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On Basicity of a Perturbed System of Cosines with Unit in Generalized Lebesgue Spaces

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Abstract. In this paper a perturbed system of cosines with a piecewise continuous phase is considered. Particular cases of these systems are eigenfunctions of second-order discontinuous differential operators. Sufficient conditions for phase jumps are found under which this system forms a basis in generalized Lebesgue spaces.

Key Words and Phrases: System of cosines, basicity, variable exponent, generalized Lebesgue space.

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1. Introduction

When solving the PDEs of mixed type by Fourier method there frequently appear systems of sines and cosines of the following form

$$\{\cos (n+\alpha)\, t\}_{n\in\mathbb{Z}_+}\,, \tag{1}$$

$$\{\sin (n+\alpha)\, t\}_{n\in\mathbb{N}}\,, \tag{2}$$

where $\alpha \in \mathbb{R}-$ is a real number(here, thereafter $\mathbb{N}-$ is the set of all natural numbers, $\mathbb{Z}_+ = \{0\}\cup\mathbb{N}$). Justification of the Fourier method requires to study the basicity properties of such systems in some function spaces. Some examples of such equations and concrete systems of trigonometric-type functions that appear after applying Fourier method can be found, for example, in [1, 2, 3, 4]. The basicity properties of the systems (1) and (2) are well studied in Lebesgue and Sobolev spaces, as well as, in their weighted settings [5, 6, 7, 8, 9, 10, 11, 12, 27, 28, 29, 30, 31].

Recently, in connection with the application in problems of mechanics and mathematical physics, interest in studying various problems in Lebesgue and Sobolev spaces with a variable exponent of summability has increased. Numerous works are devoted to this direction. Detailed information on these issues can be obtained from the monograph [13].

In this paper a perturbed system of cosines with a piecewise continuous phase is considered. Particular cases of these systems are eigenfunctions of second-order discontinuous differential operators. Sufficient conditions for phase jumps are found under which this system forms a basis in generalized Lebesgue spaces.

Notice that, similar problems for the double system of exponents with complex-valued coefficients in Lebesgue spaces with variable exponent were earlier studied in [15, 16, 17, 18, 19]. The basicity properties of the systems (1) and (2) in classical Lebesgue spaces were studied in [20, 24].

2. Preliminaries

We use the following standard denotations: $\mathbb{Z}-$the set of all integers; $\mathbb{R}-$the set of all real numbers; $\mathbb{C}-$complex plane; $(\bar{\cdot})-$complex conjugate of $(.)$; $\delta_{nk}-$Kronecker delta; $\chi_A(\cdot)-$the indicator function of the set$A$. $\omega \equiv \{z \in \mathbb{C} : |z| < 1\}-$the unit disc; $\partial\omega \equiv \{z \in \mathbb{C} : |z| = 1\}-$the unit circle.

Let $p : [-\pi, \pi] \to [1, +\infty)-$be a Lebesgue measurable function. We denote by $\mathcal{L}_0$ the set of all Lebesgue measurable functions on $[-\pi, \pi]$. Set

$$I_p(f) \overset{def}{\equiv} \int_{-\pi}^{\pi} |f(t)|^{p(t)}\, dt.$$

Let

$$\mathcal{L} \equiv \{f \in \mathcal{L}_0 : I_p(f) < +\infty\}.$$

If $p^+ = \sup\limits_{[-\pi,\pi]} vrai\, p(t) < +\infty$, then $\mathcal{L}$ is a linear space with respect to pointwise linear operations. $\mathcal{L}$ is a Banach space with respect to the norm

$$\|f\|_{p(\cdot)} \overset{def}{\equiv} \inf\left\{\lambda > 0 : I_p\left(\frac{f}{\lambda}\right) \le 1\right\},$$

and we denote it by $L_{p(\cdot)}$. Assume

$$WL \overset{def}{\equiv} \{p : p(-\pi) = p(\pi); \exists C > 0, \quad \forall t_1, t_2 \in [-\pi, \pi] : |t_1 - t_2| \le \tfrac{1}{2} \Rightarrow \\ \Rightarrow |p(t_1) - p(t_2)| \le \tfrac{C}{-\ln|t_1 - t_2|}\Big\}.$$

