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On calculation of elements of continued fractions for cubic algebraic numbers

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Abstract. In this article we consider the question on calculation of elements of continued fractions for irrationalities of degree 3. Finding of elements of continued fractions for real (positive) numbers meets great difficulties. It was constructed programme for calculation of several elements for some cubic irrationalities.

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1. Introduction

The notion of continued fractions plays an important role in many questions of the mathematics. They serves as one of representations of real numbers. Especially continued fractions have applications in the theory of approximation of real numbers. Other application of continued fractions connected with Diophantine equations. Some of such equations can be solved by using of expansion of suitable real numbers into continued fractions.

The theory continued fractions well developed for quadratic irrationalities. It is best known that such irrationalities have periodic expansions into continued fractions. But about expansions of other real numbers there is known relatively small knowledge. Simple cubic irrationalities such as $\sqrt[3]{2}$ has not an expansion into continued fractions with elements satisfying properties which have quadratic irrationalities. There is a conjecture stating that all of real irrationalities of degree greater than 2 have expansions into continued fractions with bounded elements.

There is well known class of transcendental numbers with elements in their expansions into continued fractions satisfying very simple conditions. All of such expansions have unbounded sequence of elements. Such a paradoxical situation makes use of computer methods of investigation of continued fractions very productive. In the work [1] calculus methods applied for verification of consequences of theory of continued fractions for algebraic irrationalities. The deep calculus methods applied there connected with reducing time and memory.

For investigations of teaching methods in the theory of continued fractions it possible to use more simple ways for expansions of real numbers into continued fractions. We use the fact that (see [1]) the inverse of a real non-zero algebraic number has an equation coefficients of which placed in opposite direction. This fact makes possible to construct very simple programme for calculations of elements of expansion into continued fractions.

In this article we consider the question on calculation of elements of continued fractions for irrationalities of degree 3. Finding of elements of continued fractions for real (positive) numbers meets great difficulties. Calculation of first several elements of a given irrationality, for example for $\sqrt[3]{2}$, demands a great evolutional work to be handled. The use of computer in this question lets us to simplify and preserve time for the calculation. In this article we present a programme for the calculation of elements of continued fractions for the real roots of cubic equations which are not integral. Since the continued fraction is defined by the absolute value of the number, then we shall consider the positive roots only.

Let us recall the method of expansion of a real number into continued fraction. Suppose we are given with a positive real number α being not an integer. If this number is a rational then this number could expanded into finite continued fraction by a unique way. Since the method of expansion doesn't depend of the form of the number, then we can suppose it to be an irrational. Let the number α have an integral part $q_0 = [\alpha]$. Then we have

$$0 < \alpha_1 = \{\alpha\} < 1.$$

So, $1 < \alpha_1 = \{\alpha\}^{-1} < +\infty$, and $q_1 = [\alpha_1^{-1}]$. Now we write

$$\alpha = q_0 + \frac{1}{q_1 + \left\{\alpha_1^{-1}\right\}}.$$

Continuing this process we get required expansion. The process itself provides us with the method of fining of elements of the expansion.

2. Auxiliary theoretical preliminaries

Let the real number α be a unique real positive root of the cubic equation

$$F(a_1, a_2, a_3, a_4) = a_1x^3 + a_2x^2 + a_3x + a_4 = 0.$$

Dividing both sides of the equation by $x3$ we realize that the number α^{-1} is a real root of the equation

$$F(a_4, a_3, a_2, a_1) = a_4x^3 + a_3x^2 + a_2x + a_1 = 0.$$

It is easy to see that the coefficients of this equations are ordered in the opposite direction. In accordance with the Taylor formula we have:

$$f(x) = a_1x^3 + a_2x^2 + a_3x + a_4 = f(q_0) + f'(q_0)(x - q_0) + \frac{1}{2}f''(q_0)(x - q_0)^2 +$$

