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Weighted Hardy operators in the local "complementary" weighted Morrey spaces

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Abstract. In this paper we consider local "complementary" weighted Morrey spaces ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$. We prove the boundedness of the weighted Hardy operators and their commutators in the spaces ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$.

Key Words and Phrases: Local "complementary" weighted Morrey spaces, Hardy operator, commutator, BMO space.
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1. Introduction

The classical Morrey spaces were originally introduced by Morrey in [8] to study the local behavior of solutions to second order elliptic partial differential equations. We recall its definition as

$$\mathcal{L}_{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L_p^{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{\mathcal{L}_{p,\lambda}} := \sup_{x\in\mathbb{R}^n,\, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(B(x,r))} < \infty \right\}, \tag{1.1}$$

where $0 \le \lambda \le n$, $1 \le p < \infty$. Here and everywhere in the sequel $B(x,r)$ stands for the ball in $\mathbb{R}^n$ of radius r centered at x. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$ and $|B(x,r)| = v_n r^n$, where $v_n = |B(0,1)|$. $M_{p,l}(\mathbb{R}^n)$ was an expansion of $L_p(\mathbb{R}^n)$ in the sense that $\mathcal{L}_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

Morrey found that many properties of solutions to PDE can be attributed to the boundedness of some operators on Morrey spaces. Hardy operators, Maximal functions and singular integrals play a key role in harmonic analysis since maximal functions could control crucial quantitative information concerning the given functions, despite their larger size, while singular integrals, Hilbert transform as it's prototype, nowadays intimately connected with PDE, operator theory and other fields.

Inequalities involving classical operators of harmonic analysis, such as maximal functions, fractional integrals and singular integrals of convolution type have been extensively

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investigated in various function spaces. Results on weak and strong type inequalities for operators of this kind in Lebesgue spaces are classical and can be found for example in [2, 13, 16]. Generalizations of these results to Zygmund spaces are presented in [2].

We denote by the local "complementary" Morrey spaces ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$ the space of all functions $f \in L_p(\mathbb{R}^n \backslash B(x_0, r))$, $r > 0$ with the finite norm

$$\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L_p(\mathbb{R}^n \backslash B(x_0,r))} < \infty, \quad x_0 \in \mathbb{R}^n,$$

where $1 \le p < \infty$ and $0 \le \lambda < n$. Note that ${}^{\complement}\mathcal{L}^{p,0}_{\{x_0\}}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

We consider the following Hardy operators

$$H_\beta f(x) = |x|^{\beta - n} \int_{|y| \le |x|} \frac{f(y)}{|y|^\beta} dy.$$

2. Preliminaries on Lebesgue and Morrey spaces

We use the following notation. For $1 \le p < \infty$, $L_p(\mathbb{R}^n)$ is the space of all classes of measurable functions on $\mathbb{R}^n$ for which

$$\|f\|_{L_p} = \left(\int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}} < \infty,$$

and also $WL_p(\mathbb{R}^n)$, the weak L_p space defined as the set of all measurable functions f on $\mathbb{R}^n$ such that

$$\|f\|_{WL_p} = \sup_{r>0} r \, |\{x \in \mathbb{R}^n \ : \ |f(x)| > r\}|^{1/p} < \infty.$$

For $p = \infty$ the space $L_\infty(\mathbb{R}^n)$ is defined by means of the usual modification

$$\|f\|_{L_\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^n} |f(x)|.$$

Let $L_{p,\omega}(\mathbb{R}^n)$ be the space of measurable functions on $\mathbb{R}^n$ such that

$$\|f\|_{L_{p,\omega}} = \|f\|_{L_{p,\omega}(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^p \omega(x) dx \right)^{1/p} < \infty, \quad 1 \le p < \infty.$$

and for $p = \infty$ the space $L_{\infty,\omega}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$.

Definition 2.1. The weight function ω belongs to the class $A_p(\mathbb{R}^n)$ for $1 \le p < \infty$, if the following statement

$$\sup_{x \in \mathbb{R}^n, r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \omega(y) dy \left(\frac{1}{|B(x,r)|} \int_{B(x,r)} \omega^{-\frac{1}{p-1}}(y) dy \right)^{p-1}$$

is finite and ω belongs to $A_1(\mathbb{R}^n)$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n$ and $r > 0$

$$|B(x,r)|^{-1} \int\limits_{B(x,r)} \omega(y)dy \le C \operatorname*{ess\,sup}_{y\in B(x,r)} \frac{1}{\omega(y)}.$$

Definition 2.2. We define the $BMO(\mathbb{R}^n)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO} = \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - f_{B(x,r)}|dy < \infty$$

or

$$\|f\|_{BMO} = \inf_C \sup_{x\in\mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y) - C|dy < \infty.$$

