The solution of the Bessel-Helmholtz equation

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Abstract. In this paper we construct the Green function for problem (1)–(2) and prove that the limit absorption principle holds for it.

Key Words and Phrases: Singular elliptic equations; Bessel's differential operator; Bessel-Helmholtz equation; Green's function.
2010 Mathematics Subject Classifications: Primary 35J05, 35J08, 35J75

1. Introduction

In the last ten years there is a great interest to degenerating and singular equations including the equations containing Bessel's differential operator

$$B_i = \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i}, \quad \gamma_i > 0, \ i = 1, 2.$$

Note that in these directions there exist investigations of I.A. Kiprianov [13], [14], I.A. Kiprianov, V.I. Kononenko [15], I.A. Kiprianov and M.I. Klyuchanchev [16], F.G. Mukhlisov [18], F.G. Mukhlisov, A.Sh. Khismatullin [19], A.Sh. Khusmatullin [20].

The mentioned equations are often encountered in applications, for example in axial symmetry problems of continuum mechanics. One of the intensely developing directions is investigation of elliptic equations with Bessel operator that in 1967 were called by I.A. Kiprianov B_n-elliptic. The first papers on B-elliptic equations relates to the equations of the form

$$\Delta_B u(x) = \sum_{i=1}^{n-2} \frac{\partial^2 u(x)}{\partial x_i^2} + B_{n-1}u(x) + B_n u(x) = 0.$$

I.A. Kiprianov and his colleages extensively studied the operator Δ_{B_n} and some its generalizations.

I.A. Kiprianov showed that the volume potential

$$u(x) = \int\limits_{R_+^n} |y|^{2-n-\gamma} T^y f(x)(y')^\gamma dy$$

is the solution of the B-elliptic equation

$$\Delta_B u(x) = f(x).$$

The generalized shift operator T^y is defined by

$$T^y f(x) = C_{\gamma,n} \int\limits_0^\pi \int\limits_0^\pi f\left(x' - y', (x'', y'')_\beta\right)\, d\nu(\beta)$$

where $d\nu(\beta) = \prod\limits_{i=n-1}^{n} \sin^{\gamma_i - 1} \beta_i \; d\beta_{n-1}d\beta_n$, $x' = (x_1, \ldots, x_{n-2}) \in R^{n-2}$, $x'' = (x_{n-1}, x_n) \in R^2$, $(x_i, y_i)_{\beta_i} = (x_i^2 - 2x_i y_i \cos \beta_i + y_i^2)^{\frac{1}{2}}$, $(x'', y'')_\beta = \left((x_{n-1}, y_{n-1})_{\beta_{n-1}}, (x_n, y_n)_{\beta_n}\right)$ and

$$C_{\gamma,n} = \pi^{-1} \, \Gamma^{-1}\left(\frac{|\gamma|}{2}\right) \prod_{i=n-1}^{n} \Gamma\left(\frac{\gamma_i + 1}{2}\right) = \frac{2|\gamma|}{\pi}\left(\frac{|\gamma|}{2} + 1\right)\omega(2, n, \gamma).$$

Study of wave propagation in infinite domains has a great value in physics and mechanics. Long-range wave propagation in atmosphere, sound propagation in sea, waves in pipes belong to such ones. These phenomena reduce to boundary value problems in cylindrical domains for the Helmholts equation. There is a wide references devoted to different boundary value problems for partial equations in laminated media.

For distinguishing physically interesting solutions of boundary value problems for elliptic equations in infinite domains with a spectral parameter when this parameter is a point of the continuous spectrum of the problem, three methods are used, more exactly the limit absorption principle, limit amplitude principle and Zommerfeld's radiation condition. For brevity, these principles and the Zommerfeld condition are called radiation principles.

2. Main results

Denote by $R^m(y)$, m–dimensional Euclidean space with the point $y = (y_1, \ldots, y_m)$, by $R^n(x)$ the same space with the point $x = (x_1, \ldots, x_n)$, $R_+^n(x) = \{x \in R^n(x); x = (x', x_{n-1}, x_n), x_{n-1} > 0, x_n > 0\}$, $\gamma_{n-1} > 0$, $\gamma_n > 0$.