Throughout the paper $q(\cdot)$ denotes the conjugate function of $p(\cdot)$, that is, $\frac{1}{p(t)} + \frac{1}{q(t)} \equiv 1$. Let $p^- = \inf\limits_{[-\pi,\pi]} vrai\, p(t)$. The following generalized Holder's inequality holds

$$\int_{-\pi}^{\pi} |f(t)\, g(t)|\, dt \le c\left(p^-; p^+\right) \|f\|_{p(\cdot)} \|g\|_{q(\cdot)},$$

where $c(p^-; p^+) = 1 + \frac{1}{p^-} - \frac{1}{p^+}$.

To obtain the main results, we need the following results from [32] on the basicity of the following unitary system of exponents in generalized Lebesgue spaces $L_{p(\cdot)}(0, \pi)$:

$$v_n(t) \equiv a(t)e^{int} - b(t)e^{-int}, \;\; n \in \mathbb{N},$$

where $a(t) = |a(t)|\, e^{i\alpha(t)}$, $b(t) = |b(t)|\, e^{i\beta(t)}$–are complex-valued functions on $[0,\pi]$. Suppose that the coefficients $a(\cdot)$ and $b(\cdot)$ satisfy the following conditions i)-iv):

i) $a^{\pm 1}(\cdot)\,;\, b^{\pm 1}(\cdot) \in L_\infty(0,\pi)$;

ii) $\alpha(\cdot)\,;\,\beta(\cdot)$ –are piecewise continuous functions on $(0,\pi)$ with jump points $\{t_k\}_{k\in N}$ and $\{\tau_k\}_{k\in N}$, respectively; additionally it is assumed that the set $\{\tilde{s}_k\} \equiv \{t_k\} \bigcup \{\tau_k\}$ may have $\tilde{s}_0 \in (0,\pi)$ as its only limit point and the function $\tilde{\theta}(t) \equiv \beta(t) - \alpha(t)$ has a finite right and left limits at $\tilde{s}_0$.

iii) $\sum_{k=1}^{\infty} |h(\tilde{s}_k)| < +\infty$, where $h(\tilde{s}_k) = \tilde{\theta}(\tilde{s}_k - 0) - \tilde{\theta}(\tilde{s}_k + 0)$ –is the jump of $\tilde{\theta}(\cdot)$ at $\tilde{s}_k$.

iv) The jumps $\left\{\tilde{h}_i\right\}$ hold $\left(\frac{\tilde{h}(\tilde{s}_i)}{2\pi} + \frac{1}{p(\tilde{s}_i)}\right) \notin \mathbb{Z}, \forall i \in \mathbb{N}$.

From condition iii) on jumps $h(\tilde{s}_k)$,$k \in \mathbb{N}$, it follows that there exists a number $r \in \mathbb{N}$ such that the inequalities

$$-\frac{2\pi}{p(\tilde{s}_k)} < \tilde{h}(\tilde{s}_k) < \frac{2\pi}{q(\tilde{s}_k)}, k = \overline{r,\infty},$$

hold. Let us enumerate the elements of the set $\{\tilde{s}_i\}_1^r$, in ascending order and denote it by $\{s_i\}_1^r$, i.e. $0 < s_1 < ... < s_r < \pi$. Denote the corresponding jumps by $\{h(s_i)\}_1^r$, i.e. let Suppose that for some n_0 the inequality

$$\frac{1}{p(0)} + 2(n_0 - 1) < \frac{\beta(0) - \alpha(0)}{\pi} < \frac{1}{p(0)} + 2n_0, \tag{3}$$

holds. Based on condition iv) we define integers $n_i, i = \overline{1,r}$, from the following relations

$$-\frac{1}{p(s_i)} < \frac{h(s_i)}{2\pi} + n_i - n_{i-1} < \frac{1}{q(s_i)}, i = \overline{1,r}. \tag{4}$$

So, the following main is true.

