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On the preservation of the $\mathbb{B}$-Convexity and $\mathbb{B}$-Concavity of functions by Bernstein Operators

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Abstract. In this study, we investigated whether Bernstein operators preserve $\mathbb{B}$-convexity and $\mathbb{B}$-concavity of functions. It has been proven that the Bernstein Operators do not preserve the property $\mathbb{B}$-convexity of functions, but preserve $\mathbb{B}$-concavity of functions.

Key Words and Phrases: $\mathbb{B}$-Convexity, $\mathbb{B}$-Concavity, Bernstein Operators, Shape preserving approximation

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1. Introduction

One of the major problems in approximation theory is to determine whether polynomials approximating functions have the same form as functions. This kind of approach, especially used in Computer Aided Geometric Design, Robotics and Chemistry, is called the shape preserving approach.

Let $f : [0, 1] \to \mathbb{R}$ and $n \in \mathbb{N}$. The polynomial given by

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$

is called nth Bernstein polynomial related to f. These polynomials are very useful for the shape preserving approach. It is well known that if $f \in \mathrm{C}[0, 1]$, then the sequence of the Bernstein polynomials related to f uniformly converges to f . There are many studies on whether these polynomials introduced by S.N. Bernstein [3] in 1912, preserve various convexities. Some of the studies as follows: T. Popoviciu [11] in 1937 showed that if $f : [0, 1] \to \mathbb{R}$ is a k-convex function ($k \in \mathbb{N}_0$) then $B_n(f)$ is k-convex, for each $n \in \mathbb{N}$.

Definition 1.1 ([9]). *A function $f : [0, 1] \to \mathbb{R}$ is said to be starshaped function if $f(\lambda x) \leq \lambda f(x)$ for each $x \in [0, 1]$ and $\lambda \in [0, 1]$.*

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If f differentiable on $[0,1]$, $f(0) = 0$ and $f(x) \geq 0$ for each $x \in (0,1]$, then the starshapedness of f is equivalent to inequality $xf'(x) - f(x) \geq 0$, for all $x \in (0,1]$. In 1967, L. Lupaş [9] showed that if f differentiable on $[0,1]$, $f(0) = 0$ and $f(x) \geq 0$ for each $x \in (0,1]$, then $B_n(f)(0) = 0, B_n(f)(x) \geq 0$ $(x \in (0,1])$ and $B_n(f)$ is starshaped for all $n \in \mathbb{N}$. Also, in 1996, R. Paltanea [10] proved that if f is k-quasiconvex $(k \in \mathbb{N}_0)$ then $B_n(f)$ is k-quasiconvex for each $n \in \mathbb{N}$. In 1989, T.N.T. Goodman [7] established that Bernstein polynomials preserve log-concavity and in 2020, A. Komisarski [8] get a similar result by strengthening of Goodman's result. But, in 2016, O. Vinogradov and A. Ulitskaya [12] have given an example that Bernstein Polynomials do not preserve log-convexity. For further results and detail about the shape-preserving approximation of real functions of one real variable by real polynomials, we refer to the book of S. G. Gal [6].

The aim of this paper is to investigate about whether Bernstein polynomials preserve $\mathbb{B}$-convexity and $\mathbb{B}$-concavity.

The concept of $\mathbb{B}$-convexity was defined by W. Briec and C. Horvath in 2004 [4]. Since this concept related with economy, in next studies concentrated to the set $\mathbb{R}^n_+ := \{(x_1, x_2, ..., x_n) : x_i \geq 0, \quad i = 1, 2, ..., n\}$. The properties of $\mathbb{B}$-convexity were studied widely by W. Briech, C. Horvard, A.M. Rubinov, G. Adilov and I. Yesilce (see [5, 4, 1, 2] and references therein). These topic have been studied to date and one of the results for the set $\mathbb{R}^n_+$ is the following: For $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$, let

$$x \vee y := (\max\{x_1, y_1\}, \max\{x_2, y_2\}, ..., \max\{x_n, y_n\}).$$

Proposition 1.1 ([4]). *Let $A \subset \mathbb{R}^n_+$. A is $\mathbb{B}$-convex iff $\lambda x \vee y \in A$ for all $x, y \in A$ and $\lambda \in [0,1]$.*

Definition 1.2 ([2]). *A function, $f : A \subset \mathbb{R}^n_+ \to \mathbb{R}_+$ is said to be $\mathbb{B}$-convex function, if and only if A is $\mathbb{B}$-convex and $f(\lambda x \vee y) \leq \lambda f(x) \vee f(y)$ for all $x, y \in A$ and $\lambda \in [0,1]$.*

Similarly, if A is $\mathbb{B}$-convex and $f(\lambda x \vee y) \geq \lambda f(x) \vee f(y)$ for all $x, y \in A$ and $\lambda \in [0,1]$ then f is called $\mathbb{B}$-concav function.