Let $\Lambda = R_+^n(x) \times \Omega$ be a cylindrical domain in $R_+^n(x) \times R^m(y)$, where Ω is a bounded domain in $R^m(y)$ with a smooth boundary $\partial\Omega$, where $\partial\Omega \in C^{\left[\frac{3m}{2}\right]}$.

Consider in Λ the following boundary value problem

$$(\Delta_B + k^2)u(k, x, y) = f(x, y), \tag{1}$$

$$u(k, x, y)|_\Gamma = 0, \tag{2}$$

where $f(x, y) \in C_0^\infty(\Lambda)$, k^2 is a constant number not necessarily real, Γ is the boundary of the cylinder Λ.

Note that in these directions there exist many investigations of A.N. Tikhonov, A.A. Samarskii [1], L.A. Murave [2], M.V. Federyuk [3], B.A. Iskenderov, Z.G. Abbasov, E.Kh. Eyvazov [10], B.A. Iskenderov, A.I. Mekhtieva [11].

Denote by $C_0^\infty(\Lambda)$ the space of infinitely differentiable finite functions whose support is contained in Λ, by $C^l(D)$ the space of functions continuous together with the derivatives to order l, inclusively in the domain $D \subset R_+^n \times R^m$.

Definition 2.1. *Let $Jmk^2 \neq 0$, the function $u(k, x, y) \in C^2(\Lambda) \cap C(\overline{\Lambda})$, convergent to zero as $x \to \infty$, be called the solution of problem (1)–(2) if it satisfies equation (1) in the ordinary sense and vanishes on the boundary Γ of the cylinder Λ.*

In the case when k^2 is a point of the continuous spectrum of problem (1)–(2) there exist three methods to distinguish physically interesting solutions of the problem. This is limit absorption principle, limit amplidude or at infinity radiation conditions are imposed on the solution of problem (1)–(2). We will justify these principles and study the conditions for problem (1)–(2).

We determine the Fourier-Bessel transform by the equalities

$$F_{B_n}[f(x)](s) = \int\limits_{R_+^n} f(x) \cdot j_{\frac{\gamma_{n-1}-1}{2}}(x_{n-1} \cdot s_{n-1}) j_{\frac{\gamma_n-1}{2}}(x_n \cdot s_n) e^{-i(x', s')} x_{n-1}^{\gamma_{n-1}} x_n^{\gamma_n} dx,$$

where $(x', s') = \sum\limits_{m=1}^{n-2} x_m \cdot s_m$.

Using the property of generalized shift, it is easy to show

$$F_B(f * g) = F_B(f) \cdot F_B(g).$$

Alongside with problem (1)–(2) we consider the problem

$$(\Delta_B + k^2) G(k, x, y) = \delta(x, y), \tag{3}$$

$$G(k, x, y)|_\Gamma = 0, \tag{4}$$

where $x, y \in \Lambda$, $\delta(x)$ is Dirac's delta-function.

Definition 2.2. *Let $Jmk^2 \neq 0$. Decreasing solutions of problem (3)–(4) as $x \to \infty$ (for every $y \in \Omega$) will be called the Green function of problem (1)–(2).*

Theorem 2.3. *[9]. Let*

$$F(t) = \int\limits_0^a \varphi(\tau) e^{-t\tau} d\tau, \quad a > 0,$$

where $\varphi(\tau)$ is an analytic function regular at the points of the interval $0 < \tau \leq a$, while in the vicinity of the point $\tau = 0$ is representable by the series

$$\varphi(\tau) = \tau^\alpha(a_0 + a_1\tau + \cdots), \quad \alpha > -1.$$

66

Then as $t \to +\infty$

$$F(t) = \sum_{p=1}^{\infty} \frac{\Gamma(\alpha+1)}{t^{\alpha+p}} a_{p-1},$$

where $\Gamma(\alpha)$ is Euler's gamma-function.

