Journal of Contemporary Applied Mathematics
V. 10, No 2, 2020, December
ISSN 2222-5498

Computation of Periodic Solution of Linear Constant Coefficients Ordinary Differential Equation

M. Kzaz, M. Issaoui, F.Maach

Abstract. The aim of this paper is to determinate the periodic solution of linear constant coefficients ordinary differential equation, of any order. First, we give the exact solution in the form of a sum of integrals twice the number of roots of the characteristic polynomial of the differential equation. Then, we propose a numerical method to approximate the solution, when at least one of the roots of the characteristic polynomial is not available. Finally, We will present numerical examples to illustrate the effectiveness of the proposed method.

Key Words and Phrases: Boole summation formula; Fourier series; Linear differential equation; Sidi transformation; Residue formula.

2010 Mathematics Subject Classifications: 30XX, 42A16, 1: 47E05

1. Introduction

In this paper, we are dealing with the $2C$-periodic solution of the linear differential equation with constant coefficients:

$$y^{(p)}(x) + \lambda_{p-1}y^{(p-1)}(x) + \cdots + \lambda_1 y^{'}(x) + \lambda_0 y(x) = f(x). \qquad (1)$$

Because of its extensive use by engineers of all varieties, most notably electrical engineers, several authors have been interested in this type of equation, whether from the theoretical point of view or from the numerical point of view. See [4,8,9,10,11].

By using the Fourier series, we have developed two calculation methods: the first gives us, when all the roots of the characteristic polynomial of (1) are available, the exact periodic solution of (1). The second gives us, when at least one of the roots of the characteristic polynomial is not available, a very good approximation of the periodic solution .

In fact, let $P(z) = z^p + \lambda_{p-1}z^{p-1} + \cdots + \lambda_1 z + \lambda_0$ the characteristic polynomial of degree p of (1). It is well known that, when $P(in\frac{\pi}{C}) \neq 0$ for $n = 0, \pm1, \pm2, \ldots$, the differential equation (1) has a $2C$-periodic solution given by:

$$y(x) = \sum_{n=-\infty}^{\infty} \frac{c_n}{P(in\frac{\pi}{C})} e^{in\frac{\pi}{C}x}, \text{ with } c_n = \frac{1}{2C}\int_{-C}^{C} f(u)e^{-in\frac{\pi}{C}u}du. \qquad (2)$$

It is clear that formula (2), requires the summation of a very large number of terms in order to obtain a suitable approximation of $y(x)$, and this, after having correctly calculated the coefficients c_n.

The aim of this paper is to compute, with high accurate and with minimal cost as much as possible, the periodic solution of (1). We begin by giving, when all the roots of $P(z)$ are available, the solution $y(x)$ under the form of the sum of some integrals which could by computed by appropriate integration methods.

Note that since in this case, (the roots of $P(z)$ are available), the general solution of the homogeneous differential equation is well known in the literature, see e.g. [2], the particular solution of (1) provided by the first formula proposed in this paper, will allow us to acquire all the solutions of (1).

As this formula can only be numerically practical if all the roots of the polyomial P are available, we propose a numerical method to approximate the periodic solution when the roots of the polyomial P are not all availabe.

This method will be all the more precise as the forcing term f belongs to a class of functions that will be specified.

Note also that the methods proposed in this paper can solve more general problems than those treated by Boyd in [7], where the author was interested in the approximation of Fourier series of the form $\sum_{n=-\infty}^{\infty} c_n e^{in\frac{\pi}{C}x}$, where the analytical form of c_n is given and has an asymptotic development in $\frac{1}{n^k}$.

The paper is organised as follows:

In the next section, we study the case where all the roots of the characteristic polynomial P are available and we show how to get, in a compact form, the sum of the serie (2).

In section 3, we study the case where roots of the characteristic polynomial P are not all available. We propose to split the sum into two parts. The first part, $S_M(x) = \sum_{n=-M}^{M} \frac{c_n}{P(\frac{in\pi}{c})} e^{in\frac{\pi}{C}x}$, where M is a moderate integer, will be approximated by computing the Fourier coefficients c_n by using trapezoidal rule, after introducing an appropriate variable change proposed by Sidi in [5]. The second part $\widetilde{S}_M(x) = \sum_{n=M+1}^{\infty} \left(\frac{c_n}{P(\frac{in\pi}{c})} e^{in\frac{\pi}{C}x} + \frac{c_{-n}}{P(\frac{-in\pi}{c})} e^{-in\frac{\pi}{C}x} \right)$ will be approximated by using certain numbre of the derivatives of the function f at the extermities $-C$ and C and by using the well known Boole summation formula which is given by

$$\sum_{k=M+1}^{\infty} (-1)^k g(k) =$$

$$= (-1)^{M+1}\left(\frac{g(M)}{2} + \sum_{\substack{k=1\\ k \text{ even}}}^{L} (2^k-1)\frac{B_k}{k!}g^{(k-1)}(M) - \frac{(-1)^L}{2(L-1)!}\int_M^\infty \widetilde{E}_{L-1}(t)g^{(L)}(t)dt\right) \tag{3}$$

where the B_k are the Bernoulli numbers and the $\widetilde{E}_L(t)$ are the periodic Euler functions.

In section 4, we give numerical examples which illustrat the effectiveness of the proposed formulas.

At last, we give an appendix where we give some technical results.

2. The roots of P are available

In this paragraph, we are dealing with the case where all the roots of the characteristic polynomial are available.

Let us first recall the classical result on summation of trigonometric series.

Theorem 2-1: [3, p. 270] *Let r be a rational function with a zero of order ≥ 2 at infinity and with no pole at any integer and let $0 \leq \xi \leq 2\pi$. Then*

$$\sum_{-\infty}^{\infty} r(n)e^{in\xi} = -2\pi i \sum res\left\{r(z)\frac{e^{iz\xi}}{e^{2i\pi z}-1}\right\},$$

where the sum on the right involves the residues at all poles of r.

Theorem 2-2: *The $2C$-periodic solution of the differential equation (1), with the characteristic polynomial of degree $p \geq 2$, $P(z) = \prod_{k=1}^{m}(z-z_k)^{m_k}$ with $p = \sum_{k=1}^{m} m_k \geq 2$, and where the forcing term f is supposed to be integrable on $[-C, C]$, is given by*

$$y(x) =$$

$$= -i\left(\frac{C}{i\pi}\right)^p \sum_{k=1}^{m}\frac{1}{(m_k-1)!}\left((-1)^p\int_0^{\pi-\frac{\pi}{C}x} f(x+\frac{C}{\pi}t)\beta_k(y)\,dt + \int_0^{\pi+\frac{\pi}{C}x} f(x-\frac{C}{\pi}t)\lambda_k(t)\,dt\right) \tag{4}$$

where

$$\beta_k(t) = \lim_{z\to\frac{-C}{i\pi}z_k}\frac{d^{m_k-1}}{dz^{m_k-1}}\left(\frac{1}{\prod_{\substack{l=1\\ l\neq m_k}}^{m}\left(z+\frac{C}{i\pi}z_l\right)^{m_l}}\frac{e^{izt}}{e^{2i\pi z}-1}\right),$$

$$\lambda_k(t) = \lim_{z\to\frac{C}{i\pi}z_k}\frac{d^{m_k-1}}{dz^{m_k-1}}\left(\frac{1}{\prod_{\substack{l=1\\ l\neq m_k}}^{m}\left(z-\frac{C}{i\pi}z_l\right)^{m_l}}\frac{e^{izt}}{e^{2i\pi z}-1}\right). \tag{5}$$