2. Main Results

In the sequel of the study, we consider the set $A = [0,1]$ as a $\mathbb{B}$-convex set and non-negative real valued functions defined on $[0,1]$.

Lemma 2.1. *If $f : [0,1] \to \mathbb{R}_+$ is a decreasing function, then f is a $\mathbb{B}$-convex function.*

Proof. Let $x, y \in [0,1]$ and $\lambda \in [0,1]$. There are two cases:

1. For $\lambda x \leq y$, we get $f(\lambda x \vee y) = f(y) \leq \lambda f(x) \vee f(y)$
2. For $\lambda x > y$, we get $f(\lambda x \vee y) = f(\lambda x) \leq f(y) \leq \lambda f(x) \vee f(y)$ since f is a decreasing function.

Thus from above two cases, for all $x, y \in [0,1]$ and $\lambda \in [0,1]$, $f(\lambda x \vee y) \leq \lambda f(x) \vee f(y)$. This shows that f is $\mathbb{B}$-convex. □

Corollary 2.1. *If $f : [0,1] \to \mathbb{R}_+$ is a decreasing function, then $B_n(f)$ is a $\mathbb{B}$-convex function for each $n \in \mathbb{N}$.*

Proof. Since Bernstein polynomials of a decreasing function are decreasing, proof is clear from lemma 2.1. □

Lemma 2.2. *If $f : [0,1] \to \mathbb{R}_+$ is a starshaped function, then f is a $\mathbb{B}$-convex function.*

Proof. Since f is starshaped function, the inequality $f(\lambda x) \le \lambda f(x)$ is provided for all $\lambda \in [0,1]$. Hence, for all $x, y \in [0,1]$ and $\lambda \in [0,1]$, we have $f(\lambda x \vee y) = f(y) \le \lambda f(x) \vee f(y)$ when $\lambda x \le y$ and then $f(\lambda x \vee y) = f(\lambda x) \le \lambda f(x) \le \lambda f(x) \vee f(y)$ when $\lambda x > y$. Thus f is $\mathbb{B}$-convex.

□

Corollary 2.2. *If $f : [0,1] \to \mathbb{R}_+$ is a starshaped function and let $f(0) = 0$, then Bernstein polynomials $B_n(f)$ are $\mathbb{B}$-convex.*

Lemma 2.3. *Let $f : [0,1] \to \mathbb{R}_+$ be an increasing function. Then f is a $\mathbb{B}$-convex function iff f is a starshaped function.*

Proof. If f is a $\mathbb{B}$-convex function, then, we have $f(\lambda x \vee y) \le \lambda f(x) \vee f(y)$, for all $x, y \in [0,1]$ and $\lambda \in [0,1]$. Let us take $y = 0$. Thus, the inequality $f(\lambda x \vee 0) \le \lambda f(x) \vee f(0)$ is provided for all $\lambda, x \in [0,1]$. Also, $f(\lambda x) \ge f(0)$ and $\lambda f(x) \ge f(0)$ since f is increasing. Hence, we get the inequality $f(\lambda x) \le \lambda f(x)$. for all $x \in [0,1]$ and $\lambda \in [0,1]$.

If f is a starshaped function, the $\mathbb{B}$-convexity of the function is clear from the lemma 2.3.

□

So far, we proved that decreasing and starshapedness are sufficient for $\mathbb{B}$-convexity but not necessary. To give an example which is not decreasing and starshaped function, we may use the following lemma.

Lemma 2.4. *Let $f : [0,1] \to \mathbb{R}_+$ and $c \in [0,1]$. If the function f is decreasing on $[0,c]$ and the inequality $f(\lambda x) \le \lambda f(x)$ hold for all $\lambda x \in [c,1]$ with $x, \lambda \in [0,1]$ then f is a $\mathbb{B}$-convex function.*

Proof. When the cases $c = 0$ and $c = 1$, the proof is obvious. Let $x, y \in [0,1]$ and $\lambda \in [0,1]$. If $\lambda x \le y$, then

$$f(\lambda x \vee y) = f(y) \le \lambda f(x) \vee f(y). \tag{1}$$

For $\lambda x > y$, there are three cases. If $\lambda x, y \in [c,1]$, then

$$f(\lambda x \vee y) = f(\lambda x) \le \lambda f(x) \le \lambda f(x) \vee f(y). \tag{2}$$