It holds the following asymptotic estimation of Hankel functions for large values of the argument and in what follows we will use it ([8])

$$H_{\mu}^{(1,2)}(z) = \sqrt{\frac{2}{\pi z}} e^{\pm i\left(z - \frac{\pi\mu}{2} - \frac{\pi}{4}\right)}(1 + O(z^{-1})). \tag{5}$$

As it was above noted, for $k^2 > 0$ for distinguishing physically interesting solutions of problem (1)–(2) the limit absorption principle is used.

Definition 2.4. *We say that for problem (1)–(2) the limit absorption principle holds if for the solution of this problem converging to zero at infinity, with the complex parameter $k_\varepsilon^2 = k^2 + i\varepsilon$ ($\varepsilon \neq 0$) there exists the limit*

$$u(k, x, y) = \lim_{\varepsilon \to 0} u(k_\varepsilon, x, y)$$

uniformly with respect to x, y at every compact from Λ and $u(k, x, y)$ satisfies the limit problem.

For constructing the solution of problem (1)–(2) with the complex parameter k_ε^2 we perform Fourier-Bessel transform with respect to x and get the following boundary value problem

$$[\Delta_y + (k_\varepsilon^2 - |s|^2)]\widehat{u}(k_\varepsilon, x, y) = \widehat{f}(s, y),$$

$$\widehat{u}(k, x, y)|_{\partial\Omega} = 0,$$

where $\widehat{u}(k_\varepsilon, x, y) = F_{B_n}[u(k_\varepsilon, x, y)](s)$ is the Fourier-Bessel transform with respect to x and Δ_y is the Laplace operator with respect to the variable y.

Consider the following operator $L = -\Delta_y$ with the domain of definition

$$D(L) = \left\{ v : v \in C^2(\Omega) \cap C(\overline{\Omega}), \ \Delta v \in L_2(\Omega), \ v|_{\partial\Omega} = 0 \right\}$$

For the operator L it holds the following theorem.

Theorem 2.5. *[5]. The set of eigen values $\{\lambda_\ell\}$ of the operator L is denumerable and has no finite limit points, each eigen value λ_ℓ has a finite multiplicity and the least eigen value is prime. The eigen functions $\{\varphi_\ell(y)\}$ may be chosen as real and orthonormed. Any function from $L_2(\Omega)$ expands in regularly convergent Fourier series with respect to $\{\varphi_\ell(y)\}$.*

As it follows from this theorem for eigen values of the operator L we have the chain of the inequalities:

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_\ell \leq \ldots, \quad as\, \ell \to \infty.$$

As $k_\varepsilon^2 - |s|^2 \neq \lambda_\ell$ for any $s \in R^n$ and $\widehat{u}(k_\varepsilon, x, y) \in D(L)$, then using theorem 2.3 for solving problem (1)–(2), we get

$$\widehat{u}(k_\varepsilon, x, y) = R_{k_\varepsilon^2 - |s|^2}\widehat{f}(s, y) = \sum_{l=1}^{\infty} \frac{C_\ell(s)\varphi_\ell(y)}{\lambda_\ell - k_\varepsilon^2 + |s|^2}, \tag{6}$$

where R_μ is a resolvent of the operator L,

$$C_\ell(s) = \int_\Omega \widehat{f}(s, y)\varphi_\ell(y)dy. \tag{7}$$

Note that λ_ℓ and $\varphi_\ell(y)$ are independent of s.

