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On A Jacobsthal-Like Sequence Associated with K-Jacobsthal-Lucas Sequence

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Abstract. We introduce a new generalization of the known Jacobstal sequence and we call Jacobstal-Like sequence $\langle J_{k,n}\rangle$ and then we study on Jacobstal-Like sequence$\langle J_{k,n}\rangle$ and k-Jacobsthal-Lucas sequence $\langle j_{k,n}\rangle$ side by side by introducing two special matrices for these sequences. After that, by using these matrices we obtain Binet formula for Jacobstal-Like sequence $\langle J_{k,n}\rangle$ Further, we get Cassini identity for Jacobstal-Like sequence $\langle J_{k,n}\rangle$. Also we establish some of the interesting properties of Jacobstal-Like sequence. Then we give a limit for Jacobstal-Like sequence $\langle J_{k,n}\rangle$.

Key Words and Phrases: Generalized Jacobsthal sequence, Jacobstal-Like sequence, k-Jacobsthal-Lucas sequence, Jacobsthal sequence.
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1. Introduction

The sequences of numbers and their generalizations have many applications to every field of science [4],[10],[11],[12],[13],[16],[14],[7],[8],[22],[18],[23]. Second order linear recurrences related to Fibonacci and Lucas numbers and their generalizations are studied in [1],[9],[21]. Fourth order linear recurrences and their generalizations are given in [5],[3],[17],[15]. The two most important generalizations of Fibonacci numbers are k-Fibonacci numbers $\langle F_{k,n}\rangle$ [2] and k-Lucas numbers $\langle L_{k,n}\rangle$ [6]. One of the latest works in this area is [20] where they defined Fibonacci-Like sequence $\langle V_{k,n}\rangle$ and gave some properties and theorems related to it. In this paper, we introduce Jacobstal-Like sequence $\langle J_{k,n}\rangle$ and give its some properties and theorems.

2. Preliminaries

Let $\langle G_{k,n}\rangle$ be a sequence defined by the following recurrence relation.

$$G_{k,n+2} = kG_{k,n+1} + G_{k,n}, n \geq 1 \quad \text{and} \quad G_{k,0} = a, G_{k,1} = b \tag{1}$$

where $k \in \mathbf{R}^+$ and $a, b \in \mathbf{Z}^+$.
Now we consider two cases of the sequence $\langle G_{k,n}\rangle$:

1. Jacobstal-Like sequence $\langle J_{k,n}\rangle$ is defined by

$$J_{k,n+2} = kJ_{k,n+1} + 2J_{k,n}, n \geq 0 \quad \text{and} \quad J_{k,0} = 2m, J_{k,1} = p + mk \tag{2}$$

2. $k-$jacobsthal-Lucas sequence $\langle j_{k,n}\rangle$ is defined [19] by

$$j_{k,n+2} = kj_{k,n+1} + 2j_{k,n}, n \geq 0 \quad \text{and} \quad j_{k,0} = 2, j_{k,1} = k \tag{3}$$

Both of the relations 2 and 3 have same characteristic equation $x^2 - kx - 2 = 0$.
So we have

$$r = \frac{k + \sqrt{k^2+8}}{2} \quad \text{and} \quad s = \frac{k - \sqrt{k^2+8}}{2} \tag{4}$$

where r and s are its roots.
It is easy to see the following features hold.
(i) $rs = -2, r + s = k, r - s = \sqrt{k^2+8}$
(ii) $r^2 - 2 = kr, \quad s^2 - 2 = ks$
(iii) $r^2 + 2 = kr + 4 = (r - s)r, \quad s^2 + 2 = ks + 4 = -(r - s)s$ To use in next section, we introduce two special 2×2 matrices j and J which are given by

$$J = \begin{pmatrix} k & 2 \\ 1 & 0 \end{pmatrix} \qquad \text{and} \qquad j = \begin{pmatrix} k^2+4 & 2k \\ k & 4 \end{pmatrix} \tag{5}$$