Theorem 2.1. *Let the coefficients $a(\cdot)$ and $b(\cdot)$ of system $\{v_n\}_{n\in\mathbb{N}}$ satisfy the conditions i)-iv), integers $\{n_i\}_1^r$ are defined from the relations* (3), (4). *Let*

$$\frac{\beta(\pi) - \alpha(\pi)}{2\pi} + \frac{1}{2p(\pi)} \notin \mathbb{Z}. \tag{5}$$

Then, if the inequalities

$$-\frac{1}{p(\pi)} + 2n_r < \frac{\beta(\pi) - \alpha(\pi)}{\pi} < -\frac{1}{p(\pi)} + 2(n_r + 1), \tag{6}$$

are valid, then the system $\{v_n\}_{n\in N}$ forms a basis for $L_{p(\cdot)}(0,\pi)$. Moreover, if

$$\beta(\pi) - \alpha(\pi) < -\frac{\pi}{p(\pi)} + 2n_r\pi,$$

then the system $\{v_n\}_{n\in\mathbb{N}}$ *is not complete in* $L_{p(\cdot)}(0,\pi)$*, but is minimal in it; when the following inequality holds*

$$\beta(\pi)-\alpha(\pi)>-\frac{\pi}{p(\pi)}+2(n_r+1)\pi,$$

then it is complete, but not minimal in $L_{p(\cdot)}(0,\pi)$ *.*

We also need the following result from work [33].

Consider the following system of exponents

$$\varphi_n(\theta)\equiv\exp\left[i\left(n\theta-sgnn\alpha(\theta)\right)\right],\quad n=\pm1,\pm2,...,\tag{7}$$

where $\alpha(\theta)$ is a piecewise continuous, odd function on a segment $[-\pi,\pi]$, i.e. $\alpha(-\theta)=-\alpha(\theta)$. Let $\{t_k\}_1^\infty$ be the set of first kind discontinuity points of the function $\alpha(\theta)$ on $(0,\pi)$, which has a unique limit point $t_0\in(0,\pi)$. Suppose that a function $\alpha(\theta)$ has finite right and left-side limits at a point t_0. Moreover, let the following inequalities

$$\sum_{k=1}^{\infty}\left|\alpha(t_k+0)-\alpha(t_k-0)\right|<+\infty,\tag{8}$$

hold. Assume that for any integer k the following relation

$$\frac{\alpha(t_i-0)-\alpha(t_i+0)}{\pi}\neq-\frac{1}{p(t_i)}+k,\qquad i=\overline{1,\infty},\tag{9}$$

is satisfied.

Let for some integer n_0 the inequality

$$\frac{\pi}{2p(0)}+\left(n_0-\frac{1}{2}\right)\pi<\alpha(0)<\frac{\pi}{2p(0)}+n_0\pi\tag{10}$$

be true. Denote by r the index after which the following conditions are satisfied

$$-\frac{\pi}{p(t_k)}<\alpha(t_k-0)-\alpha(t_k+0)<\frac{\pi}{q(t_k)},\tag{11}$$

$k=\overline{r,\infty}$. We enumerate the elements of the set $\{t_i\}$, $i=\overline{1,r}$ in ascending order and again denote by $\{t_i\}_1^r$, $0<t_1<...<t_r<\pi$. Define integers n_i, $i=\overline{1,r}$ from the following conditions

$$-\frac{1}{p(t_i)}<\frac{\alpha(t_i-0)-\alpha(t_i+0)}{\pi}+n_i-n_{i-1}<\frac{1}{q(t_i)},\quad i=\overline{1,r}.\tag{12}$$

The following theorem is true.