Theorem 2.6. *The Green's function of problem* (1)–(2) *is an analytic function of k, except the denumerable number of points $k = \pm\lambda_\ell^{\frac{1}{2}}$, $\ell = 1, 2, \ldots$, being the branching points and for it expansion* (19) *doesn't hold, where λ_ℓ are eigen values of the operator L, $\varphi_\ell(y)$ are appropriate eigen functions.*

Now, the solution of problem (1)–(2) is determined as Fourier-Bessel's inverse transform of $\widehat{u}(k, x, y)$

$$u(k_\varepsilon, x, y) = \int_{R_+^n} \widehat{u}(k_\varepsilon, s, y)j_{\frac{\gamma_{n-1}-1}{2}}(x_{n-1} \cdot s_{n-1})j_{\frac{\gamma_n-1}{2}}(x_n \cdot s_n)e^{-i(x', s')}s_{n-1}^{\gamma_{n-1}}s_n^{\gamma_n}ds$$

$$= \sum_{l=1}^{\infty} \varphi_\ell(y) \int_{R_+^n} \frac{C_\ell(s)j_{\frac{\gamma_{n-1}-1}{2}}(x_{n-1} \cdot s_{n-1})j_{\frac{\gamma_n-1}{2}}(x_n \cdot s_n)e^{-i(x', s')}}{\lambda_\ell - k_\varepsilon^2 + |s|^2}s_{n-1}^{\gamma_{n-1}}s_n^{\gamma_n}ds. \tag{8}$$

Here, the term by term integration is valid by uniform convergence of series (6) and its derivatives ([17]). Taking (7) into account in (8), we get

$$u(k_\varepsilon, x, y) = \sum_{l=1}^{\infty} \varphi_\ell(y)$$

$$\times \int_{R_+^n} \left[f_\ell(\xi) \int_{R_+^n} \frac{e^{i(\xi', s')} \prod_{l=n-1}^{n} j_{\frac{\gamma_l-1}{2}}(x_l s_l)j_{\frac{\gamma_l-1}{2}}(\xi_l s_l)}{\lambda_\ell - k_\varepsilon^2 + |s|^2}s_{n-1}^{\gamma_{n-1}}s_n^{\gamma_n}ds \right] \xi_{n-1}^{\gamma_{n-1}}\xi_n^{\gamma_n}d\xi, \tag{9}$$

where

$$f_\ell(\xi) = \int_\Omega f(\xi, y)\varphi_\ell(y)dy. \tag{10}$$

Denote

$$J_\ell(k_\varepsilon, x, \xi) =$$

68

$$= \lim_{N\to\infty} \int\limits_{|s|\le N} \frac{e^{i(\xi'-x',s')} \prod\limits_{l=n-1}^{n} j_{\frac{\gamma_l-1}{2}}(x_l s_l) j_{\frac{\gamma_l-1}{2}}(\xi_l s_l)}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_{n-1}^{\gamma_{n-1}} s_n^{\gamma_n} ds. \tag{11}$$

Passing in (11) to spherical coordinates, and taking into account spherical symmetry $(\lambda_\ell - k_\varepsilon^2 + |s|^2)^{-1}$, we have:

$$J_\ell(k_\varepsilon, x, \xi) = \lim_{N\to\infty} \int\limits_0^N |s'|^{n-2} \left(\int\limits_{\Omega_r} e^{i|\xi'-x'||s'|\cos\theta d\omega} \right)$$

$$\times \left(\int\limits_0^{\sqrt{N^2+|s'|^2}} \frac{\prod\limits_{l=n-1}^{n} j_{\frac{\gamma_l-1}{2}}(x_l s_l) j_{\frac{\gamma_l-1}{2}}(\xi_l s_l)}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_{n-1}^{\gamma_{n-1}} s_n^{\gamma_n} ds_{n-1} ds_n \right) d|s'|. \tag{12}$$

where Ω_r is a sphere of radius r centered at the origin, θ is an angle between the directions of the vectors $\xi' - x'$ and s', while ω-is a point of a unit sphere. As

$$\int\limits_{\Omega_r} e^{i|\xi'-x'||s'|\cos\theta} d\omega = (2\pi)^{\frac{n-2}{2}-1} (|\xi'-x'||s'|)^{1-\frac{n-2}{2}} j_{\frac{n-2}{2}-1}(|\xi'-x'||s'|), \tag{13}$$

where $j_{\frac{l}{2}}(z)$ is the Bessel function of order $\frac{l}{2}$. Substituting (13) in (12), we get