3. Main results

Theorem 1. *For $n \in \mathbf{Z}^+$ we have the following result*

$$J^n = \begin{pmatrix} \frac{2mJ_{k,n+2}-(p+mk)J_{k,n+1}}{8m^2+m^2k^2-p^2} & \frac{4mJ_{k,n+1}-2(p+mk)J_{k,n}}{8m^2+m^2k^2-p^2} \\ \frac{2mJ_{k,n+1}-(p+mk)J_{k,n}}{8m^2+m^2k^2-p^2} & \frac{4mJ_{k,n}-2(p+mk)J_{k,n-1}}{8m^2+m^2k^2-p^2} \end{pmatrix} \tag{6}$$

Proof. We use the induction method on n. Certainly the results is true for $n = 1$. Supposing that the result holds for all values $i, i \leq n$ and then

$$J^{n+1} = \begin{pmatrix} \frac{2mJ_{k,n+2}-(p+mk)J_{k,n+1}}{8m^2+m^2k^2-p^2} & \frac{4mJ_{k,n+1}-2(p+mk)J_{k,n}}{8m^2+m^2k^2-p^2} \\ \frac{2mJ_{k,n+1}-(p+mk)J_{k,n}}{8m^2+m^2k^2-p^2} & \frac{4mJ_{k,n}-2(p+mk)J_{k,n-1}}{8m^2+m^2k^2-p^2} \end{pmatrix} \begin{pmatrix} k & 2 \\ 1 & 0 \end{pmatrix}$$

$$J^{n+1} = (8m^2 + m^2k^2 - p^2)^{-1}) \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} k & 2 \\ 1 & 0 \end{pmatrix}$$

$$J^{n+1} = (8m^2 + m^2k^2 - p^2)^{-1}) \begin{pmatrix} kx_1 + x_2 & 2x_1 \\ kx_3 + x_4 & 2x_3 \end{pmatrix}$$

here

$$\begin{aligned} kx_1 + x_2 &= 2m(kJ_{k,n+2} + 2J_{k,n+1}) - (p + mk)(kJ_{k,n+1} + 2J_{k,n}) \\ &= 2mJ_{k,n+3} - (p + mk)J_{k,n+2} \end{aligned}$$

and similarly

$$kx_3 + x_4 = 2mJ_{k,n+2} - (p+mk)J_{k,n+1}$$

hence

$$J^{n+1} = \begin{pmatrix} \frac{2mJ_{k,n+3}-(p+mk)J_{k,n+2}}{8m^2+m^2k^2-p^2} & \frac{4mJ_{k,n+2}-2(p+mk)J_{k,n+1}}{8m^2+m^2k^2-p^2} \\ \frac{2mJ_{k,n+2}-(p+mk)J_{k,n+1}}{8m^2+m^2k^2-p^2} & \frac{4mJ_{k,n+1}-2(p+mk)J_{k,n}}{8m^2+m^2k^2-p^2} \end{pmatrix}$$

which is desired result. □

Corollary 1. *For* $\lim_{n\to\infty} \frac{J_{k,n+1}}{J_{k,n}} = r$, *we have*

$$r^2 - kr - 2 = 0 \tag{7}$$

Proof. To prove the required results, we use concept of limits. Since

$$\lim_{n\to\infty} \frac{J^n}{J_{k,n-1}} = \lim_{n\to\infty} \begin{pmatrix} \frac{2mJ_{k,n+2}-(p+mk)J_{k,n+1}}{(8m^2+m^2k^2-p^2)J_{k,n-1}} & \frac{4mJ_{k,n+1}-2(p+mk)J_{k,n}}{(8m^2+m^2k^2-p^2)J_{k,n-1}} \\ \frac{2mJ_{k,n+1}-(p+mk)J_{k,n}}{(8m^2+m^2k^2-p^2)J_{k,n-1}} & \frac{4mJ_{k,n}-2(p+mk)J_{k,n-1}}{(8m^2+m^2k^2-p^2)J_{k,n-1}} \end{pmatrix}$$

