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Sampling Theory Associated with q-Sturm-Liouville Operator with Discontinuity Conditions

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Abstract. In this paper, we define a q-Sturm–Liouville problem with discontinuity conditions and prove that it is self adjoint in $L_2(0, \pi)$. We show that eigenfunctions of this problem are in the form of a complete system. A sampling theorem is proved for integral transforms whose kernels are basic functions and the integral is of Jackson's type.

Key Words and Phrases: q-Sturm-Liouville operator, completeness of eigenfunctions, sampling theory, self-adjoint operator.

2010 Mathematics Subject Classifications: 34L10, 39A13, 47B25, 94A20

1. Introduction

Let us consider a q-Sturm-Liouville equation of the form

$$l(y) := -\frac{1}{q} D_{q^{-1}} D_q y(x) + u(x)y(x) = \lambda y(x), \quad 0 < x < \pi, \quad \lambda \in \mathbb{C}, \tag{1.1}$$

together with the discontinuity conditions at a point $a \in (0, \pi)$

$$y(a+0) = \gamma y(a-0), \quad D_{q^{-1}}(a+0) = \gamma^{-1} D_{q^{-1}} y(a-0), \tag{1.2}$$

and boundary conditions

$$D_{q^{-1}} y(0) = 0, \quad y(\pi) = 0, \tag{1.3}$$

where $u(x) \in L_q^2(0, \pi)$ real function, γ is real; $\gamma \neq 1$, $\gamma > 0$.

In [1], it is worth mentioning that this work based on the q-difference operator which is attributed to Jackson. In recent years, many papers subject to the boundary value problems consisting a q-Jackson derivative in the classical Sturm-Lioville problem have occured. In [2]-[4], q-Sturm-Liouville problems are investigated and a space of boundary values of the minimal operator and describe all maximal dissipative, self-adjoint, maximal accretive and other extensions of q-Sturm-Liouville operators in terms of boundary conditions are raised. A theorem on completeness of the system of eigenfunctions and associated functions of dissipative operators are proved by using the Lidskii's theorem.

Also, there are a lot of physical models involving q-difference and their related problems in [5]-[7]. The construction of expansions in q-Fourier series ([8]) was followed by the derivation of the q-sampling theorems in [10], [11]. The sampling theory associated with q-type of Sturm-Liouville equations is conceived (see [12]). In [13], Annaby and Mansour obtained asymptotic formulae for eigenvalues and eigenfunctions of q-type of Sturm-Liouville problems.

In [14]-[16], Allahverdiev and Tuna investigated the continuous spectrum of the singular q-Sturm-Liouville operators and established some criteria under which the q-Sturm-Liouville equation is of limit-point case at infinity. In [17], authors established a Parseval equality and an expansion formula in eigenfunctions for a singular q-Sturm-Liouville operator on the whole line. (Also, Allahverdiev and Tuna investigated the resolvent operator of a singular q-Dirac system (see [18])).

2. Preliminaries on q-calculus

In this section, some of the q-notations which will be used throughout the paper are given. These standard notations are founded in [20].

Let q be a positive number with $0 < q < 1$. The q-difference operator D_q is defined as

$$D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)}.$$

When required q will be replaced by q^{-1}. The following facts can be verified directly from the definition and will be used often

$$D_{q^{-1}} f(x) = (D_q f)(q^{-1}x), \quad D_q^2 f(q^{-1}x) = q D_q [D_q f(q^{-1}x)] = D_{q^{-1}} D_q f(x).$$