$$J_\ell(k_\varepsilon, x, \xi) = (2\pi)^{\frac{n-4}{2}} |\xi'-x'|^{\frac{4-n}{2}} \lim_{N\to\infty} \int\limits_0^N |s'|^{\frac{n-4}{2}} j_{\frac{n-4}{2}}(|\xi-x'||s'|) \times$$

$$\times \left(\int\limits_0^{\sqrt{N^2+|s'|^2}} \frac{\prod\limits_{l=n-1}^{n} j_{\frac{\gamma_l-1}{2}}(x_l s_l) j_{\frac{\gamma_l-1}{2}}(\xi_l s_l)}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_{n-1}^{\gamma_{n-1}} s_n^{\gamma_n} ds_{n-1} ds_n \right) d|s'| =$$

$$= (2\pi)^{\frac{n-4}{2}} |\xi'-x'|^{\frac{4-n}{2}} \prod\limits_{l=n-1}^{n} \left[2^{\frac{\gamma_l-1}{2}} \Gamma\left(\frac{\gamma_l+1}{2}\right) \right]^2 (\xi_l s_l)^{\frac{1-\gamma_l}{2}} \times$$

$$\times \lim_{N\to\infty} \int\limits_0^N \left(\int\limits_0^{\sqrt{N^2+|s'|^2}} |s'|^{\frac{n-2}{2}} J_{\frac{n-4}{2}}(|\xi-x'||s'|) \right.$$

$$\times \left. \frac{\prod\limits_{l=n-1}^{n} J_{\frac{\gamma_l-1}{2}}(x_l s_l) J_{\frac{\gamma_l-1}{2}}(\xi_l s_l)}{\lambda_\ell - k_\varepsilon^2 + |s|^2} s_{n-1}^{\gamma_{n-1}} s_n^{\gamma_n} ds_{n-1} ds_n \right) d|s'| =$$

69

$$= (2\pi)^{\frac{n-4}{2}} |\xi' - x'|^{\frac{4-n}{2}} \prod_{l=n-1}^{n} \left[2^{\frac{\gamma_l-1}{2}} \Gamma\left(\frac{\gamma_l+1}{2}\right) \right]^2 (\xi_l s_l)^{\frac{1-\gamma_l}{2}} \lim_{N\to\infty} \int_0^N \frac{r^{\frac{n+1}{2}}}{\lambda_\ell - k_\varepsilon^2 + r^2} \times$$

$$\times \left(\int_0^{\frac{\pi}{2}} J_{\frac{n-4}{2}}(|\xi' - x'|r\sin\varphi) \prod_{l=n-1}^{n} J_{\frac{\gamma_l-1}{2}}(rx_l\cos\varphi) J_{\frac{\gamma_l-1}{2}}(r\xi_l\cos\varphi) \sin^{\frac{n-1}{2}}\varphi \cos\varphi d\varphi \right) dr. \tag{14}$$

The internal integral in (14) is Sonin's second determined integral. Therefore we have

$$J_\ell(k_\varepsilon, x, \xi) =$$

$$= (2\pi)^{\frac{n-4}{2}} \prod_{l=n-1}^{n} 2^{\frac{\gamma_l-1}{2}} \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)} (\xi_l s_l)^{\frac{1-\gamma_l}{2}} (\xi_l s_l)^{\frac{\gamma_l-1}{2}} \lim_{N\to\infty} \int_0^N \frac{r^{n+|\gamma|-4}}{\lambda_\ell - k_\varepsilon^2 + r^2}$$

$$\times \left(\int_0^\pi \prod_{l=n-1}^{n} \frac{J_{\frac{n+\gamma_l-2}{2}}\left(\sqrt{|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi}\right) \sin^{\gamma_l-1}\varphi d\varphi}{\left(|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi\right)^{\frac{n+\gamma_l-2}{4}}} \right) dr$$

$$= 2^{\frac{n+|\gamma|-2}{2}} \pi^{\frac{n-4}{2}} \prod_{l=n-1}^{n} \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)} \lim_{N\to\infty} \int_0^N \frac{r^{\frac{n+|\gamma|}{2}-3}}{\lambda_\ell - k_\varepsilon^2 + r^2}$$

$$\times \left(\int_0^\pi \prod_{l=n-1}^{n} \frac{J_{\frac{n+\gamma_l-2}{2}}\left(\sqrt{|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi}\right) \sin^{\gamma_l-1}\varphi d\varphi}{\left(|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi\right)^{\frac{n+\gamma_l-2}{4}}} \right) dr.$$