now

$$\lim_{n\to\infty} \frac{2mJ_{k,n+2} - (p+mk)J_{k,n+1}}{J_{k,n-1}} = r^2(2mr - p - mk)$$

$$\lim_{n\to\infty} \frac{2mJ_{k,n+1} - (p+mk)J_{k,n}}{J_{k,n-1}} = r(2mr - p - mk)$$

$$\lim_{n\to\infty} \frac{4mJ_{k,n+1} - 2(p+mk)J_{k,n}}{J_{k,n-1}} = 2r(2mr - p - mk)$$

$$\lim_{n\to\infty} \frac{4mJ_{k,n} - 2(p+mk)J_{k,n-1}}{J_{k,n-1}} = 2(2mr - p - mk)$$

Therefore

$$\lim_{n\to\infty} \frac{J^n}{J_{k,n-1}} = (8m^2 + m^2k^2 - p^2)^{-1} \begin{pmatrix} r^2(2mr-p-mk) & 2r(2mr-p-mk) \\ r(2mr-p-mk) & 2(2mr-p-mk) \end{pmatrix}$$

$$= (8m^2 + m^2k^2 - p^2)^{-1} \begin{pmatrix} (kr+2)(2mr-p-mk) & 2r(2mr-p-mk) \\ r(2mr-p-mk) & 2(2mr-p-mk) \end{pmatrix}$$

If we equate determinant on both sides, we obtain

$$4 + 2kr - 2r^2 = 0$$

If we simplify it:

$$r^2 - kr - 2 = 0$$

as desired. □

Theorem 2. *For $n \in \mathbf{N}$ the following result holds*

$$J^n = \begin{matrix} (r-s)^{n-1}U^n, & if \quad n \quad odd \\ (r-s)^n J^n, & if \quad n \quad even \end{matrix} \tag{8}$$

where $U^n = \begin{pmatrix} j_{k,n+1} & 2k_{k,n} \\ j_{k,n} & 2j_{k,n-1} \end{pmatrix}$.

Proof. To prove the theorem, we use the induction method on n. First, we consider odd n. Let $n = 1$, we have

$$j = \begin{pmatrix} k^2+4 & 2k \\ k & 4 \end{pmatrix} = \begin{pmatrix} j_{k,2} & 2j_{k,1} \\ j_{k,1} & 2j_{k,0} \end{pmatrix}$$

Now, assume that the result holds for all odd values $i, i \leq n$, and then

$$\begin{aligned} j^{n+2} &= (r-s)^{n-1}U^n j^2 \\ &= (r-s)^{n-1}U^n J^2 (r-s)^2 \\ &= (r-s)^{n+1}U^n U^2 \\ &= (r-s)^{n+1}U^{n+2} \end{aligned}$$

as required. Now we consider even n. Let $n = 2$, we have

$$j^2 = (k^2+8)\begin{pmatrix} k^2+2 & 2k \\ k & 2 \end{pmatrix} = \begin{pmatrix} k^4+10k^2+16 & 2k^3+16k \\ k^3+8k & 2k^2+16 \end{pmatrix}$$

Assume that the result holds for all even values $0i, i \leq n$, and then

$$\begin{aligned} j^{n+2} &= (r-s)^n J^n j^2 \\ &= (r-s)^n J^n J^2 (r-s)^2 \\ &= (r-s)^{n+2} J^{n+2} \end{aligned}$$