Definition 2.3. We define the $BMO_{p,\omega}(\mathbb{R}^n)$ $(1 \le p < \infty)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO_{p,\omega}} = \sup_{x\in\mathbb{R}^n,\, r>0} \frac{\|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)}}{\|\chi_{B(x,r)}\|_{L_{p,\omega}(\mathbb{R}^n)}} < \infty.$$

Theorem 2.4. *[7, Theorem 4.4] Let $1 \le p < \infty$ and ω be a Lebesgue measurable function. If $\omega \in A_p(\mathbb{R}^n)$, then the norms $\|\cdot\|_{BMO_{p,\omega}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

We find it convenient to define the local "complementary" weighted Morrey spaces in the form as follows.

Definition 2.5. Let $1 \le p < \infty$. The local "complementary" weighted Morrey space $\mathcal{M}^{p,\omega}(\mathbb{R}^n)$ is defined by the norms

$$\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L_{p,\omega}(\mathbb{R}^n \setminus B(x_0,r))} < \infty, \quad x_0 \in \mathbb{R}^n.$$

3. Weighted Hardy operator and commutators in the spaces ${}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)$

The proof of the main result of this section presented in Theorem 3.1 is based on the estimate given in the following preliminary theorem.

Theorem 3.1. *Let $1 < p < \infty$, $0 \le \lambda < n$, $\omega \in A_p(\mathbb{R}^n)$. Then the weighted Hardy operator H_β is bounded from the space ${}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)$ to the space ${}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)$.*

Proof. Let $f \in {}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)$, $\delta > 0$. We have

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz = \int\limits_{B(0,r)} \frac{|f(z)|}{|z|^\beta} dz$$

$$= \delta \int_{B(0,r)} \frac{|f(z)|}{|z|^{\beta+\delta}} \left(\int_0^{|z|} s^{\delta-1} ds \right) dz$$

$$= \delta \int_0^r s^{\delta-1} \left(\int_{\{z\in\mathbb{R}^n : s<|z|<r\}} \frac{|f(z)|}{|z|^{\beta+\delta}} dz \right) ds.$$

Applying Hölder inequality, we get

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz \le C \int_0^r \frac{1}{s^\beta} \|f\|_{L_{p,\omega}(\mathbb{R}^n \setminus B(0,s))} \|\omega^{-1}\|_{L_{p'}(B(0,r))} \frac{ds}{s}.$$

Then we have

$$\int\limits_{|z|<r} \frac{|f(z)|}{|z|^\beta} dz \le C \, \|\omega^{-1}\|_{L_{p'}(B(0,r))} \int_0^r \frac{1}{s^\beta} \|f\|_{L_{p,\omega}(\mathbb{R}^n \setminus B(0,s))} \frac{ds}{s} \tag{3.1}$$

Therefore by (3.1) we have

$$\begin{aligned}
&\|H_\beta f\|_{L_{p,\omega}(\mathbb{R}^n \setminus B(0,t))} \\
&\le C \left\| \|\omega^{-1}\|_{L_{p'}(B(0,|\cdot|))} |\cdot|^{\beta-n} \int_0^{|\cdot|} \frac{1}{s^\beta} \|f\|_{L_{p,\omega}(\mathbb{R}^n \setminus B(0,s))} \frac{ds}{s} \right\|_{L_{p,\omega}(\mathbb{R}^n \setminus B(0,t))} \\
&\le C \|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)} \left\| \frac{|\cdot|^\beta}{\|\omega\|_{L_p(B(0,|\cdot|))}} \int_0^{|\cdot|} \frac{s^{-\frac{\lambda}{p'}-\beta}}{\|\omega\|_{L_{p'}(B(0,s))}} \frac{ds}{s} \right\|_{L_{p,\omega}(\mathbb{R}^n \setminus B(0,t))} \\
&\le C \|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)} \left\| \frac{|\cdot|^{-\frac{\lambda}{p'}}}{\|\omega\|_{L_{p'}(B(0,|\cdot|))} \|\omega\|_{L_p(B(0,|\cdot|))}} \right\|_{L_{p,\omega}(\mathbb{R}^n \setminus B(0,t))} \\
&\le C \|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)} \left\| \frac{|\cdot|^{-\frac{\lambda}{p'}}}{\|\omega\|_{L_p(B(0,|\cdot|))} \|\omega\|_{L_{p'}(B(0,|\cdot|))}} \right\|_{L_{p,\omega}(\mathbb{R}^n \setminus B(0,t))} \\
&\le C \|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)} \int_0^t \frac{s^{-\frac{\lambda}{p'}}}{\|\omega\|_{L_{p'}(B(0,s))}} \frac{ds}{s} \\
&\le C \|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)} \frac{t^{-\frac{\lambda}{p'}}}{\|\omega\|_{L_{p'}(B(0,s))}}.
\end{aligned}$$

□

Theorem 3.2. *Let $1 < p < \infty$, $0 \le \lambda < n$, $\omega \in A_p(\mathbb{R}^n)$, $b \in BMO_{p',\omega^{-1}}(\mathbb{R}^n)$.*

Then the commutator of weighted Hardy operator H_β, $[b, H_\beta]$, is bounded from the space ${}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)$ to the space ${}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)$.