Let $n + |\gamma|$ be an odd number. Then $z^{\frac{n+|\gamma|}{2}} J_{\frac{n+|\gamma|}{2}-1}(z)$ is an even function. Therefore

$$J_{\ell_N}(k_\varepsilon, x, \xi) = 2^{\frac{n+|\gamma|-2}{2}} \pi^{\frac{n-4}{2}} \prod_{l=n-1}^{n} \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)} \lim_{N\to\infty} \int_{-N}^N \frac{r^{\frac{n+|\gamma|}{2}-3}}{\lambda_\ell - k_\varepsilon^2 + r^2} \times$$

$$\times \left(\int_0^\pi \prod_{l=n-1}^{n} \frac{J_{\frac{n+\gamma_l-2}{2}}\left(\sqrt{|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi}\right) \sin^{\gamma_l-1}\varphi d\varphi}{\left(|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi\right)^{\frac{n+\gamma_l-2}{4}}} \right) dr.$$

Expressing the Bessel function by the Hankel function (see [8])

$$J_\nu(z) = \frac{1}{2}\left(H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right), \tag{15}$$

we get

$$J_{\ell_N}(k_\varepsilon, x, \xi) = 2^{\frac{n+|\gamma|}{2}-4} \pi^{\frac{n-4}{2}} \prod_{l=n-1}^{n} \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)} \lim_{N\to\infty} \int_{-N}^N \frac{r^{\frac{n+|\gamma|}{2}-3}}{\lambda_\ell - k_\varepsilon^2 + r^2} \times$$

$$
\times \left(\int_0^\pi \prod_{l=n-1}^n \frac{H^{(1)}_{\frac{n+\gamma_l-2}{2}}(\sqrt{|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi})\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi)^{\frac{n+\gamma_l-2}{4}}} + \right.
$$

$$
\left. + \int_0^\pi \prod_{l=n-1}^n \frac{H_{\frac{n+\gamma_l-2}{2}}(\sqrt{|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi})\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi'-x'|^2 r^2 + \xi_l^2 r^2 + x_l^2 r^2 - 2r^2 x_l \xi_l \cos\varphi)^{\frac{n+\gamma_l-2}{4}}} \right) dr
$$

$$
\equiv J_\ell^{(1)}(k_\varepsilon, x, \xi) + J_\ell^{(2)}(k_\varepsilon, x, \xi). \tag{16}
$$

Taking into account the analyticity of the integrand function in (14) and asymptotic (15) of the Hankel function as $|s| \to +\infty$, applying the reside theorem and tending $N \to +\infty$, we get

$$
J_\ell^{(1)}(k_\varepsilon, x, \xi) = 2^{\frac{n+|\gamma|}{2}-4} \pi^{\frac{n-4}{2}} \prod_{l=n-1}^n \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)} \lim_{N\to\infty} \int_{-N}^N \frac{r^{\frac{n+|\gamma|}{2}-3}}{\lambda_\ell - k_\varepsilon^2 + r^2} \times
$$

$$
\times \left(\int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma_l-2}{2}}(r\sqrt{|\xi'-x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi})\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi'-x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi)^{\frac{n+\gamma_l-2}{4}}} \right) =
$$

$$
= 2^{\frac{n+|\gamma|}{2}-4} \pi^{\frac{n-4}{2}} \prod_{l=n-1}^n \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)} \times
$$

$$
\times \lim_{N\to\infty} \int_0^\pi \frac{\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi'-x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi)^{\frac{n+\gamma_l-2}{4}}} \times
$$