As required. □

Proposition 1. *For $n \geq 0$ we have*

$$J^n \binom{J_{k,1}}{J_{k,0}} = \binom{J_{k,n+1}}{J_{k,n}}.$$

Proof. We prove the proposition by the induction method on n. Indeed the result is true for $n = 0$. Now, assume that the result holds for all values $i, i \leq n$, and then

$$\begin{aligned} J^{n+1}\binom{J_{k,1}}{J_{k,0}} &= J^n J \binom{J_{k,1}}{J_{k,0}} \\ &= \begin{pmatrix} k & 2 \\ 1 & 0 \end{pmatrix} \binom{J_{k,n+1}}{J_{k,n}} \end{aligned}$$

$$= \begin{pmatrix} kJ_{k,n+1} + 2J_{k,n} \\ J_{k,n+1} \end{pmatrix}$$
$$= \begin{pmatrix} J_{k,n+2} \\ J_{k,n+1} \end{pmatrix}$$

As required. We give Binet formula for Jacobstal-Like sequence $\langle J_{k,n} \rangle$ in the following theorem. □

Theorem 3. *For $n \in \mathbf{Z_0}$, the n^{th} terms for $\langle J_{k,n} \rangle$ given by*

$$J_{k,n} = p\frac{r^n - s^n}{r - s} + m(r^n + s^n).$$

Proof. If n is even then again by (8) we achieve

$$J^n = (r-s)^{-1} \begin{pmatrix} r^{n+1} - s^{n+1} & 2(r^n - s^n) \\ r^n - s^n & 2(r^{n-1} - s^{n-1}) \end{pmatrix} \tag{9}$$

By using Proposition 1, we obtain

$$\begin{pmatrix} J_{k,n+1} \\ J_{k,n} \end{pmatrix} = (r-s)^{-1} \begin{pmatrix} r^{n+1} - s^{n+1} & 2(r^n - s^n) \\ r^n - s^n & 2(r^{n-1} - s^{n-1}) \end{pmatrix} \begin{pmatrix} p + mk \\ 2m \end{pmatrix}$$

$$\begin{pmatrix} J_{k,n+1} \\ J_{k,n} \end{pmatrix} = (r-s)^{-1} \begin{pmatrix} b_1 & b_2 \\ r^n - s^n & 2(r^{n-1} - s^{n-1}) \end{pmatrix} \begin{pmatrix} p + mk \\ 2m \end{pmatrix}$$

where b_1 and b_2 are the corresponding terms of the matrix. Thus

$$\begin{pmatrix} J_{k,n+1} \\ J_{k,n} \end{pmatrix} = (r-s)^{-1} \left(b_1(p+mk) + 2mb_2(p+mk)(r^n - s^n) + 2m(2(r^{n-1} - s^{n-1})) \right)$$

$$= (r-s)^{-1} \left(b_1(p+mk) + 2mb_2(p(r^n - s^n) + mkr^n) + 4mr^{n-1} - mks^n - 4ms^{n-1} \right)$$
$$= (r-s)^{-1} \left(b_1(p+mk) + 2mb_2p(r^n - s^n) + mr^{n-1}(kr+4) - ms^{n-1}(ks+4) \right)$$
$$= (r-s)^{-1} \left(b_1(p+mk) + 2mb_2p(r^n - s^n) + mr^{n-1}(r^2+2) - ms^{n-1}(s^2+2) \right)$$
$$= (r-s)^{-1} \left(b_1(p+mk) + 2mb_2p(r^n - s^n) + mr^n(r-s) + ms^n(r-s) \right)$$

Equating corresponding terms on both sides, we get

$$J_{k,n} = p\frac{r^n - s^n}{r - s} + m(r^n + s^n).$$

This is Binet's formula for Jacobsthal-like sequence. Now we give a relation between Jacobsthal-like sequence and $k-$jacobsthal-Lucas sequence. □

Corollary 2. *For $n \in \mathbf{Z}^+$, we have*

$$2mJ_{k,n+2} - (p+mk)J_{k,n+1} + 4mJ_{k,n} - 2(p+mk)J_{k,n-1} = (8m^2 + m^2k^2 - p^2)j_{k,n}.$$