Theorem 2.2. *Let $\alpha(t)$ be real-valued, piecewise continuous, odd function on a segment $[-\pi,\pi]$, concerning discontinuities of which the above stated conditions* (8)-(10) *are true. Integers n_i, $i=\overline{1,r}$, are determined from conditions* (10) *and* (12). *Suppose that*

$$\alpha(\pi)\neq-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi.$$

Then, in order for the system of exponents (7) *to be a basis in space $L_{p(\cdot)}(-\pi,\pi)$, it is sufficient that the inequality*

$$-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+(n_r+1)\pi \tag{13}$$

is valid. Moreover, if $\alpha(\pi)<-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi$, then the system (7) *is not complete in$L_{p(\cdot)}(-\pi,\pi)$, but is minimal; for $\alpha(\pi)\geq-\frac{\pi}{2p(\pi)}+(n_r+1)\pi$, it is complete in $L_{p(\cdot)}(-\pi,\pi)$, but not minimal.*

Before proving this theorem, we state some corollaries that follow directly from Theorem 2.1.

Let $\alpha(t)$ be a piecewise continuous function on $[-\pi,\pi]$ with respect to discontinuities of which the above formulated conditions are valid.

Corollary 2.3. Let for some integer n_0

$$\frac{\pi}{2p(0)}+(n_0-1)\pi<\alpha(0)<\frac{\pi}{2p(0)}+n_0\pi. \tag{14}$$

Numbers n_r are defined from the conditions (14), (12) and in addition $\alpha(\pi)\neq-\frac{\pi}{2p(\pi)}+n_r\pi$. Then the system of sines $\sin(nt-\alpha(t))$, $n=\overline{1,\infty}$, forms a basis for $L_{p(\cdot)}(0,\pi)$, if and only if the inequalities

$$-\frac{\pi}{2p(\pi)}+n_r\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+(n_r+1)\pi,$$

are true; if $\alpha(\pi)<-\frac{\pi}{2p(\pi)}+n_r\pi$, then the system is not complete in $L_{p(\cdot)}(0,\pi)$, but is minimal; for $\alpha(\pi)\geq-\frac{\pi}{2p(\pi)}+(n_r+1)\pi$ is complete, but is not minimal.

Regarding the cosine system, we have the following

Corollary 2.4. Let

$$\frac{\pi}{2p(0)}+\left(n-\frac{1}{2}\right)\pi<\alpha(0)<\frac{\pi}{2p(0)}+\left(n+\frac{1}{2}\right)\pi. \tag{15}$$

Numbers n_r are defined from the conditions (15), (12) and in this case $\alpha(\pi)\neq-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi$. Then the system of cosines $\cos(nt-\alpha(t))$, $n=\overline{1,\infty}$, forms a basis for $L_{p(\cdot)}(0,\pi)$ if and only if the inequalities

$$-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{3}{2}\right)\pi;$$

are true, if $\alpha(\pi) < -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi$, then the system is not complete in $L_{p(\cdot)}(0,\pi)$, but is minimal; for $\alpha(\pi) > -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{3}{2}\right)\pi$, it is complete, but is not minimal in $L_{p(\cdot)}(0,\pi)$.

3. Basicity of the system of cosines

Consider the following system of cosines

$$\{1, \cos(nt - \alpha(t))\}_{n=1}^{\infty}, \tag{16}$$

where $\alpha(t)$ is real piecewise continuous function on a segment $[0,\pi]$. Let $\{t_k\}_1^{\infty}$ be the set of first kind discontinuity points on $(0,\pi)$, which has a unique limit point $t_0 \in (0,\pi)$. Suppose that a function $\alpha(t)$ has finite right and left-side limits at a point t_0. Moreover, it holds

$$\sum_{n=1}^{\infty} |\alpha(t_k + 0) - \alpha(t_k - 0)| < +\infty. \tag{17}$$

Let an index r and numbers $n_i,\ i = \overline{1,r}$, be defined from the conditions (11) and (12), where n_0 some integer for which

$$\frac{\pi}{2p(0)} + \left(n_0 - \frac{1}{2}\right)\pi < \alpha(0) < \frac{\pi}{2p(0)} + \left(n_0 + \frac{1}{2}\right)\pi.$$

The following theorem is valid.