Associated with this operator there is a nonsymmetric formula for the
q-differentiation of a product

$$D_q[f(x)g(x)] = f(qx) D_q g(x) + g(x) D_q f(x). \tag{2.1}$$

The q-integral usually associated with the name of Jackson is defined in the interval $(0, \pi)$, as

$$\int_0^\pi f(x) d_q x = (1-q) \sum_{n=0}^{\infty} f(\pi q^n) \pi q^n.$$

Let $L_q^2(0, \pi)$ be the space of all complex-valued functions defined on $(0, \pi)$, such that

$$\|f\| = \left(\int_0^\pi |f(x)|^2 d_q x \right)^{\frac{1}{2}} < \infty.$$

The space $L_q^2(0, \pi)$ is a separable Hilbert space (see [9]) with the inner product

$$< f, g >= \int_0^\pi f(x) \overline{g(x)} d_q x.$$

If f and g are both q-regular at zero, there is a rule of q-integration by parts given by

$$\int_0^\pi g(x)D_qf(x)d_qx = (fg)(\pi) - (fg)(0) - \int_0^\pi D_qg(x)f(qx)d_qx. \quad (2.2)$$

The q appearing in the argument of f in the right-hand side integrand is another manifestation of the symmetry that is everywhere present in q-calculus. As an important special case, we have

$$\int_0^\pi D_qf(x)d_qx = (f)(\pi) - (f)(0). \quad (2.3)$$

Theorem 2.1. *(see [3]) The eigenvalues problem (1.1)-(1.3) form an infinite sequence of real numbers which can be ordered in an ascending way. Moreover, the set of all normalized eigenfunctions of (1.1)-(1.3) forms an orthonormal basis for $L_q^2(0,\pi)$.*

Essential in our discussion will be the q-Wronskian of two functions f and g defined as

$$W_q(f,g)(x) = f(x)D_qg(x) - g(x)D_qf(x). \quad (2.4)$$

Theorem 2.2. *(Kramer's Lemma see [21]) Let $I \subset \mathbb{R}$ be a bounded interval. $K(x,t)$ be a kernel belonging to $L^2(I)$ for each fixed t in a suitable subset D of $\mathbb{R}$. Suppose also that, for some sequence of points belonging to D, $\{K(x,\lambda_n)\}$ is an orthogonal basis for $L^2(I)$. Under these conditions, every function f written in the form*

$$f(t) = \int_I g(x)K(x,t)d_qx,$$

admits the sampling expansion

$$f(t) = \sum_{n=0}^{\infty} f(\lambda_n)\frac{\int_I \overline{K(x,\lambda_n)}K(x,t)d_qx}{\int_I |K(x,t)|^2 d_qx},$$

the sampling series converges absolutely and uniformly on every set $C \subset D$ for which $\|K(x,t)\|$ is bounded.

Lemma 2.1. *(see [2]) Let $f(.)$, $g(.)$ in $L_q^2(0,\pi)$ be defined on $[0,q^{-1}]$. Then, for $x \in (0,\pi]$ we have*

$$< D_qf, g >= f(\pi)\overline{g(\pi q^{-1})} - \lim_{n\to\infty} f(\pi q^n)\overline{g(\pi q^{n-1})} + < f, -\frac{1}{q}D_{q^{-1}}g >, \quad (2.5)$$

$$< -\frac{1}{q}D_{q^{-1}}f, g >= \lim_{n\to\infty} f(\pi q^{n-1})\overline{g(\pi q^n)} - f(piq^{-1})\overline{g(\pi)} + < f, D_qg > . \quad (2.6)$$

3. Self-Adjoint Problem

Theorem 3.1. *The q-Sturm-Liouville eigenvalue problem (1.1)-(1.3) is self-adjoint on* $C_q^2(0) \cap L_q^2(0,\pi)$.