Proof. If we equate corresponding terms of matrices in the (9), we get

$$\frac{2mJ_{k,n+2}-(p+mk)J_{k,n+1}}{8m^2+m^2k^2-p^2}=\frac{r^{n+1}-s^{n+1}}{r-s}$$

and

$$\frac{4mJ_{k,n}-2(p+mk)J_{k,n-1}}{8m^2+m^2k^2-p^2}=\frac{2(r^{n-1}-s^{n-1})}{r-s}$$

Therefore,

$$\frac{2mJ_{k,n+2}-(p+mk)J_{k,n+1}+4mJ_{k,n}-2(p+mk)J_{k,n-1}}{8m^2+m^2k^2-p^2}=\frac{r^{n-1(2+r^2)-s^{n-1}(2+s^2)}}{r-s}$$

$$=\frac{r^n(r+2/r)-s^n(s+2/s)}{r-s}=\frac{r^n(r-s)+s^n(r-s)}{r-s}=r^n+s^n=j_{k,n+1}$$

Thus

$$2mJ_{k,n+2}-(p+mk)J_{k,n+1}+4mJ_{k,n}-2(p+mk)J_{k,n-1}=(8m^2+m^2k^2-p^2)j_{k,n}.$$

Hence the result. We obtain Cassini's identity for Jacobsthal- Like sequence by using matrix J. □

Theorem 4. *For $n \geq 1$ we have*

$$J_{k,n}^2-J_{k,n+1}J_{k,n-1}=(-2)^{n-1}(8m^2+m^2k^2-p^2). \tag{10}$$

Proof.

$$det(J^n)=\begin{vmatrix} \frac{2mJ_{k,n+2}-(p+mk)J_{k,n+1}}{8m^2+m^2k^-p^2} & \frac{4mJ_{k,n+1}-2(p+mk)J_{k,n}}{8m^2+m^2k^-p^2} \\ \frac{2mJ_{k,n+1}-(p+mk)J_{k,n}}{8m^2+m^2k^-p^2} & \frac{4mJ_{k,n+1}-2(p+mk)J_{k,n-1}}{8m^2+m^2k^-p^2} \end{vmatrix}$$

$$\begin{aligned} det(J^n)=(8m^2{+}m^2k^-p^2)^{-2}8m^2(kJ_{k,n+1}+2J_{k,n})-4m(p+mk)(kJ_{k,n+1}+2J_{k,n})J_{k,n-1} \\ -4m(p+mk)J_{k,n}J_{k,n+1}+2(p+mk)^2J_{k,n-1}J_{k,n+1}-8m^2J_{k,n+1}^2 \\ +4m(p+mk)J_{k,n}J_{k,n+1}+4m(p+mk)J_{k,n}J_{k,n+1}-2(p+mk)^2J_{k,n}^2 \end{aligned} \tag{11}$$

Here,

$$J_{k,n+2}=kJ_{k,n+1}+2J_{k,n}$$

$$J_{k,n+1}=kJ_{k,n}+2J_{k,n-1}$$

$$J_{k,n+1}^2=kJ_{k,n}J_{k,n+1}+2J_{k,n+1}J_{k,n-1}$$

By using the equations above,

$$J_{k,n}^2(16m^2-2(p+mk)^2)+J_{k,n+1}J_{k,n-1}(-4mk(p+mk)+2(p+mk)^2)$$

$$+ J_{k,n}J_{k,n+1}(8m^2k + 4m(p+mk)) + J_{k,n+1}^2(-8m^2) + J_{k,n}J_{k,n-1}(-8m(p+mk))$$

$$= (kJ_{k,n}^2 + 2J_{k,n-1}J_{k,n})(8m^2k + 4m(p+mk))$$
$$+ 2J_{k,n-1}J_{k,n}(8m^2 + 4m(p+mk)) + J_{k,n}J_{k,n-1}(-8m(p+mk)) + J_{k,n+1}^2(-8m^2)$$