Theorem 3.1. *Let the above formulated conditions be valid with respect to a function* $\alpha(\theta)$ *and numbers* $n_i,\ i = \overline{1,r}$ *determined from conditions* (11) *and* (12)*. Let* $\alpha(\pi) \neq -\frac{\pi}{2p(\pi)} + \left(n_r - \frac{1}{2}\right)\pi$*. Then the system of cosines* (16) *forms a basis for* $L_{p(\cdot)}(0,\pi)$ *if and only if*

$$-\frac{\pi}{2p(\pi)} + \left(n_r - \frac{1}{2}\right)\pi < \alpha(\pi) < -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi;$$

if $\alpha(\pi) < -\frac{\pi}{2p(\pi)} + \left(n_r - \frac{1}{2}\right)\pi$*, then the system* (16) *is not complete in* $L_{p(\cdot)}(0,\pi)$*, but it is minimal; for* $\alpha(\pi) > -\frac{\pi}{2p(\pi)} + \left(n_r + \frac{1}{2}\right)\pi$*, it is complete, but not minimal in* $L_{p(\cdot)}(0,\pi)$*.*

Proof. Denote by $c_n,\ n = \overline{0,\infty}$, the following numbers

$$\int_0^{\pi} \bar{h}_n^c(\theta)\, d\theta = c_n, \quad n = 0, 1, ...$$

where $\bar{h}_n^c(\theta),\ n = \overline{0,\infty}$, is biorthogonally conjugated system to system $\cos(nt - \alpha(t)),\ n = \overline{0,\infty}$. Let us show that $c_0 \neq 0$. An analytical expression for this system was derived in [32], where $a(t) = e^{i\tilde{\alpha}(t)}$, $b(t) = e^{i\tilde{\beta}(t)}$ and

$$\tilde{\alpha}(t) = \frac{\pi}{2} - t - \alpha(t), \quad \tilde{\beta}(t) = -\tilde{\alpha}(t).$$

Thus, following the work [32], we have

$$\bar{h}_n^c(\sigma) = -\frac{i}{e^{I(\sigma)}} \sum_{k=1}^{n+1} b_{n+1-k} \sin k\sigma, \qquad n = \overline{0,\infty},$$

where

$$I(\sigma) = -\frac{i}{4\pi} \int_0^\pi \frac{\cos\frac{\theta}{2}}{\cos\frac{\sigma}{2}} \ln \frac{b(\theta)}{a(\theta)} \left[\frac{1}{\sin\frac{\theta-\sigma}{2}} + \frac{1}{\sin\frac{\theta+\sigma}{2}} \right] d\theta =$$

$$= \frac{1}{2\pi} \int_0^\pi \tilde{\beta}(\theta) \frac{\cos\frac{\theta}{2}}{\cos\frac{\sigma}{2}} \left[\frac{1}{\sin\frac{\theta-\sigma}{2}} + \frac{1}{\sin\frac{\theta+\sigma}{2}} \right] d\theta.$$

Hence it follows that $I(\sigma)$ is real-valued function depending on σ on segment $[0,\pi]$. Moreover

$$\bar{h}_0^c(\sigma) = -ib_0 e^{-I(\sigma)} \sin\sigma.$$

Since $\sin\sigma > 0$ on $(0,\pi)$, it follows that

$$A_0 = \int_0^\pi \bar{h}_0^c(\sigma)\, d\sigma \neq 0.$$

Denote

$$b_n = \frac{c_n}{c_0}, \quad n = \overline{1,\infty}.$$

Define the system $H_n^A(\theta), \quad n = \overline{0,\infty}$ as follows

$$\left. \begin{array}{l} H_0^A(\theta) = \frac{1}{c_0} \bar{h}_0^c(\theta), \\ H_n^A(\theta) = \bar{h}_n^c(\theta) - b_n \bar{h}_0^c(\theta), \quad n = \overline{1,\infty} \end{array} \right\} \tag{18}$$