Proof: Since $degP \geq 2$, the serie $\sum_{n=-\infty}^{\infty}\frac{e^{in\frac{\pi}{C}(x-t)}}{P(\frac{in\pi}{c})}$ est uniformly convergent on $[-C, C]$. Moerever, f is integrable on $[-C, C]$. Therefore, the solution $y(x)$ can be written in the form:

$$y(x) = \frac{1}{2C}\int_{-C}^{C} f(t) \sum_{n=-\infty}^{\infty} \frac{e^{in\frac{\pi}{C}(x-t)}}{P(in\frac{\pi}{c})} dt.$$

Setting $L_m(z) = \prod_{l=1}^{m} \left(z - \frac{C}{i\pi}z_l\right)^{m_k}$, and making the change of variable $u = x - t$, the expression of $y(x)$ becomes:

$$y(x) = \frac{1}{2C}\left(\frac{C}{i\pi}\right)^p \int_{x-C}^{x+C} f(x-u) \sum_{n=-\infty}^{\infty} \frac{e^{in\frac{\pi}{C}u}}{L_m(n)} du.$$

Making the change of variable $t = \frac{-\pi}{C}u$ on $[x - C, 0]$ and the change of variable $t = \frac{\pi}{C}u$ on $[0, x + C]$, the precedent expression of $y(x)$ becomes:

$$y(x) =$$

$$= \frac{1}{2\pi}\left(\frac{C}{i\pi}\right)^p \left(\int_0^{\pi-\frac{\pi}{C}x} f(x+\frac{C}{\pi}t) \sum_{n=-\infty}^{\infty} \frac{\exp(int)}{L_m(-n)} dt + \int_0^{\pi+\frac{\pi}{C}x} f(x-\frac{C}{\pi}t) \sum_{n=-\infty}^{\infty} \frac{\exp(int)}{L_m(n)} dt\right). \tag{6}$$

Now, since for all x in $[-C, C]$, $0 \le \pi - \frac{\pi}{C}x \le 2\pi$ and $0 \le \pi + \frac{\pi}{C}x \le 2\pi$, we get from Theorem 2-1,
$\sum_{n=-\infty}^{\infty} \frac{\exp(int)}{L_m(n)} = -2\pi i \sum res\left\{\frac{1}{L_m(z)}\frac{e^{izt}}{e^{2i\pi z}-1}\right\}$ and $\sum_{n=-\infty}^{\infty} \frac{\exp(int)}{L_m(-n)} = -2\pi i \sum res\left\{\frac{1}{L_m(-z)}\frac{e^{izt}}{e^{2i\pi z}-1}\right\}$.

Or, from residue's definition, we have $res\left\{\frac{1}{L_m(z)}\frac{e^{izt}}{e^{2i\pi z}-1}\right\} = \frac{\lambda_k(t)}{(m_k-1)!}$ and $res\left\{\frac{1}{L_m(-z)}\frac{e^{izt}}{e^{2i\pi z}-1}\right\} = (-1)^p \frac{\beta_k(t)}{(m_k-1)!}$ where $\lambda_k(t)$ and $\beta_k(t)$ are given by (5). Hence, we get the result with (6).

■

Remark 2-3: In the particular case where all the roots of the characteristic polynomial are simple, the expression of $y(x)$ is reduced to the following expression:

$$y(x) = \frac{C}{\pi}\sum_{k=1}^{p} \frac{1}{(1-e^{2Cz_k})\prod_{\substack{l=1\\ l\neq k}}^{p}(z_k - z_l)} \times$$

$$\times\left(e^{2Cz_k}\int_0^{\pi-\frac{\pi}{C}x} f(x+\frac{C}{\pi}y)e^{-\frac{C}{\pi}z_k y}dy + \int_0^{\pi+\frac{\pi}{C}x} f(x-\frac{C}{\pi}y)e^{\frac{C}{\pi}z_k y}dy\right). \tag{7}$$

Remark 2-4: The integrals appearing in (4) and in (7), can be calculated using any suitable numerical integration method.

3. The roots of P are not all available

In this paragraph, we are dealing with the case where the roots of the characteristic polynomial P of the differential equation (1) are not all available. We propose to split the serie (2) in the two following parts:

$$y(x) = S_M(x) + \widetilde{S}_M(x)$$

where

$$S_M(x) = \sum_{n=-M}^{M} \frac{c_n}{P(in\frac{\pi}{C})} e^{in\frac{\pi}{C}x}$$

and

$$\widetilde{S}_M(x) = \sum_{n=M+1}^{\infty} \left(\frac{c_n}{P(in\frac{\pi}{C})} e^{in\frac{\pi}{C}x} + \frac{c_{-n}}{P(-in\frac{\pi}{C})} e^{-in\frac{\pi}{C}x} \right).$$

Each of the sums $S_M(x)$ and $\widetilde{S}_M(x)$ will be approximated in a different way. The integer M is chosen moderate and fixed.

This method is proposed in the case where the forcing term f is of class $C^L\left[-C, C\right]$, for a given L.

3.1. Computation of the sum $S_M(x)$

Let M a moderate and fixed integer. ($M \simeq 10$). To compute effectively the Fourier coefficients $(c_n)_{-M \le n \le M}$, we can use any efficient and suitable numerical method. (See [1]). Among efficient methods for moderate M, we can use certain transformations $(\Psi_m(t))_m$, given by Sidi in [5] and defined by:

$$\left.\begin{array}{lcl} \Psi_0(t) & = & t \\ \Psi_1(t) & = & \frac{1}{2}(1-\cos \pi t) \\ \Psi_m(t) & = & \Psi_{m-2}(t) - \frac{\Gamma(\frac{m}{2})}{2\sqrt{\pi}\Gamma(\frac{m+1}{2})}(\sin \pi t)^{m-1}\cos(\pi t) \end{array}\right. , \ t \in [0,1]. \tag{8}$$

By returning to the interval [0,1] via the change of variable $u = 2Ct - C$, we obtain

$$S_M(x) = \int_0^1 f(2Ct - C)\left(\sum_{n=-M}^{M} \frac{(-1)^n}{P(in\frac{\pi}{C})} e^{in(\frac{\pi}{C}x - 2\pi t)} \right) dt.$$

Now, using the Sidi transformation given by (8), the sum $S_M(x)$ becomes:

$$S_M(x) = \int_0^1 f(2C\Psi_m(t) - C)\Psi'_m(t) G_{M,m}(t,x)dt \tag{9}$$

with $G_{M,m}(t,x) = \sum\limits_{n=-M}^{M} (-1)^n \frac{e^{-2in\pi\Psi_m(t)}}{P(in\frac{\pi}{C})} e^{in\frac{\pi}{C}x}$.