$$= kJ_{k,n}^2(8m^2+4m(p+mk)) + J_{k,n+1}^2(-8m^2) + J_{k,n}J_{n-1}(-8m(p+mk) + 16m^2k + 8m(p+mk))$$

$$= kJ_{k,n}^2(12m^2k + 4mp) + J_{k,n+1}^2(-8m^2) + J_{k,n}J_{k,n-1}(16m^2k)$$

$$= J_{k,n}^2(10m^2k^2 + 4mpk + 16m^2 - 2p^2 - 4mpk - 2m^2k^2) + J_{k,n+1}J_{k,n-1}(-4mpk - 4m^2k^2$$
$$+ 2p^2 + 4mpk + 2m^2k^2 - 16m^2) + J_{k,n+1}^2(-8m^2) + J_{k,n}J_{k,n-1}(16m^2k)$$

$$= J_{k,n}^2(2m^2k^2 + 16m^2 - 2p^2) + J_{k,n-1}J_{k,n+1}(-16m^2 + 2p^2 - 2m^2k^2)$$

$$= (-16m^2 + 2m^2k^2 - 2p^2)(J_{k,n}^2 - J_{k,n-1}J_{k,n+1})(8m^2 + m^2k^2 - p^2)^{-2}2(8m^2 + m^2k^2 - p^2)$$

Since $det(J^n) = (-2)^n$, we have

$$= 2(J_{k,n}^2 - J_{k,n-1}J_{k,n+1})(8m^2 + m^2k^2 - p^2)^{-1} = (-2)^n$$

$$(J_{k,n}^2 - J_{k,n-1}J_{k,n+1}) = (-2)^{n-1}(8m^2 + m^2k^2 - p^2)$$

Hence the result. From the proof of this theorem, we conclude that

$$[2mJ_{k,n+2} - (p+mk)J_{k,n+1}][4mJ_{k,n} - (p+mk)J_{k,n-1}] -$$
$$- [4mJ_{k,n+1} - (p+mk)J_{k,n}][2mJ_{k,n+1} - (p+mk)J_{k,n}]$$
$$= (-2)^{n-1}(8m^2 + m^2k^2 - p^2) \quad (12)$$

□

Theorem 5. *For $n \in \mathbf{Z_0}$, the characteristic equation of J^n is given by*

$$x^2 - xj_{k,n} + (-2)^{n-1} = 0$$

Proof. Since J^n is a square matrix then by Cayley Hamilton theorem, we have

$$det(J^n - xI_2) = 0$$

here

$$det(J^n - xI_2) = \begin{vmatrix} \frac{2mJ_{k,n+2} - (p+mk)J_{k,n+1}}{8m^2+m^2k-p^2} - x & \frac{4mJ_{k,n+1} - 2(p+mk)J_{k,n}}{8m^2+m^2k-p^2} \\ \frac{2mJ_{k,n+1} - (p+mk)J_{k,n}}{8m^2+m^2k-p^2} & \frac{4mJ_{k,n+1} - 2(p+mk)J_{k,n-1}}{8m^2+m^2k-p^2} - x \end{vmatrix}$$

$$= (8m^2 + m^2k^2 - p^2)^{-2}\Big([2mJ_{k,n+2} - (p+mk)J_{k,n+1}][4mJ_{k,n} - 2(p+mk)J_{k,n-1}]$$
$$- x(8m^2 + m^2k^2 - p^2)[2mJ_{k,n+2} - (p+mk)J_{k,n+1}]$$
$$- x(8m^2 + m^2k^2 - p^2)[4mJ_{k,n} - 2(p+mk)J_{k,n-1}] + x^2(8m^2 + m^2k^2 - p^2)^2$$
$$- (4mJ_{k,n+1} - 2(p+mk)J_{k,n})(2mJ_{k,n+1} - (p+mk)J_{k,n})\Big)$$