Proof. We first prove that $y(.)$, $z(.)$ in $L_q^2(0,\pi)$, we have the following q-Lagrange's identity

$$\int_0^\pi \left(ly(x)\overline{z(x)} - y(x)\overline{lz(x)} \right) d_q x = [y,z](\pi) - \lim_{n\to\infty} [y,z](\pi q^n), \tag{3.1}$$

where

$$[y,z](x) := y(x)\overline{D_{q^{-1}} z(x)} - D_{q^{-1}} y(x)\overline{z(x)}. \tag{3.2}$$

Applying (2.6) with $f(x) = D_q y(x)$ and $g(x) = z(x)$, we obtain

$$< -\frac{1}{q} D_{q^{-1}} D_q y(x), z(x) >$$

$$= -(D_q y)(\pi q^{-1})\overline{z(\pi)} + \lim_{n\to\infty} (D_q y)(\pi q^{n-1})\overline{z(\pi q^n)} + < D_q y, D_q z >$$

$$= -D_{q^{-1}} y(\pi)\overline{z(\pi)} + \lim_{n\to\infty} D_{q^{-1}} y(\pi q^n)\overline{z(\pi q^n)} + < D_q y, D_q z > . \tag{3.3}$$

Applying (2.5) with $fx) = y(x)$, $g(x) = D_q z(x)$, we obtain

$$\begin{aligned} < D_q y, D_q z > &= y(\pi)\overline{D_q z(\pi q^{-1})} - \lim_{n\to\infty} y(\pi q^n)\overline{D_q z(\pi q^{n-1})} \\ &+ < y, -\frac{1}{q} D_{q^{-1}} D_q z > \\ &= y(\pi)\overline{D_{q^{-1}} z(\pi)} - \lim_{n\to\infty} y(\pi q^n)\overline{D_{q^{-1}} z(\pi q^n)} \\ &+ < y, -\frac{1}{q} D_{q^{-1}} D_q z > . \end{aligned}$$

Therefore,

$$< -\frac{1}{q} D_{q^{-1}} D_q y(x), z(x) > = [y,z](\pi) - \lim_{n\to\infty} [y,z](\pi q^n) + < y, -\frac{1}{q} D_{q^{-1}} D_q z > . \tag{3.4}$$

Lagrange's identity (3.1) results from (3.4) and the reality of $u(x)$. Letting $y(.), z(.)$ in $C_q^2(0)$ and assuming the that they satisfy (1.2)-(1.3),we obtain

$$D_{q^{-1}} y(0) = 0, \quad D_{q^{-1}} z(0) = 0. \tag{3.5}$$

The continuity of $y(.), z(.)$ at zero implies that $\lim_{n\to\infty} [y,z](\pi q^n) = [y,z](0)$. Then (3.4) will be

$$< -\frac{1}{q} D_{q^{-1}} D_q y, z > = [y,z](\pi) - [y,z](0) + < y, -\frac{1}{q} D_{q^{-1}} D_q z > .$$

From (3.5), we have

$$[y,z](0) = y(0)\overline{D_{q^{-1}} z(0)} - D_{q^{-1}} y(0)\overline{z(0)} = 0.$$

Similarly,

$$[y,z](\pi) = y(\pi)\overline{D_{q^{-1}}z(\pi)} - D_{q^{-1}}y(\pi)\overline{z(\pi)} = 0.$$

Since $u(x)$ is real valued, then

$$\begin{aligned} < l(y), z > &= < -\frac{1}{q}D_{q^{-1}}D_q y(x) + u(x)y(x), z(x) > \\ &= < -\frac{1}{q}D_{q^{-1}}D_q y(x), z(x) > + < u(x)y(x), z(x) > \\ &= < y, -\frac{1}{q}D_{q^{-1}}D_q z > + < y(x), u(x)z(x) > \\ &= < y, l(z) >, \end{aligned}$$

i.e. l is a self-adjoint operator. □

A λ defines as an eigenvalue of the problem (1.1)-(1.3) if there is a non-trivial solution $\eta(.)$ which satisfies the problem at this λ. $\eta(.)$ is an eigenfunction of the problem (1.1)-(1.3) corresponding to the eigenvalue λ. The multiplicity of an eigenvalue is defined to be the number of linearly independent solutions corresponding to it. In particular an eigenvalue is simple if only if it has only one linearly independent solution.