$$
\times \left(\int_{-N}^N \frac{r^{\frac{n+|\gamma|}{2}-3} H^{(1)}_{\frac{n+\gamma_l-2}{2}}(r\sqrt{|\xi'-x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi})}{\lambda_\ell - k_\varepsilon^2 + r^2} dr \right) d\varphi
$$

$$
= i2^{\frac{n+|\gamma|}{2}-4} \pi^{\frac{n-1}{2}} \prod_{l=n-1}^n \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)} \left(i\sqrt{\lambda_\ell - k_\varepsilon^2} \right)^{\frac{n+|\gamma|-6}{2}} \times
$$

$$
\times \int_0^\pi \frac{H^{(1)}_{\frac{n+\gamma_l-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi'-x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi})\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi'-x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi)^{\frac{n+\gamma_l-2}{4}}}
$$

and

$$
J_\ell^{(2)}(k_\varepsilon, x, \xi) = -i2^{\frac{n+|\gamma|}{2}-4} \pi^{\frac{n-1}{2}} \prod_{l=n-1}^n \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)} \left(-i\sqrt{\lambda_\ell - k_\varepsilon^2} \right)^{\frac{n+|\gamma|-6}{2}} \times
$$

$$\times \int\limits_0^\pi \frac{H^{(2)}_{\frac{n+\gamma_l-2}{2}}(-i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi})\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi)^{\frac{n+\gamma_l-2}{4}}}$$

$$= -i2^{\frac{n+|\gamma|}{2}-4}\pi^{\frac{n-1}{2}}\prod_{l=n-1}^n \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+|\gamma|-6}{2}} \times$$

$$\times \int\limits_0^\pi \frac{H^{(1)}_{\frac{n+\gamma_l-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi})\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_n\cos\varphi)^{\frac{n+\gamma_l-2}{4}}}$$

As (see [8])

$$H^{(2)}_{\frac{n+\gamma_l-2}{2}}(-z) = (-1)^{\frac{n+\gamma_l-2}{2}}H^{(1)}_{\frac{n+\gamma_l-2}{2}}(z),$$

then from (14)–(16) it follows that

$$J_\ell(k_\varepsilon, x, \xi) = i2^{\frac{n+|\gamma|}{2}-3}\pi^{\frac{n-1}{2}}\prod_{l=n-1}^n \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)}\left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+|\gamma|-6}{2}} \times$$

$$\times \int\limits_0^\pi \frac{H^{(1)}_{\frac{n+\gamma_l-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi})\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi)^{\frac{n+\gamma_l-2}{4}}} \qquad (17)$$

Behaving in the same way as above, at even n for $J_\ell(k_\varepsilon, x, \xi)$ we get formula (7). Substituting (17) in (9), we get

$$u(k_\varepsilon, x, y) = i2^{\frac{n+|\gamma|}{2}-3}\pi^{\frac{n-1}{2}}\prod_{l=n-1}^n \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)}\sum_{\ell=1}^\infty \left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+|\gamma|-6}{2}}\varphi_\ell(y)\times$$

$$\times \int\limits_{R_+^n}\int\limits_0^\pi \frac{H^{(1)}_{\frac{n+\gamma_l-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi})\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi)^{\frac{n+\gamma_l-2}{4}}}f_\ell(\xi)\xi_l d\xi \quad (18)$$

Now, substituting the value of $f_\ell(\xi)$ from (10) in (18) and distinguishing the function $f(x)$, we get

$$u(k_\varepsilon, x, y) = i2^{\frac{n+\gamma}{2}-3}\pi^{\frac{n-1}{2}}\prod_{l=n-1}^n \frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)}\sum_{\ell=1}^\infty \left(i\sqrt{\lambda_\ell - k_\varepsilon^2}\right)^{\frac{n+|\gamma|-6}{2}}\varphi_\ell(y)\times$$

$$\times \int\limits_{R_+^n}\int\limits_0^\pi \frac{H^{(1)}_{\frac{n+\gamma_l-2}{2}}(i\sqrt{\lambda_\ell - k_\varepsilon^2}\sqrt{|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi})\sin^{\gamma_l-1}\varphi d\varphi}{(|\xi' - x'|^2 + \xi_l^2 + x_l^2 - 2x_l\xi_l\cos\varphi)^{\frac{n+\gamma_l-2}{4}}} \times$$

