$$

$$= \int_\Omega \cdots \int_\Omega e^{2\pi i(c_1 q(f_1(\bar{x}_1)+\cdots+f_1(\bar{x}_h))+\cdots+c_N q(f_N(\bar{x}_1)+\cdots+f_N(\bar{x}_h)))} d\bar{x}_1 \cdots d\bar{x}_h. \tag{5}$$

Making under the multiple integral exchange of variables we get, in accordance with the lemmas from the works [5,6]:

$$\int_\Omega \cdots \int_\Omega e^{2\pi i(c_1 q(f_1(\bar{x}_1)+\cdots+f_1(\bar{x}_h))+\cdots+c_N q(f_N(\bar{x}_1)+\cdots+f_N(\bar{x}_h)))} d\bar{x}_1 \cdots d\bar{x}_h =$$

$$= \int_0^h \cdots \int_0^h du_1 \cdots du_N e^{2\pi i q(c_1 u_1+\cdots+c_N u_N)} \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}}, \tag{6}$$

where $\Pi(\bar{u})$ denotes the Gram determinant of the functions standing on the right hand side of the system's equations.

$$f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h) = u_j,\ j = 1, ..., N,$$

and G denotes the Gram determinant of gradients of functions standing on the left hand sides of these equations. Representing the integral on the right side of the relation (6) as a sum of Fourier coefficients of pieces of inner surface integral which is taken in improper meaning defined in above works. For this purpose let's dissect the integral at the right side of (6) into the sum of integrals taken along unite cubes:

$$\int_0^h \cdots \int_0^h du_1 \cdots du_N e^{2\pi i(c_1 q u_1+\cdots+c_N q u_N)} \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} =$$

$$= \sum_{h_1=1}^{h} \cdots \sum_{h_N=1}^{h} \int_{h_1-1}^{h_1} \cdots \int_{h_N-1}^{h_N} g(u_1, ..., u_N) e^{2\pi i(c_1 q u_1+\cdots+c_N q u_N)} du_1 \cdots du_N,$$

where we have denoted

$$g(u_1, ..., u_N) = \left\{ \begin{array}{c} \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}}, \text{if } \Pi(\bar{u}) \text{ is non-empty,} \\ 0, \quad \text{if else.} \end{array} \right\}$$

Consequently, at every fixed q we have the following relation, due to said above:

$$\left| \left(\int_\Omega e^{2\pi i(c_1 q f_1(\bar{x})+\cdots+c_N q f_N(\bar{x}))} d\bar{x} \right)^h \right|^2 \le h^N \times$$

$$\sum_{h_1=1}^{h} \cdots \sum_{h_N=1}^{h} \left| \int_{h_1-1}^{h_1} \cdots \int_{h_N-1}^{h_N} g(u_1, ..., u_N) e^{2\pi i(c_1 q u_1+\cdots+c_N q u_N)} du_1 \cdots du_N \right|^2 .$$

Therefore,

$$(\mu(q))^{2h} \le \mu_q << q^{-1-N\delta} \sum_{h_1=1}^{h} \cdots \sum_{h_N=1}^{h} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} 1 \times$$

$$\times \left| \int_{h_1-1}^{h_1} \cdots \int_{h_N-1}^{h_N} g(u_1, ..., u_N) e^{2\pi i(c_1 q u_1+\cdots+c_N q u_N)} du_1 \cdots du_N \right|^2$$

It is obvious that at the right hand side the multiple sum over $(c_1, ..., c_N)$ doesn't exceed the sum of all squares of absolute values for the Fourier coefficients of the surface integral. Then, by the Parseval's equality:

$$(\mu(q))^{2h} \leq \mu_q << q^{-1-N\delta} \times$$

$$\times \sum_{h_1=1}^{h} \cdots \sum_{h_N=1}^{h} \int_{h_1-1}^{h_1} \cdots \int_{h_N-1}^{h_N} du_1 \cdots du_N \left(\int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} \right)^2 =$$

$$= q^{-1-N\delta} \int_0^h \cdots \int_0^h du_1 \cdots du_N \left(\int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} \right)^2$$

As it was established in [5, p. 78], the multiple integral at the right side is a special integral of Terry's problem with the polynomial

$$P(x_1, x_2, ..., x_k) = \sum_{\substack{0 \leq i_1, \cdots, 0 \leq i_k \\ i_1 + i_2 + \cdots + i_k \leq n}} a_{i_1 i_2 \cdots i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}.$$

and here $x^{(j)} = x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$. This special integral can be written as

$$(2\pi)^{-N} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_0^1 \cdots \int_0^1 e^{2\pi i \left(\sum_{j=1}^{N} \alpha_j x^{(j)}\right)} dx_1 \cdots dx_k \right|^{2k} d\alpha_1 \cdots d\alpha_N.$$

As it was established in [1], the special integral converges if $h > N \max(n_1, n_2, ..., n_k)$. Then, the series

$$\sum_{q=1}^{\infty} \mu_q$$

converges. Since the $\delta > 0$ is any, then the theorem follows from the lemma 1. The theorem is proven.

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