Proof. Let $f \in {}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)$, $\delta > 0$. We have

$$\int\limits_{|z|<r} \frac{|b(z)-b(x)|\,|f(z)|}{|z|^{\beta}}dz = \int\limits_{B(0,r)} \frac{|b(z)-b(x)||f(z)|}{|z|^{\beta}}dz$$
$$= \delta \int_{B(0,r)} |z|^{-\delta-\beta}|b(z)-b(x)||f(z)| \left(\int_0^{|z|} s^{\delta-1}ds\right) dz$$

$$= \delta \int_0^r s^{\delta-1} \left(\int_{\{z\in\mathbb{R}^n : s<|z|<r\}} |z|^{-\delta-\beta}|b(z)-b(x)||f(z)|dz\right) ds$$
$$= \delta \int_0^r s^{-\beta-1} \left(\int_{\{z\in\mathbb{R}^n : s<|z|<r\}} |b(z)-b(x)||f(z)|dz\right) ds.$$

Applying Hölder inequality, we get

$$\int\limits_{|z|<r} \frac{|b(z)-b(x)|\,|f(z)|}{|z|^{\beta}}dz \le C\int_0^r t^{-\beta-1}\,\|f\|_{L_{p,\omega}(\mathbb{R}^n\setminus B(0,t))}\,\|b(\cdot)-b(x)\|_{L_{p',\omega^{-1}}(B(0,r))}dt.$$

Then, we obtain

$$\int\limits_{|z|<r} \frac{|b(z)-b(x)|\,|f(z)|}{|z|^{\beta}}dz$$
$$\le C\|b\|_{BMO_{p',\omega^{-1}}}\,\|\omega^{-1}\|_{L_{p'}(B(0,r))}\int_0^r t^{-\beta-1}\,\|f\|_{L^{p,\omega}(\mathbb{R}^n\setminus B(0,t))}dt. \tag{3.2}$$

Therefore by (3.2), we get

$$\|[b,H_\beta]f\|_{L_{p,\omega}(\mathbb{R}^n\setminus B(0,\rho))} \le C\|b\|_{BMO_{p',\omega^{-1}}}$$
$$\times \left\| |\cdot|^{\beta-n}\|\omega^{-1}\|_{L_{p'}(B(0,|\cdot|))}\int_0^{|\cdot|} t^{\beta}\,\|f\|_{L_{p,\omega}(\mathbb{R}^n\setminus B(0,t))}\frac{dt}{t}\right\|_{L_{p,\omega}(\mathbb{R}^n\setminus B(0,\rho))}$$
$$\le C\|b\|_{BMO_{0,p',\omega^{-1}}}\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)}\left\|\frac{|\cdot|^{\beta}}{\|\omega\|_{L_p(B(0,|\cdot|))}}\int_0^{|\cdot|}\frac{t^{-\frac{\lambda}{p'}-\beta}}{\|\omega\|_{L_{p'}(B(0,t))}}\frac{dt}{t}\right\|_{L_{p,\omega}(\mathbb{R}^n\setminus B(0,\rho))}$$
$$\le C\|b\|_{BMO_{0,p',\omega^{-1}}}\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)}\left\|\frac{|\cdot|^{-\frac{\lambda}{p'}}}{\|\omega\|_{L_p(B(0,|\cdot|))}\|\omega\|_{L_{p'}(B(0,|\cdot|))}}\right\|_{L_{p,\omega}(\mathbb{R}^n\setminus B(0,\rho))}$$
$$\le C\|b\|_{BMO_{0,p',\omega^{-1}}}\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)}\int_0^{\rho}\frac{t^{-\frac{\lambda}{p'}}}{\|\omega\|_{L_{p'}(B(0,t))}}\frac{dt}{t}$$
$$\le C\|b\|_{BMO_{0,p',\omega^{-1}}}\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda,\omega}_{\{0\}}(\mathbb{R}^n)}\frac{\rho^{-\frac{\lambda}{p'}}}{\|\omega\|_{L_{p'}(B(0,\rho))}}.$$

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