$$= (8m^2 + m^2k^2 - p^2)^{-2}\Big(x^2(8m^2 + m^2k^2 - p^2)^2$$
$$- x(8m^2 + m^2k^2 - p^2)[2mJ_{k,n+2} - (p+mk)J_{k,n+1} + 4mJ_{k,n} - 2(p+mk)J_{k,n-1}]$$
$$+ (2mJ_{k,n+2} - (p+mk)J_{k,n+1})(4mJ_{k,n} - 2(p+mk)J_{k,n-1})$$
$$- (4mJ_{k,n+1} - 2(p+mk)J_{k,n})(2mJ_{k,n+1} - (p+mk)J_{k,n})\Big)$$

If we consider Corollary 1 and equation (12) we get

$$det(J^n - xI_2) = (8m^2 + m^2k^2 - p^2)^{-2}\Big(x^2(8m^2 + m^2k^2 - p^2)^2 - xj_{k,n}(8m^2 + m^2k^2 - p^2)^2 +$$
$$+ (-2)^{n-1}(8m^2 + m^2k^2 - p^2)^2\Big) = x^2 - xj_{k,n} + (-2)^{n-1} = 0$$

Hence the result. □

Theorem 6. *For any integer $n \geq 0$,*

$$\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} k^i J_{k,n} = J_{k,2n}$$

Proof. If we use Binet's formula, we obtain

$$\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} k^i \Big(p\frac{r^i - s^i}{r - s} + m(r^i + s^i)\Big)$$

$$= \frac{p}{r-s}\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} k^i (r^i - s^i) + m\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} k^i (r^i + s^i)$$

$$= \frac{p}{r-s}\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} ((kr)^i - (ks^i)) + m\sum_{i=0}^{n} \binom{n}{i} 2^{n-i} ((kr)^i + (ks^i))$$

$$= \frac{p}{r-s}\sum_{i=0}^{n} ((2+kr)^i - (2+ks)^i) + m\sum_{i=0}^{n} ((2+kr)^i + (2+ks)^i)$$

$$= \frac{p}{r-s}((2+kr)^n - (2+ks)^n) + m((2+kr)^n + (2+ks)^n)$$

$$= \frac{p}{r-s}(r^{2n} - s^{2n}) + m(r^{2n} + s^{2n})$$

This completes the proof. □

Theorem 7.

$$\lim_{n\to\infty} \frac{J_{k,n}}{J_{k,n-1}} = s.$$

Proof. If we use Binet's formula, we get

$$\lim_{n\to\infty} \frac{\frac{p}{r-s}(r^n - s^n) + m(r^n + s^n)}{\frac{p}{r-s}(r^{n-1} - s^{n-1}) + m(r^{n-1} + s^{n-1})}$$

For $r \geq s$ and taking into account that

$$\lim_{n\to\infty} (\frac{r}{s})^n = 0$$

$$\lim_{n\to\infty} \frac{\frac{p}{r-s}(\frac{r^n}{s^n} - 1) + m(\frac{r^n}{s^n} + 1)}{\frac{p}{r-s}(\frac{r^{n-1}}{s^n} - \frac{1}{s}) + m(\frac{r^{n-1}}{s^n} + \frac{1}{s})} = \frac{-\frac{p}{r-s} + m}{-\frac{p}{s(r-s)} + \frac{m}{s}} = s$$

This completes the proof. □

4. Conclusion

We gave a new generalization of the known Jacobstal sequence and we called Jacobstal-Like sequence $\langle J_{k,n}\rangle$ and then we studied on Jacobstal-Like sequence $\langle J_{k,n}\rangle$ and $k-$Jacobsthal-Lucas sequence $\langle j_{k,n}\rangle$. By using two special matrices we found Binet formula and Cassini identity for Jacobstal-Like sequence $\langle J_{k,n}\rangle$. Also we obtained some of the interesting properties of Jacobstal-Like sequence and finally we introduced a limit for Jacobstal-Like sequence $\langle J_{k,n}\rangle$.

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