Applying the trapezoidal rule to order N, (because the transformation of Sidi has been proposed to make this rule optimal), we obtain the following approximation:

$$S_M(x) = S_{M,N}(x,m) + R_{M,N}(x,m) \tag{10}$$

with

$$S_{M,N}(x,m) = \frac{1}{N}\sum_{j=1}^{N-1} f(2C\Psi_m(\frac{j}{N}) - C)\Psi'_m(\frac{j}{N})G_{M,m}(\frac{j}{N},x) \tag{11}$$

and

$$R_{M,N}(x,m) = O\left(N^{-(m+1)\inf((\delta_0+1),(\gamma_0+1))}\right) \text{ as } N \to \infty. \tag{12}$$

This, if we suppose

i) $\left(\sum\limits_{n=-M}^{M} \frac{(-1)^n}{P(inb)} e^{in(bx-2\pi t)}\right) f(2Ct - C) \sim \sum\limits_{0\le s} m_s t^{\delta_s}, \quad t \to 0^+$

ii) $\left(\sum\limits_{n=-M}^{M} \frac{(-1)^n}{P(inb)} e^{in(bx-2\pi t)}\right) f(2Ct - C) \sim \sum\limits_{0\le s} n_s (1-t)^{\gamma_s}, \quad t \to 1^-$

where the γ_s and δ_s are distinct complex numbers that satisfy

$$\begin{aligned} -1 &< \Re\gamma_0 \le \Re\gamma_1 \le \Re\gamma_2 \le \cdots, \lim_{s\to\infty} \Re\gamma_s = +\infty, \\ -1 &< \Re\delta_0 \le \Re\delta_1 \le \Re\delta_2 \le \cdots, \lim_{s\to\infty} \Re\delta_s = +\infty. \end{aligned}$$

The estimate of the quadrature error given by (12) is deduced from Theorem 4-1 given by Sidi in [5, p. 335]. Better still, the estimate (12) can be improved, again according to Theorem 4-1 of Sidi, by obtaining

$$R_{M,N}(x,m) = O\left(N^{-(m+1)\inf((\delta_1+1),(\gamma_1+1))}\right) \text{ as } N \to \infty$$

when we consider the transformation $\Psi_m(t)$, where m is an integer that can be written in the form:

$$m = \frac{q - \inf(\delta_0,\gamma_0)}{1 + \inf(\delta_0,\gamma_0)}$$

where q is an even positive integer.

Let us also note that if the forcing term f is highly oscillatory, there is a large litteraure offering suitable methods for computing $S_M(x)$. See e.g. [6].

3.2. Computation of the sum $\widetilde{S}_M(x)$

Let us begin by giving the following notations.

Notation: In the whole of this subsection, we are using the following notations:

Let $I = \{j \in \{0,...,p-1\}, \lambda_j \neq 0\}$ where the λ_j are the coefficients of the characteristic polynomial P and let $i_p = \min I$.

Let us note Q the function $Q(z)=z^{-p}P(z)-1$ and let $\delta_k=\sum\limits_{j+i=k} d_j\zeta_i$, $k=0,..,K(p-i_p)$ where the d_j are the coefficients of the polynomial : $\sum\limits_{k=0}^{K}(-1)^k Q^k\left(\frac{1}{z}\right)=\sum\limits_{k=0}^{K(p-i_p)} d_k z^k$ and $\zeta_i=\begin{cases} f^{(i)}(C)-f^{(i)}(-C) & , \quad 0\le i\le K \\ 0 & , \quad K+1\le i\end{cases}$.

Let us set

$$\widetilde{R}_{M,K,L}(x)=\frac{1}{2C}R_{M,K}(x)+\frac{1}{2C}\sum_{k=0}^{K(p-i_p)}\frac{\delta_k}{(ib)^{k+p+1}}R_k\left(MK,L,x\right) \tag{13}$$

with $b=\frac{\pi}{C}$ and

$$R_k\left(MK,L,x\right)=\frac{(-1)^{MK}}{2(2L-1)!}\int_0^\infty \widetilde{E}_{2L-1}(t)\left(h_k^{(2L)}(MK+t)+h_k^{(2L)}(-MK-t)\right)dt \tag{14}$$

where $h_k(t)=\frac{e^{ibxt}}{t^{k+p+1}}$, $\widetilde{E}_{2L-1}(t)$ the periodic Euler function and with

$$\begin{aligned} R_{M,K}(x)= \; & (-1)^K\sum_{k=0}^{K}\zeta_k\sum_{n=M+1}^{\infty}\frac{(-1)^n}{(inb)^{k+p+1}}\left(\frac{Q^{K+1}(inb)}{1+Q(inb)}e^{inbx}-(-1)^{k+p}\frac{Q^{K+1}(-inb)}{1+Q(-inb)}e^{-inbx}\right) \\ & +\int_{-C}^{C}f^{(1+K)}(y)\sum_{k=0}^{K(p-i_p)}d_k\left(\sum_{n=M+1}^{\infty}\frac{e^{inb(x-y)}-(-1)^{k+p+K}e^{-inb(x-y)}}{(inb)^{k+p+1+K}}\right)dy \\ & -(-1)^K\int_{-C}^{C}f^{(1+K)}(y)\sum_{n=M+1}^{\infty}\frac{\frac{Q^{K+1}(inb)}{1+Q(inb)}e^{inb(x-y)}-(-1)^{p+K}\frac{Q^{K+1}(-inb)}{1+Q(-inb)}e^{-inb(x-y)}}{(inb)^{p+1+K}}dy \end{aligned} \tag{15}$$

Proposition 3-1: *With the notations given above, we have*

$$\widetilde{S}_M(x)=\widetilde{S}_{M,K,L}(x)+\widetilde{R}_{M,K,L}\left(x\right) \tag{16}$$

with $\widetilde{R}_{M,K,L}\left(x\right)$ *given by (13), and*

$$\widetilde{S}_{M,K,L}(x)=-\frac{1}{2C\left(ib\right)^{p+1}}\sum_{k=0}^{K(p-i_p)}\frac{\delta_k}{(ib)^k}S_{M,K,L}(k,x) \tag{17}$$

where the $S_{M,K,L}(k,x)$ *are given by*

$$\begin{aligned} S_{M,K,L}(k,x)= & \sum_{n=M+1}^{MK}\frac{(-1)^n}{n^{k+p+1}}\left(e^{ibxn}-(-1)^{k+p}e^{-ibxn}\right) \\ & -\frac{(-1)^{MK}}{2\left(MK\right)^{k+p+1}}\left(e^{ibxMK}-(-1)^{k+p}e^{-ibxMK}+\frac{1}{(k+p)!}\sum_{j=1}^{L}(4^j-1)\frac{B_{2j}}{j}U_{M,K}(k,x,j)\right) \end{aligned} \tag{18}$$

where the B_{2j} are the Bernoulli numbers and with

$$U_{M,K}(k,x,j)=\sum_{l=0}^{2j-1}\frac{(k+p+l)!}{l!(2j-1-l)!}\frac{1}{(MK)^{l}}\left((-1)^{l}\,e^{ibxMK}+(-1)^{k+p}\,e^{-ibxMK}\right). \tag{19}$$