Lemma 3.1. *Let f and g be q-regular at zero. The Wronskian $W_q(f,g)(x)$ of the q-Sturm-Liouville problem (1.1) does not defend on x.*

Proof. The proof can be done similar to [12]. □

Let $\eta(x,\lambda)$ be the solution of equation (1.1) with discontinuity conditions (1.2) and initial conditions

$$\eta(0,\lambda) = 1, \quad D_{q^{-1}}\eta(0,\lambda) = 0, \tag{3.6}$$

and $\xi(x,\lambda)$ be the solution of equation (1.1) with discontinuity conditions (1.2) and

$$\xi(\pi,\lambda) = 0, \quad D_{q^{-1}}\xi(\pi,\lambda) = 1. \tag{3.7}$$

Since the q-Wronskian is independent of x, we can evaluate it at $x = \pi$ and use the above conditions on ξ in order to write

$$W_q(\eta,\xi)(\lambda) = W_q(\lambda) = -\eta(\pi,\lambda) = D_{q^{-1}}\xi(0,\lambda). \tag{3.8}$$

It follows from the condition (1.3) that $W_q(\lambda) = 0$ if and only if λ is an eigenvalue of q-Sturm-Liouville problem (1.1).

Lemma 3.2. *The eigenvalues and eigenfunctions of the q-Sturm-Liouville problem (1.1)-(1.3) have the following properties:*

i. The eigenvalues are real.

ii. Eigenfunctions that belong to different eigenvalues are orthogonal.

iii. All eigenvalues are simple.

Proof. i. Let λ_0 be an eigenvalue with an eigenfunction $\eta_0(.)$. Then,

$$< l(\eta_0), \eta_0 >=< \eta_0, l(\eta_0) > . \tag{3.9}$$

Since $l(\eta_0) = \lambda_0\eta_0$, then

$$(\lambda_0 - \overline{\lambda_0}) \int_0^{\pi} |\eta_0(x)|^2 d_qx. \tag{3.10}$$

Since $\eta_0(.)$is nontrivial then $\lambda_0 = \overline{\lambda_0}$, which proves **i**.

ii. Let λ, μ be two distinct eigenvalues with corresponding eigenfunctions $\eta(.)$, $\xi(.)$, respectively. Then,

$$(\lambda - \mu) \int_0^{\pi} \eta(x)\overline{\xi(x)} d_qx = 0.$$

Since $\lambda \equiv \mu$,then $\eta(.)$ and $\xi(.)$ are orthogonal.

iii. Let λ_0 be an eigenvalue with two eigenfunctions $\eta_1(.)$ and $\eta_2(.)$. From [2] (see Corollary 2.15) we can prove that the functions $\{\eta_1(.), \eta_2(.)\}$ are linearly dependent by proving that their q-Wronskian vanishes at $x = 0$. Indeed,

$$\begin{aligned} W_q(\eta_1, \eta_2)(0) &= \eta_1(0)D_q\eta_2(0) - \eta_2(0)D_q\eta_1(0) \\ &= \eta_1(0)\overline{D_{q^{-1}}\eta_2(0)} - \eta_2(0)\overline{D_{q^{-1}}\eta_1(0)} = [\eta_1, \eta_2] = 0, \end{aligned}$$

since both η_1 and η_2 satisfy (1.3).

□

4. The q-sampling theory

Theorem 4.1. *Let $\eta(x, \lambda)$ and $\xi(x, \lambda)$ be the solutions of (1.1) selected as above. Then every function f of the form*

$$f(\lambda) = \int_0^{\pi} u(x)\eta(x, \lambda)d_qx, \quad u \in L_q^2(0, \pi), \tag{4.1}$$

can be written as the Lagrange-type sampling expansion

$$f(\lambda) = \sum_{n=0}^{\infty} f(\lambda_n) \frac{W_q(\lambda)}{\dot{W}_q(\lambda_n)(\lambda_n - \lambda)}, \tag{4.2}$$

where $W_q(\lambda)$ is the q-Wronskian of the functions $\eta(x, \lambda)$ and $\xi(x, \lambda)$.