Let us show that this system is biorthogonal to the system (16). Indeed, we have

$$\left(H_n^A, \cos(m\theta - \alpha(\theta))\right) = \left(\bar{h}_0^c(\theta), \cos(m\theta - \alpha(\theta))\right) -$$

$$-b_n \left(\bar{h}_0^c(\theta), \cos(m\theta - \alpha(\theta))\right) = \delta_{nm}, \quad n, m = \overline{1,\infty},$$

and

$$\left(\bar{h}_n^c - b_n \bar{h}_0^c, 1\right) = c_n - b_n c_0 = 0.$$

Thus, the system (16) and (18) are biorthogonal.

Let$\alpha(\pi) < -\frac{\pi}{2p(\pi)} + \left(n_r - \frac{1}{2}\right)\pi$. Under this condition the system $\cos(nt - \alpha(t))$, $n = \overline{0,\infty}$, is not complete in $L_{p(\cdot)}(0,\pi)$. So, there exists a function $\psi(\theta) \in L_{q(\cdot)}(0,\pi)$, $\frac{1}{p(\cdot)} + \frac{1}{q(\cdot)} = 1$ which is not identically equal to zero, for which

$$\left(\bar{h}_n^c, \psi\right) = 0, \qquad n = \overline{0,\infty}.$$

Hence, it follows that$(H_n^c,\psi)=0, \qquad n=\overline{0,\infty}$, i.e. the system (16) is not complete. For $\alpha(\pi)>-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi$, the minimality of the system (16) is obvious. So, let

$$-\frac{\pi}{2p(\pi)}+\left(n_r-\frac{1}{2}\right)\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi.$$

In this case the system $\cos(nt-\alpha(t))$, $n=\overline{0,\infty}$, forms a basis for $L_{p(\cdot)}(0,\pi)$. Take any function $\psi(t)\in L_{p(\cdot)}(0,\pi)$ and make the following partial sum

$$S_N=\sum_{n=0}^{N}(H_n^c,\psi)\ \varphi_n(t),$$

where

$$\varphi_n(t)=\cos(nt-\alpha(t)),\ \varphi_0(t)\equiv 1, \quad n=\overline{1,\infty}.$$

Let us show $S_N\to\psi, \quad$ as $N\to\infty \quad$ in $L_{p(\cdot)}(0,\pi)$. We have

$$\|S_N-\psi\|_{L_{p(\cdot)}}=\left\|\frac{1}{c_0}\left(\bar h_0^c,\psi\right)+\sum_{n=1}^{N}\left(\bar h_n^c-b_n\bar h_0^c,\psi\right)\ \varphi_n(t)-\psi\right\|_{L_{p(\cdot)}}=$$

$$=\left\|\frac{1}{c_0}\left(\bar h_0^c,\psi\right)+\sum_{n=1}^{N}\left(\bar h_n^c,\psi\right)\ \varphi_n-\left(\bar h_0^c,\psi\right)\sum_{n=1}^{N}b_n\varphi_n-\psi\right\|_{L_{p(\cdot)}}=$$

$$=\left\|\frac{1}{c_0}\left(\bar h_0^c,\psi\right)+\sum_{n=0}^{N}\left(\bar h_n^c,\psi\right)\ \varphi_n-\left(\bar h_0^c,\psi\right)\left[\sum_{n=1}^{N}b_n\varphi_n+\cos(\alpha(t))\right]-\psi\right\|_{L_{p(\cdot)}}\le$$

$$=\left\|\sum_{n=0}^{N}\left(\bar h_n^c,\psi\right)\ \varphi_n-\psi\right\|_{L_{p(\cdot)}}+\left\|\frac{1}{c_0}\left(\bar h_0^c,\psi\right)-\left(\bar h_0^c,\psi\right)\left[\sum_{n=1}^{N}b_n\varphi_n+\cos(\alpha(t))\right]\right\|_{L_{p(\cdot)}}.$$