Proof: From Lemma A-1, we have:

$$\begin{aligned}\frac{c_n}{P(inb)} &= \frac{(-1)^{n+1}}{2C(inb)^{p+1}}\sum_{k=0}^{K(p-i_p)}\frac{\delta_k}{(inb)^k}+\frac{(-1)^{K+n}}{2C(inb)^{p+1}}\frac{Q^{K+1}(inb)}{1+Q(inb)}\sum_{k=0}^{K}\frac{\zeta_k}{(inb)^k}\\ &+\frac{1}{2C}\frac{1}{(inb)^{K+p+1}}\sum_{k=0}^{K(p-i_p)}\frac{d_k}{(inb)^k}\int_{-C}^{C}f^{(1+K)}(u)e^{-inbu}du\\ &-\frac{(-1)^K}{2C(inb)^{K+p+1}}\frac{Q^{K+1}(inb)}{1+Q(inb)}\int_{-C}^{C}f^{(1+K)}(u)e^{-inbu}du.\end{aligned}$$

Since $\sum\limits_{n=M+1}^{\infty}(-1)^n e^{inbx}\sum\limits_{k=0}^{K(p-i_p)}\frac{\delta_k}{(inb)^{k+p+1}}=\sum\limits_{k=0}^{K(p-i_p)}\delta_k\sum\limits_{n=M+1}^{\infty}(-1)^n\frac{e^{inbx}}{(inb)^{k+p+1}}$, we get

$$\begin{aligned}\sum_{n=M+1}^{\infty}\frac{c_n}{P(inb)}e^{inbx} &= -\frac{1}{2C}\sum_{k=0}^{K(p-i_p)}\delta_k\sum_{n=M+1}^{\infty}(-1)^n\frac{e^{inbx}}{(inb)^{k+p+1}}\\ &+\frac{(-1)^K}{2C}\sum_{k=0}^{K}\zeta_k\sum_{n=M+1}^{\infty}(-1)^n\frac{Q^{K+1}(inb)}{1+Q(inb)}\frac{e^{inbx}}{(inb)^{k+p+1}}\\ &+\frac{1}{2C}\int_{-C}^{C}f^{(1+K)}(y)\sum_{k=0}^{K(p-i_p)}d_k\left(\sum_{n=M+1}^{\infty}\frac{e^{inb(x-y)}}{(inb)^{k+p+1+K}}\right)dy\\ &-\frac{(-1)^K}{2C}\int_{-C}^{C}f^{(1+K)}(y)\left(\sum_{n=M+1}^{\infty}\frac{e^{inb(x-y)}}{(inb)^{p+1+K}}\frac{Q^{K+1}(inb)}{1+Q(inb)}\right)dy.\end{aligned}$$

Replacing n by $-n$ in the previous expression, we obtain the expression of $\sum\limits_{n=M+1}^{\infty}\frac{c_{-n}}{P(-inb)}e^{-inbx}$. Hence, with $R_{M,K}(x)$ defined by (15), the expression of $\widetilde{S}_M(x)$ becomes:

$$\widetilde{S}_M(x)=-\frac{1}{2C}\sum_{k=0}^{K(p-i_p)}\frac{\delta_k}{(ib)^{k+p+1}}\sum_{n=M+1}^{\infty}(-1)^n\frac{e^{inbx}-(-1)^{k+p}\,e^{-inbx}}{n^{k+p+1}}+\frac{1}{2C}R_{M,K}(x).$$

With $h_k(z)=\frac{e^{ibxz}}{z^{k+p+1}}$ and with The Boole summation formula recalled in (3), we get:

$$\begin{aligned}\sum_{n=MK+1}^{\infty}(-1)^n h_k(n) &= -(-1)^{MK}\left(\frac{h_k(MK)}{2}+\sum_{j=1}^{L}(2^{2j}-1)\frac{B_{2j}}{2j!}h_k^{(2j-1)}(MK)\right)\\ &\quad+\frac{(-1)^{MK}}{2(2L-1)!}\int_0^{\infty}\widetilde{E}_{2L-1}(t)h_k^{(2L)}(MK+t)dt.\end{aligned}$$

Similarly, we get by remplacing in the previous equality, n by $-n$, we obtain the corresponding equality for $\sum\limits_{n=M+1}^{\infty}(-1)^n\frac{(-1)^{k+p+1}}{n^{k+p+1}}e^{-inbx}$.

Summing now the two obtained equalities, we get:

$$\sum_{n=MK+1}^{\infty} (-1)^n (h_k(n) + h_k(-n)) = \frac{(-1)^{MK+1}}{2} (h_k(MK) + h_k(-MK))$$
$$-(-1)^{MK}\sum_{j=1}^{L}(4^j - 1)\frac{B_{2j}}{2j!}\left(h_k^{(2j-1)}(MK) - h_k^{(2j-1)}(-MK)\right) + R_k(MK, L, x)$$

where $R_k(MK, L, x)$ is given by (14). Thus, by setting

$$\widetilde{R}_{M,K,L}(x) = \frac{1}{2C} R_{M,K}(x) + \frac{1}{2C} \sum_{k=0}^{K(p-i_p)} \frac{\delta_k}{(ib)^{k+p+1}} R_k(MK, L, x)$$

and

$$\begin{aligned} S_{M,K,L}(k, x) \quad = \quad & \sum_{n=M+1}^{MK} (-1)^n (h_k(n) + h_k(-n)) \\ & -\tfrac{(-1)^{MK}}{2} (h_k(MK) + h_k(-MK)) \\ & -(-1)^{MK} \sum_{j=1}^{L} (4^j - 1) \tfrac{B_{2j}}{2j!} \left(h_k^{(2j-1)}(MK) - h_k^{(2j-1)}(-MK)\right) \end{aligned}$$

we get the result by applying Leibniz formula to the $h_k^{(2j-1)}$.

■

3.3. Error analysis of the solution

In this subsection, we will give an error bound due to the approximation of $\widetilde{S}_M(x)$ by $\widetilde{S}_{M,K,L}(x)$.

Theorem 3-2: *Let $P(z) = z^p + \lambda_{p-1}z^{p-1} + \cdots + \lambda_1 z + \lambda_0$ the characteristic polynomial of the differential equation (1).*

Let $I = \{j \in \{0, ..., p-1\}, \lambda_j \neq 0\}$ and let $j_p = \max I$ and $i_p = \min I$.

Let $Q(z) = z^{-p}P(z) - 1$ and let M and K such that:

1) $\forall n \geq M+1$, $|Q(inb)| < 1$ and $|1 + Q(inb)| \geq \frac{1}{2}$.

2) $\exists A > 0, \forall n \geq M+1 : |Q(inb)| \leq \frac{A|\lambda_{j_p}|}{(nb)^{p-j_p}}$.

3) $\exists B > 0$, $\int_{-C}^{C} \left|f^{(1+K)}(y)\right| dy \leq M^K B$.