Proof. Multiply equation (1.1) by $\eta(x, \lambda_n)$. Then consider again equation (1.1), but replace λ by λ_n and multiply this last equation by $\eta(x, \lambda)$. Subtracting the two results yields

$$(\lambda - \lambda_n)\eta(x, \lambda)\eta(x, \lambda_n) = D_q^2\eta(q^{-1}x, \lambda_n)\eta(x, \lambda) - D_q^2\eta(q^{-1}x, \lambda)\eta(x, \lambda_n).$$

From the rule for the q-differentiation of a product (2.1), we can write

$$(\lambda-\lambda_n)\eta(x,\lambda)\eta(x,\lambda_n)=D_q\left[D_q\eta(q^{-1}x,\lambda_n)\eta(x,\lambda)-D_q\eta(q^{-1}x,\lambda)\eta(x,\lambda_n)\right].$$

Performing a q-integration by means of (2.3) gives

$$\begin{aligned}&(\lambda-\lambda_n)\int_0^\pi \eta(x,\lambda)\eta(x,\lambda_n)d_qx\\ =&\int_0^\pi D_q\left[D_q\eta(q^{-1}x,\lambda_n)\eta(x,\lambda)-D_q\eta(q^{-1}x,\lambda)\eta(x,\lambda_n)\right]d_qx\\ =&D_q\eta(q^{-1}\pi,\lambda_n)\eta(\pi,\lambda)-D_q\eta(q^{-1}\pi,\lambda)\eta(\pi,\lambda_n)\\ &-\left(D_q\eta(q^{-1}0,\lambda_n)\eta(0,\lambda)-D_q\eta(q^{-1}0,\lambda)\eta(0,\lambda_n)\right).\end{aligned}$$

From (1.3) and (3.6) conditions, we get

$$\begin{aligned}(\lambda-\lambda_n)\int_0^\pi \eta(x,\lambda)\eta(x,\lambda_n)d_qx =&D_q\eta(q^{-1}\pi,\lambda_n)\eta(\pi,\lambda)-D_q\eta(q^{-1}\pi,\lambda)\eta(\pi,\lambda_n)\\ =&\eta(\pi,\lambda)D_{q^{-1}}\eta(\pi,\lambda_n)-\eta(\pi,\lambda_n)D_{q^{-1}}\eta(\pi,\lambda).\end{aligned}$$

From (3.8), we have

$$(\lambda-\lambda_n)\int_0^\pi \eta(x,\lambda)\eta(x,\lambda_n)d_qx=-W_q(\lambda)D_{q^{-1}}\eta(\pi,\lambda_n)+W_q(\lambda_n)D_{q^{-1}}\eta(\pi,\lambda).$$

From λ_n eigenvalues be zeros of $W_q(\lambda)$ characteristic function of q-Sturm-Liouville problem (1.1)-(1.3), we obtain $W_q(\lambda_n)=0$. Then, we have

$$(\lambda-\lambda_n)\int_0^\pi \eta(x,\lambda)\eta(x,\lambda_n)d_qx=-W_q(\lambda)D_{q^{-1}}\eta(\pi,\lambda_n).$$

As a result

$$\int_0^\pi \eta(x,\lambda)\eta(x,\lambda_n)d_qx=\frac{-W_q(\lambda)D_{q^{-1}}\eta(\pi,\lambda_n)}{(\lambda-\lambda_n)},$$

and taking the limit a $\lambda\to\lambda_n$ gives

$$\int_0^\pi |\eta(x,\lambda_n)|^2d_qx=-\dot{W}_q(\lambda_n)D_{q^{-1}}\eta(\pi,\lambda_n).$$

We can therefore apply Kramer's lemma and write an integral transform of the form (4.1) as

$$f(\lambda)=\sum_{n=0}^{\infty} f(\lambda_n)\frac{W_q(\lambda)}{\dot{W}_q(\lambda_n)(\lambda_n-\lambda)}. \tag{4.3}$$

□

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