Here the first term tends to zero as $N\to\infty$. The second term can be rewritten as

$$\left|\frac{1}{c_0}\left(\bar h_0^c,\psi\right)\right|\left\|c_0\left[\sum_{n=1}^{N}b_n\varphi_n+\cos(\alpha(t))\right]-1\right\|_{L_{p(\cdot)}}=$$

$$=\left|\frac{1}{c_0}\left(\bar h_0^c,\psi\right)\right|\left\|\sum_{n=0}^{N}\left(\bar h_n^c,1\right)-1\right\|_{L_{p(\cdot)}}\to 0, \quad N\to\infty.$$

So, it is proved that

$$\|S_N-\psi\|_{L_{p(\cdot)}}\to 0, \qquad N\to\infty.$$

The theorem is proved. □

Considering Corollary 2.3 and Theorem 3.1, the following theorem can be easily proved.

Theorem 3.2. *Let all the conditions of Theorem 2.2 hold with respect to a function* $\alpha(\theta)$*. Integers* $n_i, \ \ i=\overline{1,r}$ *are defined from the conditions* (11), (12) *in which* n_0 *is some number, that*

$$\frac{\pi}{2p(0)}+\left(n_0-\frac{1}{2}\right)\pi<\alpha(0)<\frac{\pi}{2p(0)}+n_0\pi.$$

Suppose also that $\alpha(\pi)\neq-\frac{\pi}{2p(\pi)}+n_r\pi$*. Then the system*

$$\exp\left[i\left(n\theta-sgnn\alpha(\theta)\right)\right] \ \ , \ \ n=0,\pm1,... \tag{19}$$

forms a basis for $L_{p(\cdot)}(-\pi,\pi)$ *if and only if*

$$-\frac{\pi}{2p(\pi)}+n_r\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi;$$

if $\alpha(\pi)<-\frac{\pi}{2p(\pi)}+n_r\pi$*, then this system is not complete in* $L_{p(\cdot)}(-\pi,\pi)$*, but is minimal; for* $\alpha(\pi)>-\frac{\pi}{2p(\pi)}+\left(n_r+\frac{1}{2}\right)\pi$*, it is complete, but is not minimal in* $L_{p(\cdot)}(-\pi,\pi)$*.*

In conclusion, we provide one more theorem, which is equivalent in a sense to Theorem 3.1.

Theorem 3.3. *Let the function* $\alpha(\theta)$ *satisfy all the conditions of Theorem 3.1. Integers* $n_i, \ i=\overline{1,r}$ *are determined from conditions* (11)*, and* (12)*. Moreover,* $\alpha(\pi)\neq-\frac{\pi}{2p(\pi)}+\left(n_r-\frac{1}{2}\right)\pi$*. The system of functions*

$$\left\{\cos\left(n\theta-\alpha(\theta)\right);\sin\left(n\theta-\alpha(\theta)\right)\right\}_{n=0}^{\infty}, \tag{20}$$

forms a basis for $L_{p_{(\cdot)}}(-\pi,\pi)$*, if and only if*

$$-\frac{\pi}{2p(\pi)}+\left(n_r-\frac{1}{2}\right)\pi<\alpha(\pi)<-\frac{\pi}{2p(\pi)}+n_r\pi.$$

is hold. And, if $\alpha(\pi)<-\frac{\pi}{2p(\pi)}+\left(n_r-\frac{1}{2}\right)\pi$*, then the system* (20) *is not complete in* $L_{p(\cdot)}(-\pi,\pi)$*, but is minimal; for* $\alpha(\pi)>-\frac{\pi}{2p(\pi)}+n_r\pi$*, it is complete, but is not minimal in* $L_{p(\cdot)}(-\pi,\pi)$*.*

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