With $\mu = p + (p - j_p)(K+1)$,we have the two following error bounds:

$x \neq 0$,

$$\begin{aligned} \left|\widetilde{S}_M(x) - \widetilde{S}_{M,K,L}(x)\right| \quad \leq \quad & \tfrac{4M}{(Mb)^{\mu-K}} \left(\tfrac{A|\lambda_{j_p}|}{bM}\right)^{K+1} \left(\tfrac{B}{(K+\mu)b^K} + \sum_{k=0}^{K} \tfrac{|\zeta_k|}{(k+\mu)(Mb)^k}\right) \\ & + \tfrac{2B}{(Mb)^p b^{K+1}} \sum_{k=0}^{K} \tfrac{|d_k|}{(k+p+K)(Mb)^k} \\ & + \tfrac{8}{(3C)^{2L}} \tfrac{b}{(bMK)^{p+2}} \sum_{k=0}^{K(p-i_p)} \tfrac{1}{(k+p)!} \tfrac{|\delta_k|}{(bMK)^k} \sum_{l=0}^{2L} C_{2L}^{l} \tfrac{(k+p+l)!}{k+p+2+l} \tfrac{x^{2L-l}}{(MKb)^l}. \end{aligned} \tag{20}$$

$x=0$,

$$\begin{aligned}\left|\widetilde{S}_M(x)-\widetilde{S}_{M,K,L}(x)\right| &\leq \tfrac{4M}{(Mb)^{\mu-K}}\left(\tfrac{A\left|\lambda_{j_p}\right|}{bM}\right)^{K+1}\left(\tfrac{B}{(K+\mu)b^K}+\sum_{k=0}^{K}\tfrac{|\zeta_k|}{(k+\mu)(Mb)^k}\right)\\ &+\tfrac{2B}{(Mb)^p b^{K+1}}\sum_{k=0}^{K}\tfrac{|d_k|}{(k+p+K)(Mb)^k}\\ &+\tfrac{4}{(3\pi MK)^{2L}}\tfrac{1}{(MKb)^p}\tfrac{1}{b}\sum_{k=0}^{K(p-i_p)}\left(1-(-1)^{k+p}\right)\tfrac{(k+p+2L-1)!}{(k+p)!}\tfrac{|\delta_k|}{(bMK)^k}.\end{aligned} \tag{21}$$

Proof: We have from (16) and (13),

$$\widetilde{S}_M(x)-\widetilde{S}_{M,K,L}(x)=\frac{1}{2C}R_{M,K}(x)+\frac{1}{2C}\sum_{k=0}^{K(p-i_p)}\frac{\delta_k}{(ib)^{k+p+1}}\delta_k R_k\left(MK,L,x\right),$$

where $R_{M,K}(x)$ is given by (15) and $R_k\left(MK,L,x\right)$ is given by (14).

1) with (15), we get

$$\begin{aligned}|R_{M,K}(x)| &\leq \sum_{k=0}^{K}|\zeta_k|\sum_{n=M+1}^{\infty}\tfrac{1}{(nb)^{k+p+1}}\left(\left|\tfrac{Q^{K+1}(inb)}{1+Q(inb)}\right|+\left|\tfrac{Q^{K+1}(-inb)}{1+Q(-inb)}\right|\right)\\ &+2\left(\sum_{k=0}^{K(p-i_p)}|d_k|\sum_{n=M+1}^{\infty}\tfrac{1}{(nb)^{k+p+1+K}}\right)\int_{-C}^{C}\left|f^{(1+K)}(y)\right|dy\\ &+\sum_{n=M+1}^{\infty}\tfrac{1}{(nb)^{p+1+K}}\left(\left|\tfrac{Q^{K+1}(inb)}{1+Q(inb)}\right|+\left|\tfrac{Q^{K+1}(-inb)}{1+Q(-inb)}\right|\right)\int_{-C}^{C}\left|f^{(1+K)}(y)\right|dy.\end{aligned}$$

On one hand, we have: $\exists A>0, \forall n\geq M+1: |Q\left(inb\right)|\leq\frac{A\left|\lambda_{j_p}\right|}{(nb)^{p-j_p}}$ and $|1+Q\left(inb\right)|\geq\frac{1}{2}$. Thus: $\forall n\geq M+1$, $\left|\frac{Q^{K+1}(inb)}{1+Q(inb)}\right|\leq 2\frac{A^{K+1}\left|\lambda_{j_p}\right|^{K+1}}{(nb)^{(p-j_p)(K+1)}}$.

On the other hand, the forcing term f verifies: $\exists B>0$, $\int_{-C}^{C}\left|f^{(1+K)}(y)\right|dy\leq M^K B$. Hence,

$$\begin{aligned}|R_{M,K}(x)| &\leq 4\tfrac{A^{K+1}\left|\lambda_{j_p}\right|^{K+1}}{b^{1+\mu}}\sum_{k=0}^{K}\tfrac{|\zeta_k|}{b^k}\sum_{n=M+1}^{\infty}\tfrac{1}{n^{k+1+\mu}}\\ &+2\tfrac{M^K B}{b^{p+1+K}}\sum_{k=0}^{K}\tfrac{|d_k|}{b^k}\sum_{n=M+1}^{\infty}\tfrac{1}{n^{k+p+1+K}}\\ &+4\tfrac{M^K B}{b^{1+K+\mu}}A^{K+1}\left|\lambda_{j_p}\right|^{K+1}\sum_{n=M+1}^{\infty}\tfrac{1}{n^{1+K+\mu}}.\end{aligned}$$

Moerever, we have for all $j>0$, $\sum_{n=M+1}^{\infty}\frac{1}{n^{j+1}}\leq\frac{1}{jM^j}$. Thus,

$$\begin{aligned}|R_{M,K}(x)| &\leq \tfrac{4M}{(Mb)^{\mu-K}}\left(\tfrac{A\left|\lambda_{j_p}\right|}{bM}\right)^{K+1}\left(\sum_{k=0}^{K}\tfrac{|\zeta_k|}{(k+\mu)(Mb)^k}+\tfrac{B}{(K+\mu)b^K}\right)\\ &+\tfrac{2B}{(Mb)^p b^{K+1}}\sum_{k=0}^{K}\tfrac{|d_k|}{(k+p+K)(Mb)^k}.\end{aligned} \tag{22}$$

2) Let us now give an upper bound to the term $\left|\sum_{k=0}^{K(p-i_p)} \frac{\delta_k}{(ib)^{k+p+1}} R_k(MK,L,x)\right|$ where $R_k(MK,L,x)$ is given by (14).

The Fourier expansion for the periodic Euler function is given by:

$$\widetilde{E}_{2L-1}(t) = (-1)^{L+1} \frac{4.\,(2L-1)!}{\pi^{2L}} \sum_{k=1}^{\infty} \frac{\cos((2k+1)\pi t)}{(2k+1)^{2L}}.$$

Thus, we obtain:

$$|R_k(MK,L,x)| \leq \frac{4}{(3\pi)^{2L}} \int_0^{\infty} \left(\left| h_k^{(2L)}(MK+t) + h_k^{(2L)}(-MK-t) \right| \right) dt. \quad (23)$$

To give an upper bound to the term $|R_k(MK,L,x)|$, we are led to distinguish two cases: $x \neq 0$ and $x = 0$.