72

$$\times \int\limits_{\Omega} f(\xi,z)\varphi_\ell(z)dz\xi_l d\xi = i2^{\frac{n+|\gamma|}{2}-3}\pi^{\frac{n-1}{2}}\frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)}\times$$

$$\times \int\cdots\int\limits_{\Lambda}\sum_{\ell=1}^{\infty}\left(i\sqrt{\lambda_\ell-k_\varepsilon^2}\right)^{\frac{n+|\gamma|-6}{2}}\varphi_\ell(y)\varphi_\ell(z)\times$$

$$\times\int\limits_0^{\pi}\frac{H^{(1)}_{\frac{n+\gamma_l-2}{2}}\left(i\sqrt{\lambda_\ell-k_\varepsilon^2}\sqrt{|\xi'-x'|^2+\xi_l^2+x_l^2-2x_l\xi_l\cos\varphi}\right)\sin^{\gamma_l-1}\varphi d\varphi}{\left(|\xi'-x'|^2+\xi_l^2+x_l^2-2x_l\xi_l\cos\varphi\right)^{\frac{n+\gamma_l-2}{4}}}f(\xi,z)\varphi_\ell(z)d\Lambda.$$

The function

$$G(k_\varepsilon,x-\xi,y,z)=i2^{\frac{n+|\gamma|}{2}-3}\pi^{\frac{n-1}{2}}\prod_{l=n-1}^{n}\frac{\Gamma^2\left(\frac{\gamma_l+1}{2}\right)}{\Gamma\left(\frac{\gamma_l}{2}\right)\Gamma\left(\frac{1}{2}\right)}\sum_{\ell=1}^{\infty}\left(i\sqrt{\lambda_\ell-k_\varepsilon^2}\right)^{\frac{n+|\gamma|-6}{2}}\varphi_\ell(y)\varphi_\ell(z)\times$$

$$\times\int\limits_0^{\pi}\frac{H^{(1)}_{\frac{n+\gamma_l-2}{2}}\left(i\sqrt{\lambda_\ell-k_\varepsilon^2}\sqrt{|\xi'-x'|^2+\xi_l^2+x_l^2-2x_l\xi_l\cos\varphi}\right)\sin^{\gamma_l-1}\varphi d\varphi}{\left(|\xi'-x'|^2+\xi_l^2+x_l^2-2x_l\xi_l\cos\varphi\right)^{\frac{n+\gamma_l-2}{4}}} \qquad (19)$$

is Green's function of problem (1)–(2) with a complex parameter k_ε^2. The theorem is proved.

From this theorem it follows

Theorem 2.7. *The solution of problem* (1)–(2) *with a complex parameter k_ε^2 is represented in the form*

$$u(k_\varepsilon,x,y)=\int\cdots\int\limits_{\Lambda}G(k_\varepsilon,x-\xi,y,z)f(\xi,z)d\Lambda, \qquad (20)$$

this solution is unique, where $G(k_\varepsilon,x-\xi,y,z)$ is determined by formula (19).

Taking into account asymptotics (5) as $z\to\infty$ of the function $H^{(1)}_{\frac{n+\gamma_l}{2}-1}(z)$ we get that series (19) and its derivatives contained in equation (1), converge uniformly with respect to ε for $|x-\xi|>0$ and therefore passing in (20) and in its derivatives to limit as $\varepsilon\to 0$, we get that $u(k,x,y)$ is the solution of problem (1)–(2). Thus, we proved the following theorem.

Theorem 2.8. *For problem* (1)–(2) *it holds the limit absorption principle (for $n=1,2$; $k=\pm\lambda_\ell^{\frac{1}{2}}$, $l=1,2,\ldots$).*

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