$$+\frac{1}{6}f'''(q_0)(x - q_0)^3.$$

If we consider the polynomial

$$F(\frac{1}{3!}f'''(q_0), \frac{1}{2!}f''(q_0), f'(q_0), f(q_0))$$

with integral coefficients instead of $F(a_1, a_2, a_3, a_4)$ then this polynomial has a zero$\{\alpha\} = \alpha - q_0$ .Then the polynomial $F(f(q_0), f'(q_0), \frac{1}{2!}f''(q_0), \frac{1}{3!}f'''(q_0),)$ must have a zero $(\alpha - q_0)^{-1} > 1$. To continue finding of sequel element we must calculate an integral part of this number. We can define this number as a first natural number k for which the considered polynomial has different sings at the points k and $k+1$.

Described above method of finding of the elements for continued fractions used in construction of a programme in the language Turbo-Pascal.

```
Var i, j, k, m, l: real;
Var a1, a2, a3, a4, b1, b2, b3, b4: real;
Label; 10, 20, 30 ,40, 50, 60, 70, 80;
Begin
Writeln ('input real coefficients of the polynomial');
Writeln ('a1*x^3+a2*x^2+a3*x+a4, without integral roots');
Writeln ('a1 and a4 must be non-zero');
10: write(a1='); read (a1);
write(a2='); read (a2);
write(a3='); read (a3);
write(a4='); read (a4);
if a1*a4=0 then goto 10;
20: writeln('input the number of incomplete quotients');
read (m);
if m<1 then goto 20;
i:=0; j:=1; b1:=a1; b2:=a2; b3:=a3; b4:=a4;
if a4/(a1+a2+a3+a4)<0 then goto 30; goto 50;
30: writeln ('q(0)=0'); i:=i+1;
40: a1:=b4; a2:=b3; a3:=b2; a4:=b1;
50: if (a1*j*j*j+a2*j*j+a3*j+a4)/( a1*(j+1)*(j+1)*(j+1)+
+a2*(j+1)*(j+1)+a3*(j+1)+a4)>0
then goto 60; goto 70;
60: j:=j+1; goto 50;
70: write ('q('); write (i); write (')='); write (j); i:=i+1;
If i>m then goto 80; writeln (' '); b1:=a1; b2:=3*a1*j+a2;
b3:=3*a1*j*j+2*a2*j+a3; b4:=a1*j*j*j+a2*j*j+a3*j+a4; j:=1; goto 40
80: end.
```

3. Examples

Consider several examples.

1.Let's compute the first 16 elements of continued fraction for the number $\sqrt[3]{14}$. This number is a real root of the polynomial $x^3 - 14$.

$$q(0) = 2, q(1) = 2, q(2) = 2, q(3) = 3, q(4) = 1, q(5) = 1, q(6) = 5, q(7) = 2,$$

$$q(8) = 2, q(9) = 2, q(10) = 2, q(11) = 2, q(12) = 2, q(13) = 2, q(14) = 2, q(15) = 2,$$

2. Let's compute the first 16 elements of continued fraction for the number $\sqrt[3]{13}$. This number is a real root of the polynomial $x^3 - 13$.

$$q(0) = 2, q(1) = 2, q(2) = 1, q(3) = 5, q(4) = 1, q(5) = 1, q(6) = 4, q(7) = 3, q(8) = 2,$$

$$q(9) = 1, q(10) = 1, q(11) = 3, q(12) = 1, q(13) = 7, q(14) = 1, q(15) = 1.$$

Analogically we find following expansions.

3. $\sqrt[3]{126} = [5, 75, 5, 112, 1, 2, 1, 133, 1, 2, 2, ...]$

4. $\sqrt[3]{124} = [4, 1, 7, 3, 1, 3, 1, 111, 4, 133, 7, 37, ...]$

5. $\sqrt[3]{1001} = [10, 300, 10, 450, 8, 1, 5, 1, 599, 2, 3, 4, 1, 1, 5, 6, 3, 4, 4, 2...]$

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