First case: $x \neq 0$

Using Leibniz formula, we get, $\left|h_k^{(2L)}(MK+t)\right| \leq \frac{1}{(k+p)!} \sum_{l=0}^{2L} C_{2L}^l \frac{(k+p+l)!}{(MK+t)^{k+p+1+l}} (bx)^{2L-l}$, which gives by using (23), $|R_k(MK,L,x)| \leq 8\left(\frac{bx}{3\pi}\right)^{2L} \frac{1}{(k+p)!} \frac{1}{(MK)^{k+p+2}} \sum_{l=0}^{2L} C_{2L}^l \frac{(k+p+l)!}{k+p+2+l} \frac{1}{(MKbx)^l}$. Hence, we obtain:

$$\left|\sum_{k=0}^{K(p-i_p)} \frac{\delta_k}{(ib)^{k+p+1}} R_k(MK,L,x)\right| \leq$$

$$\leq \frac{8}{(3C)^{2L}} \frac{b}{(bMK)^{p+2}} \sum_{k=0}^{K(p-i_p)} \frac{1}{(k+p)!} \frac{|\delta_k|}{(bMK)^k} \sum_{l=0}^{2L} C_{2L}^l \frac{(k+p+l)!}{k+p+2+l} \frac{x^{2L-l}}{(MKb)^l}.$$

Thus, we obtain (20) with (22) and the last inequality.

Second case: $x = 0$. We have $h_k^{(j)}(y) = \frac{(-1)^j}{(k+p)!} \frac{(k+p+j)!}{y^{k+p+1+j}}$ which gives:

$$\left|h_k^{(2L)}(MK+t) + h_k^{(2L)}(-MK-t)\right| = \frac{\left(1-(-1)^{k+p}\right)}{(k+p)!} \frac{(k+p+2L)!}{(MK+t)^{k+p+1+2L}}.$$

Hence, $|R_k(MK,L,x)| \leq \frac{4\left(1-(-1)^{k+p}\right)}{(3\pi)^{2L}} \frac{(k+p+2L-1)!}{(k+p)!} \frac{1}{(MK)^{k+p+2L}}$ which gives,

$$\left|\sum_{k=0}^{K(p-i_p)} \frac{\delta_k}{(ib)^{k+p+1}} R_k(MK,L,0)\right| \leq$$

$$\leq \frac{4}{(3\pi MK)^{2L}} \frac{1}{b(MKb)^p} \sum_{k=0}^{K(p-i_p)} \frac{|\delta_k|}{b^k} \left(1-(-1)^{k+p}\right) \frac{(k+p+2L-1)!}{(k+p)!} \frac{1}{(MK)^k}.$$

Hence, we obtain (21) with (22) and the last inequality.

■

Remark 3-3: We note from the inequalities (20) and (21), that the error of approximation will be all the smaller the smaller the terms $\frac{A\left|\lambda_{j_p}\right|}{bM}$ and $\frac{B}{b^K}$.

4. Numerical examples

In this paragraph, we will consider 2 differential equations. In practice, we will approximate $y\,(x)$, not by $\widetilde{y}(x) = S_{M,N}(x,m) + \widetilde{S}_{M,K,L}(x)$, but by

$$\widetilde{y}(x,l) = S_{M,N}(x,m) + \widetilde{S}_{M,K,L}(x,l), \tag{24}$$

where $S_{M,N}(x,m)$ given by (11), $l \in \{0,...,K(p-i_p)\}$, and

$$\widetilde{S}_{M,K,L}(x,l) = -\frac{1}{2C\,(ib)^{p+1}}\sum_{k=0}^{l}\frac{\delta_k}{(ib)^k}S_{M,K,L}(k,x).$$

The reason is that the error due to the approximation of $\widetilde{S}_M(x)$ by $\widetilde{S}_{M,K,L}(x)$ in (16), can quickly become small compared to the error due to the approximation of $S_M(x)$ by $S_{M,N}(x,m)$ in (10), because this latter approximation will depend closely on the numerical integration method used to approximate the coefficients c_n (except in the case of course where the exact values of c_n are known).

In the following, we will note

$$E(x,l) = \left|y\,(x) - \widetilde{y}(x,l)\right|.$$

and we will consider two differential equations:

$$y^{(3)}(x) + y(x) = f(x) \tag{25}$$

and

$$y^{(3)}(x) - 3y^{'}(x) + 2y(x) = f(x) \tag{26}$$

where f is a 2-periodic function.

For the first equation (25), we have $C = 1$, $P(z) = z^3 + 1 = (z+1)(z - e^{-i\frac{\pi}{3}})(z - e^{i\frac{\pi}{3}})$, $i_p = 0$ and $p = 3$. The exact solution is then given from (7) by:

$$\begin{aligned} y(x) \quad = \quad & \tfrac{1}{2\pi(1+\cos(\frac{\pi}{3}))(e^2-1)}\left(\int_0^{\pi(1-x)} f(x+\tfrac{t}{\pi})e^{\frac{t}{\pi}}dt + e^2\int_0^{\pi(x+1)} f(x-\tfrac{t}{\pi})e^{\frac{-t}{\pi}}dt\right) \\ & + \tfrac{2}{\pi}\,\mathrm{Real}\left(\tfrac{1}{e^{-2z}-1}\int_0^{\pi(1-x)} f(x+\tfrac{t}{\pi})e^{-\frac{z}{\pi}t}dt - \tfrac{1}{e^{2z}-1}\int_0^{\pi(x+1)} f(x-\tfrac{t}{\pi})e^{\frac{z}{\pi}t}dt\right) \end{aligned}$$

with $z = e^{-i\frac{\pi}{3}}$.

For the second equation (26), we have $C = 1$, $P(z) = z^3 - 3z + 2 = (z+2)(z-1)^2$, $i_p = 0$ and $p = 3$. The exact solution is then given from (4) by:

$$\begin{aligned} y(x) &= \tfrac{1}{9\pi(e^4-1)}\left(\int_0^{\pi(1-x)} f(x+\tfrac{1}{\pi}t)e^{\frac{2}{\pi}t}dt + e^4\int_0^{\pi(x+1)} f(x-\tfrac{1}{\pi}t)e^{\frac{-2}{\pi}t}dt\right) \\ &+ \tfrac{e^2}{9\pi^2(e^2-1)^2}\int_0^{\pi(1-x)} f(x+\tfrac{1}{\pi}t)\left(\pi(e^2+5)+3t(e^2-1)\right)e^{-\frac{t}{\pi}}dt \\ &+ \tfrac{1}{9\pi^2(e^2-1)^2}\int_0^{\pi(x+1)} f(x-\tfrac{1}{\pi}t)\left(\pi(7e^2-1)+3(1-e^2)t\right)e^{\frac{t}{\pi}}dt. \end{aligned}$$

As first example, we consider $f(x) = e^{ax}$ on $[-1,1]$, and as second example, we consider $f(x) = \frac{1}{x+a}$ on $[-1,1]$. For these two examples, we have:

a) $S_{M,N}(x,m) = \frac{1}{N}\sum_{j=1}^{N-1} f(2\Psi_m(\frac{j}{N})-1)\Psi'_m(\frac{j}{N})G_{M,m}(\frac{j}{N},x)$ with $\Psi_m(t)$ given by (8) and $G_{M,m}(\frac{j}{N},x) = \sum_{n=-M}^{M}(-1)^n \frac{e^{-2in\pi\Psi_m(t)}}{1-in^3\pi^3}e^{in\pi x}$ for the first equation and $G_{M,m}(\frac{j}{N},x) = \sum_{n=-M}^{M}(-1)^n \frac{e^{-2in\pi\Psi_m(t)}}{2-3in\pi-in^3\pi^3}e^{in\pi x}$ for the second equation.