If $\lambda x, y \in [0,c]$, then

$$f(\lambda x \vee y) = f(\lambda x) \le f(y) \le \lambda f(x) \vee f(y). \tag{3}$$

If $\lambda x \in [c, 1]$ and $y \in [0, c]$, then

$$f(\lambda x \vee y) = f(\lambda x) \leq \lambda f(x) \leq \lambda f(x) \vee f(y). \tag{4}$$

From (1), (2), (3) and (4) we obtain the inequality $f(\lambda x \vee y) \leq \lambda f(x) \vee f(y)$ for all $x, y \in [0, 1]$ and $\lambda \in [0, 1]$. This shows that f is a $\mathbb{B}$-convex function. □

Remark 2.1. *The Bernstein Operators do not preserve the property $\mathbb{B}$-convexity of functions.*

Example 2.1. *Let $f : [0, 1] \to \mathbb{R}_+$ be defined by*

$$f(x) = \begin{cases} 1 - x, & if \quad 0 \leq x \leq \frac{1}{2} \\ x, & if \quad \frac{1}{2} < x \leq 1 \end{cases}$$

From lemma 2.6 this function is $\mathbb{B}$-convex. However, the Bernstein polynomials related to the funtion f do not provide the $\mathbb{B}$-convexity. For example, for $n = 2$

$$B_2(f)(x) = \sum_{k=0}^{2} \binom{2}{k} x^k (1-x)^{2-k} f\left(\frac{k}{2}\right) = 1 - x + x^2$$

If we take $\lambda = \frac{3}{4}$, $x = 1$ and $y = 1/2$, we have $\lambda x = \frac{3}{4} > \frac{1}{2} = y$ and

$$B_2(f)\left(\lambda x = \frac{3}{4}\right) = \frac{13}{16}, \quad B_2(f)\left(y = \frac{1}{2}\right) = \frac{3}{4} \quad and \quad \frac{3}{4} B_n(f)(x = 1) = \frac{3}{4}.$$

As a result of these equalities, we get the inequality

$$B_2(f)\left(\frac{3}{4} \vee \frac{1}{2}\right) = B_2(f)\left(\frac{3}{4}\right) > \frac{3}{4} B_2(f)(1) \vee B_2(f)\left(\frac{1}{2}\right).$$

Thus $B_2(f)$ is not $\mathbb{B}$-convex

Lemma 2.5. *Let $f : [0, 1] \to \mathbb{R}_+$ be a differentiable function on $[0, 1]$. Then, the inequality $f(\lambda x) \geq \lambda f(x)$ holds for all $x \in [0, 1]$ and $\lambda \in [0, 1]$ iff the inequality $x f'(x) - f(x) \leq 0$ holds for all $x \in [0, 1]$.*

Proof. In the case $x = 0$, the proof is clear. Let $x > 0$ and (λ_n) be a sequence in $[0, 1]$ with $\lim_{n\to\infty} \lambda_n = 1$. Since $f(\lambda_n x) \geq \lambda_n f(x)$ for each $n \in \mathbb{N}$, then we have

$$\frac{f(\lambda_n x) - f(x)}{(\lambda_n - 1)x} \leq \frac{f(x)}{x}.$$

Thus, we get

$$f'(x) = \lim_{n\to\infty} \frac{f(\lambda_n x) - f(x)}{(\lambda_n - 1)x} \leq \frac{f(x)}{x}$$

This shows that the inequality $x f'(x) - f(x) \leq 0$ holds.

Now, if f is differentiable on $(0,1]$ then $g(x) = \frac{f(x)}{x}$ is a decreasing function on $(0,1]$ since $\left(\frac{f(x)}{x}\right)' = xf'(x) - f(x) \leq 0$. From this reason, we have $g(\lambda x) \geq g(x)$ i.e. $\frac{f(\lambda x)}{\lambda x} \geq \frac{f(x)}{x}$. Consequently we have the inequality $f(\lambda x) \geq \lambda f(x)$.