b) $\widetilde{S}_{M,K,L}(x,l) = -\frac{1}{2\pi^4}\sum_{k=0}^{l}\frac{\delta_k}{(i\pi)^k}S_{M,K,L}(k,x)$ with $S_{M,K,L}(k,x)$ given by (18), and $\delta_k = (e^a - e^{-a})\sum_{l+j=k} a^j d_l$ for the first example, and $\delta_k = \sum_{l+j=k}(-1)^j j!\left(\frac{1}{(a+1)^{j+1}} - \frac{1}{(a-1)^{j+1}}\right)d_l$, for the second case, where the d_l, for the two examples, are the coefficients of the polynomial $\sum_{l=0}^{K}(-1)^l z^{3l}$ for the first equation and the d_l are the coefficients of the polynomial $\sum_{l=0}^{K}(-1)^l(2z-3)^l z^{2l}$ for the second equation.

We get the following results with: $M = K = 10$, $L = 5$, $\Psi_m(t) = \Psi_4(t) = t - \frac{2}{3\pi}\sin(2\pi t) + \frac{1}{12\pi}\sin(4\pi t)$, with $N = 60$ and for differents values of l, $0 \le l \le 3K$.

For $f(t) = e^{at}$ on $[-1,1]$, we get for the first equation

	$a=1$			$a=\pi$		
	$E(0,l)$	$E(\frac{1}{2},l)$	$E(\frac{9}{10},l)$	$E(0,l)$	$E(\frac{1}{2},l)$	$E(\frac{9}{10},l)$
$l=0$	$8.7\ 10^{-10}$	$3.3\ 10^{-8}$	$8.5\ 10^{-8}$	$8.3\ 10^{-8}$	$9.7\ 10^{-7}$	$2.7\ 10^{-6}$
$l=3$	$1.3\ 10^{-15}$	$2.9\ 10^{-14}$	$6.6\ 10^{-11}$	$7.1\ 10^{-10}$	$4.9\ 10^{-10}$	$8.3\ 10^{-10}$
$l=6$	$6.6\ 10^{-16}$	$2.9\ 10^{-14}$	$6.6\ 10^{-11}$	$5.3\ 10^{-14}$	$3.0\ 10^{-13}$	$6.3\ 10^{-10}$

and for the second equation

	$a=1$			$a=\pi$		
	$E(0,l)$	$E(\frac{1}{2},l)$	$E(\frac{9}{10},l)$	$E(0,l)$	$E(\frac{1}{2},l)$	$E(\frac{9}{10},l)$
$l=0$	$3.4\ 10^{-9}$	$3.1\ 10^{-8}$	$9.1\ 10^{-8}$	$1.0\ 10^{-7}$	$9.5\ 10^{-7}$	$2.8\ 10^{-7}$
$l=3$	$8.5\ 10^{-12}$	$4.9\ 10^{-12}$	$4.7\ 10^{-11}$	$9.8\ 10^{-10}$	$6.7\ 10^{-10}$	$2.0\ 10^{-9}$
$l=6$	$4.2\ 10^{-16}$	$2.9\ 10^{-14}$	$6.6\ 10^{-11}$	$7.3\ 10^{-14}$	$6.2\ 10^{-13}$	$8.4\ 10^{-13}$

For $f(t) = \frac{1}{a+t}$ on $[-1,1]$, we get for the first equation

	$a=2$			$a=20$		
	$E(0,l)$	$E(\frac{1}{2},l)$	$E(\frac{9}{10},l)$	$E(0,l)$	$E(\frac{1}{2},l)$	$E(\frac{9}{10},l)$
$l=0$	$7.0\ 10^{-10}$	$1.3\ 10^{-8}$	$2.9\ 10^{-8}$	$2.7\ 10^{-14}$	$7.1\ 10^{-12}$	$1.7\ 10^{-11}$
$l=3$	$7.9\ 10^{-12}$	$3.2\ 10^{-12}$	$1.3\ 10^{-13}$	$1.6\ 10^{-16}$	$1.5\ 10^{-16}$	$1.4\ 10^{-13}$
$l=6$	$9.9\ 10^{-15}$	$6.1\ 10^{-14}$	$1.9\ 10^{-11}$	$4.5\ 10^{-19}$	$6.1\ 10^{-17}$	$1.4\ 10^{-13}$

and for the second equation

	$a=2$			$a=20$		
	$E(0,l)$	$E(\frac{1}{2},l)$	$E(\frac{9}{10},l)$	$E(0,l)$	$E(\frac{1}{2},l)$	$E(\frac{9}{10},l)$
$l=0$	$1.4\ 10^{-9}$	$1.3\ 10^{-8}$	$2.7\ 10^{-10}$	$5.5\ 10^{-12}$	$1.0\ 10^{-11}$	$4.0\ 10^{-12}$
$l=3$	$1.2\ 10^{-11}$	$5.2\ 10^{-12}$	$9.4\ 10^{-12}$	$1.5\ 10^{-14}$	$8.5\ 10^{-15}$	$1.0\ 10^{-13}$
$l=6$	$1.0\ 10^{-14}$	$6.7\ 10^{-14}$	$1.9\ 10^{-11}$	$1.5\ 10^{-19}$	$6.5\ 10^{-17}$	$1.4\ 10^{-13}$

Remark 4-1: When x is close to the extremities C or $-C$, we can get better results by increasing the value of N and the values of L and l. Indeed, we obtain with $N = 120$ in (11) and $L = 100$ in (18), the following values for $E(\frac{9}{10}, l)$ for different values of l,

	Equation (25)			
	$f(t) = e^{at}$		$f(t) = \frac{1}{a+t}$	
l	$a=1$	$a=\pi$	$a=2$	$a=20$
0	$8.5\ 10^{-8}$	$2.7\ 10^{-6}$	$2.9\ 10^{-8}$	$1.7\ 10^{-11}$
5	$1.3\ 10^{-15}$	$1.1\ 10^{-11}$	$4.4\ 10^{-13}$	$2.9\ 10^{-18}$
10	$1.1\ 10^{-18}$	$9.5\ 10^{-18}$	$5.0\ 10^{-17}$	$4.7\ 10^{-22}$

	Equation (26)			
	$f(t) = e^{at}$		$f(t) = \frac{1}{a+t}$	
l	$a=1$	$a=\pi$	$a=2$	$a=20$
0	$9.0\ 10^{-8}$	$2.8\ 10^{-6}$	$2.7\ 10^{-8}$	$4.1\ 10^{-12}$
5	$3.9\ 10^{-14}$	$1.5\ 10^{-11}$	$5.2\ 10^{-12}$	$9.5\ 10^{-17}$
10	$4.9\ 10^{-17}$	$3.1\ 10^{-15}$	$3.1\ 10^{-16}$	$1.6\ 10^{-18}$

5. Conclusion

As we can see, the two results given in this paper, do not only apply to the approximation of the periodic solution of the differential equation (1), but they can also be useful in solving any problem whose solution is written in the form (2). It also remains to propose other methods when, simultaneously, at least one of the roots of the characteristic polynomial is not available and the function is not differentiable or its order of differentiability is very low.

6. Appendix

Lemma A-1: *Let* $K \in \mathbb{N}^*$, *let* $f \in C^K[-C, C]$ *and let* $c_n = \frac{1}{2C}\int_{-C}^{C} f(u)e^{-inbu}du$ *with* $b = \frac{\pi}{C}$.

Let P *a polynomial with* $\deg P = p$ *and* i_p *the degree of its lowest degree monomial and let* Q *the function given by:* $Q(z) = z^{-p}P(z) - 1$.

Let $M \in \mathbb{N}^*$ *such that:* $\forall n \geq M+1$, $|Q(inb)| < 1$.

Let $\delta_k = \sum\limits_{j+i=k} d_j\zeta_i$, $k = 0, .., K(p - i_p)$ *where* $\zeta_i = \begin{cases} f^{(i)}(C) - f^{(i)}(-C) & , \ 0 \leq i \leq K \\ 0 & , \ K+1 \leq i \end{cases}$ and *the* d_j *are the coefficients of the polynomial* $\sum\limits_{k=0}^{K} (-1)^k Q^k\left(\frac{1}{z}\right) = \sum\limits_{k=0}^{K(p-i_p)} d_k z^k$. *We have:*

$$\begin{aligned} \frac{c_n}{P(inb)} &= \frac{(-1)^{n+1}}{2C(inb)^{p+1}} \sum_{k=0}^{K(p-i_p)} \frac{\delta_k}{(inb)^k} + \frac{(-1)^{K+n}}{2C(inb)^{p+1}} \frac{Q^{K+1}(inb)}{1+Q(inb)} \sum_{k=0}^{K} \frac{\zeta_k}{(inb)^k} \\ &+ \frac{1}{2C} \frac{1}{(inb)^{K+p+1}} \sum_{k=0}^{K(p-i_p)} \frac{d_k}{(inb)^k} \int_{-C}^{C} f^{(1+K)}(u)e^{-inbu}du \\ &- \frac{(-1)^K}{2C(inb)^{K+p+1}} \frac{Q^{K+1}(inb)}{1+Q(inb)} \int_{-C}^{C} f^{(1+K)}(u)e^{-inbu}du. \end{aligned}$$

Proof: On one hand, we have by simple integration by parts:

$$c_n = \frac{(-1)^{n+1}}{2C} \sum_{k=0}^{K} \frac{\zeta_k}{(inb)^{k+1}} + \frac{1}{2C\,(inb)^{K+1}} \int_{-C}^{C} f^{(K+1)}(u)e^{-inbu}du.$$

On the other hand, we have $P(z) = z^p\,(1 + Q(z))$, which gives for $n \geq M+1$,

$$\frac{1}{P(inb)} = \frac{1}{(inb)^p}\left(\sum_{k=0}^{K} (-1)^k Q^k(inb) + (-1)^{K+1}\frac{Q^{K+1}(inb)}{1+Q(inb)}\right).$$

Hence,

$$\begin{aligned} (inb)^{p+1}\frac{c_n}{P(inb)} &= \frac{(-1)^{n+1}}{2C} \sum_{k=0}^{K} (-1)^k Q^k(inb) \sum_{k=0}^{K} \frac{\zeta_k}{(inb)^k} + \frac{(-1)^{K+n}}{2C} \frac{Q^{K+1}(inb)}{1+Q(inb)} \sum_{k=0}^{K} \frac{\zeta_k}{(inb)^k} \\ &+ \frac{1}{2C} \frac{1}{(inb)^K} \sum_{k=0}^{K} (-1)^k Q^k(inb) \int_{-C}^{C} f^{(1+K)}(u)e^{-inbu}du \\ &- \frac{(-1)^K}{2C(inb)^K} \frac{Q^{K+1}(inb)}{1+Q(inb)} \int_{-C}^{C} f^{(1+K)}(u)e^{-inbu}du. \end{aligned}$$

Or, with $\sum\limits_{k=0}^{K} (-1)^k Q^k(inb) = \sum\limits_{k=0}^{K(p-i_p)} \frac{d_k}{(inb)^k}$, we get, $\sum\limits_{k=0}^{K} (-1)^k Q^k(inb) \sum\limits_{k=0}^{K} \frac{\zeta_k}{(inb)^k} = \sum\limits_{k=0}^{K(p-i_p)} \frac{\delta_k}{(inb)^k}$ with $\delta_k = \sum\limits_{j+i=k} d_j\zeta_i$, $k = 0, .., K(p-i_p)$. Thus, we get the result.

■

References

[1] P.J. Davis, P. Rabinowitz, W. Rheinbolt, Methods of numerical integration, Academic Press, New York, 1984

[2] M. Tenenbaum, H. Pollard, Ordinary differential equations, 1985, Amazon

[3] P. Henrici, Applied and computational complex analysis, V1, J. Wiley & Sons, New york, 1988

[4] C.L.-Y. Wang, Computer implementation of a series solution for constant coefficient Ordinary Differential Equations. Computers & Mathematics with Applications, 31(3), 43-60, 1996

[5] A. Sidi, Extension of a class of Periodizing Variable Transformations for Numerical Integration. Math. Comp., 75(253), 327-343, 2005

[6] D. Huybrechs, S. Vandewalle, On the evaluation of Highly Oscillatory Integrals by analytic continuation. SIAM J. Numer. Anal., 44, 1026-1048, 2006

[7] J. P. Boyd, Acceleration of algebraically-converging Fourier Series when the coefficients have series in powers of 1/n, J. Comput. Phys. 228, 1404-1411, 2009

[8] R. Camporesi, Linear Ordinary Differential Equations with constant coefficients. Revisiting the impulsive response method using factorization, International journal of mathematical education in science and technology. 44(4), 497-514, 2011

[9] V.V. Karachik, Method for constructing solutions of Linear 0rdinary Differential Equations with constant coefficients. Comput. Math. Math. Phys. 52, 219-234, 2012

[10] L.I. W. Roeger, R.E. Mickens, Exact Finite Difference Scheme for Linear Differential Equation with constant coefficients. J. Diff. Equ. Appl., 19(10), 1663-1670, 2013

[11] Soon-Mo Jung, Jaiok Roh, The Linear Differential Equations with complex constant coefficients and Schrödinger equations. J. Appl. Math. Letters, 66, 23-29, 2017

M. Kzaz
University Cadi-Ayyad, Faculty of Sciences and Technics Guelitz,
Departement of Mathematics, B.P. 517, Marrakech, Morocco
E-mail: m.kzaz@uca.ma

M. Issaoui
University Cadi-Ayyad, Faculty of Sciences and Technics Guelitz,
Departement of Mathematics, B.P. 517, Marrakech, Morocco
*E-mail:*m.issaoui@gmail.com

F.Maach
University Cadi-Ayyad, Faculty of Sciences and Technics Guelitz,
Departement of Mathematics, B.P. 517, Marrakech, Morocco
*E-mail:*f.maach@uca.ma

Received 29 September 2020
Accepted 14 November 2020