□

Theorem 2.1. *Let $f : [0,1] \to \mathbb{R}_+$. Then f is a $\mathbb{B}$-concav function iff f is increasing and the inequaliy $f(\lambda x) \geq \lambda f(x)$ holds for all $x, \lambda \in [0,1]$.*

Proof. Since f is $\mathbb{B}$-concav, the inequality $f(\lambda x \vee y) \geq \lambda f(x) \vee f(y)$ is provided for all $x, y \in [0,1]$ and $\lambda \in [0,1]$. Let $\lambda = 1$. Thus, if $x > y$, then we have

$$f(x \vee y) = f(x) \geq f(x) \vee f(y) \geq f(y).$$

This shows that f is an increasing function on $[0,1]$. Also, if we take $y = 0$, then we obtain

$$f(\lambda x \vee 0) = f(\lambda x) \geq \lambda f(x) \vee f(0) \geq \lambda f(x)$$

for all $x, \lambda \in [0,1]$.

For inverse, let $x, y \in [0,1]$ and $\lambda \in [0,1]$. Since f is increasing,

$$\lambda x \leq y \quad \Rightarrow \quad f(\lambda x \vee y) = f(y) \geq \lambda f(x) \vee f(y) \tag{5}$$

$$\lambda x > y \quad \Rightarrow \quad f(\lambda x \vee y) = f(\lambda x) \geq \lambda f(x) \vee f(y). \tag{6}$$

Consequently, from (5) and (6), we get the inequality $f(\lambda x \vee y) \geq \lambda f(x) \vee f(y)$ for all $x, y \in [0,1]$ and $\lambda \in [0,1]$. Hence f is a $\mathbb{B}$-concav function. □

Corollary 2.3. *Let $f : [0,1] \to \mathbb{R}_+$ be a differentiable function on $[0,1]$. Then f is a $\mathbb{B}$-concav function iff f is increasing and the inequality $xf'(x) - f(x) \leq 0$ holds for all $x \in [0,1]$.*

Theorem 2.2. *If $f : [0,1] \to \mathbb{R}_+$ is a $\mathbb{B}$-concav function then $B_n(f)$ is $\mathbb{B}$-concav for each $n \in \mathbb{N}$.*

Proof. Let $n \in \mathbb{N}$. For prove to the theorem, it is sufficient to show that the nth Bernstein polynomial of the function is increasing and satisfies the inequality $xB_n'(f)(x) - B_n(f)(x) \leq 0$ for all $x \in [0,1]$. For the derivate of the nth Bernstein Polynomial, we have the following equality

$$B_n'(f)(x) = \sum_{k=0}^{n-1} n \binom{n-1}{k} \left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right] x^k (1-x)^{n-k-1}.$$

Moreover $x^k(1-x)^{n-k} \geq 0$ for all $x \in [0,1]$, $k = 0, 1, ..., n$ and $\left[f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right)\right] \geq 0$ since f is increasing (due to $\mathbb{B}$-concavity of f). Thus, $B_n'(f)(x) \geq 0$ for all $x \in [0,1]$. Consequently, $B_n(f)$ is increasing for each $n \in \mathbb{N}$.

For second part of proof, we can easily see the following equalities with simple algebraic operations:

$$\frac{B_n(f)(x)}{x} = \sum_{k=0}^{n-1} \left(\frac{n}{k+1}\right) \binom{n-1}{k} f\left(\frac{k+1}{n}\right) x^k (1-x)^{n-k-1}$$

and

$$B_n^{'}(f)(x) - \frac{B_n(f)(x)}{x} = \sum_{k=0}^{n-1} n \binom{n-1}{k} \left[\left(\frac{k}{k+1}\right) f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \right] x^k (1-x)^{n-k-1}.$$

Since f provides the inequality $f(\lambda x) \geq \lambda f(x)$ for all $x, \lambda \in [0,1]$, we obtain

$$\left(\frac{k}{k+1}\right) f\left(\frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) \leq f\left(\frac{k}{k+1} \frac{k+1}{n}\right) - f\left(\frac{k}{n}\right) = f\left(\frac{k}{n}\right) - f\left(\frac{k}{n}\right) = 0 \quad .$$

Considering this inequality and $\binom{n-1}{k} x^k (1-x)^{n-k-1} \geq 0 \quad \big(x \in [0,1]\big)$, consequently Bernstein polynomial provide the condition

$$x B_n^{'}(f)(x) - B_n(f)(x) \leq 0$$

for all $x \in [0,1]$. In this case $B_n(f)$ is $\mathbb{B}$-concav for each $n \in \mathbb{N}$.

□

Corollary 2.4. *If $f : [0,1] \to \mathbb{R}_+$ is a $\mathbb{B}$-concav function, the it is approached uniformly to f on $[0,1]$ by $\mathbb{B}$-concav polynomials.*

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