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On Spectral Properties of the Family of Fourth Order Differential Operators with Finite and Exponentially Decreasing Coefficients

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Abstract. In the space $L_2(0,\infty)$ we consider a pencil of fourth order differential operators when the principle characteristical polynomial has a root (imaginary unit) of multiplicity three and a simple root $(-i)$ with continuous boundary conditions at zero whose number of conditions depends on the location of spectral parameter λ in a complex plane.The pencil has a characteristical property that its principal part is not self-adjoint and has a threefold continuous spectrum. The linearity or exponential decrease conditions imposed on the coefficients of differential expression ensures the existence of transformation operator that transforms the solution of the equation under consideration to the solution of the equation containing only principal terms. It was proved that if the pencil has finitely many nonreal eigen-values and has no spectral properties, then the finite function $f(x)$ smooth to the 7-th order and whose support does not contain a zero point we have a formula of spectral expansion of a continuous trace in eigen functions of discrete spectrum and in principal functions uniformly convergent for all $x \in [0,\infty)$.

Key Words and Phrases: spectrum, spectral expansion, resolvent, adjoint operator.

2010 Mathematics Subject Classifications: 34L05, 47E05, 47A40, 34L25

In the space $L_2(0,\infty)$ we consider a pencil of differential operators L_λ^α, generated by the differential equation

$$l\left(x,\frac{d}{dx},\lambda\right)y \equiv \left(\frac{d}{dx}-i\lambda\right)^3\left(\frac{d}{dx}+i\lambda\right)y + r(x)\frac{dy}{dx} + (\lambda p(x)+q(x))\,y = 0, \qquad (1)$$

and boundary conditions

$$U_v(y) = \alpha_{v0}y(0,\lambda) + \alpha_{v1}y'(0,\lambda) + \alpha_{v2}y''(0,\lambda) + \alpha_{v3}y'''(0,\lambda) = 0,\ v=\overline{1,3} \qquad (2)$$

where λ is a spectral parameter, $r(x), p(x), q(x)$ are complex valued functions; a) determined and continuous on the finite interval $[0,a]$ with continuous derivatives to the third, fourth and fifth orders, respectively and vanish outside the interval $[0,a]$; b) determined and continuous, exponentially decreasing on the interval $[0,\infty)$;α_{vk} are fixed complex numbers such that the forms $U_v(y)$ are linearly independent, the number of boundary conditions change depending on location of the parameter λ in the complex plane, here $v = \overline{1,3}$, $k = \overline{0,3}$.

Different issues in the aspect of direct spectral analysis and inverse problems related to theory of ordinary differential pencils given on finite intervals [2-6,10,15,17] and infinite intervals [1,2,8,9,11-14,16,18-22] both in the case of simple roots [3,4,8,9,12] and in the case of multiple roots [1,2,6-9,13-18] of the principal characteristical polynomial with coefficients satisfying certain conditions of algebraic relations [6,8,10], with integrability conditions [13,14,16,19,20] and also with periodic and almost periodic coefficients and their small perturbations [5,8,12,18] were studied intensively subsequently up to now. As a result, a certain spectral theory of regular and singular differential pencils was created only in the case of simple roots of the principal characteristical polynomial. A well-defined spectral theory of singular pencils with multiple characteristical roots was not qet created. The study of spectral expansions in the elements of the solutions of singular spectral problems with multiple characteristics is special in difficulty and little-studied section related to generalized theory of series of expansions in eigen-functions of a discrete spectrum and in principle function of the continuous spectrum. In special case of twofold roots of a principal characteristical polynomial, the multiple spectral expansion formulas were obtained in [13, 14].

The present paper is the continuation of the studies [16,19,20], where the equation (1) is studied and the transformation operators transforming the solution of the equation $\left(\frac{d}{dx} - i\lambda\right)^3 \left(\frac{d}{dx} + i\lambda\right) y = 0$ to the solution of the equation (1), were constructed. In [19] it was shown that the pencil L_λ^α with certain conditions of integrability of the coefficients of equation (1), may have in open lower and open upper semipeanes finite or denumerable number of eigenvalues, while the continuous spectrum files the real axis with spectral properties. In [20] being the continuation of [19] it is proved that if a pencil has a finitely many nonreal eigenvalues and has no spectral properties, then for the function $f(x)$ smooth to the 7-th order, finite in the vicinity of a zero and infinity we have a formula of spectral expansion in eigen functions of a discrete spectrum and in principal functions of the continuous function, uniformly convergent for all $x \in [0, \infty)$.

Note that the pencil L_λ^α has a characteristical property: the principle part is a not self-adjoint and has a three-fold continuous spectrum.

In the present paper, at first we assume that: a) the coefficients of the functions $r(x), p(x), q(x)$ in the pencil L_λ^α vanish outside the interval $[0, a]$. Then we study the case b) when the coeffcients of the equation are exponentially decreasing functions on the semi-axis $[0, \infty)$. Here all the notations from [19] are hold.

Let us consider the case a) The functions $r(x), p(x), q(x)$ equal zero outside the interval $[0, a]$. Then for $y_j^+ (x, \lambda, a) = x^{j-1} e^{i\lambda x}, j = \overline{1,3}$, $y_4^- (x, \lambda, a) = e^{-i\lambda x}$ $x > a$. Then the functions $y_j^+ (x, \lambda, a), j = \overline{1,3}$; $y_4^- (x, \lambda, a)$ are determined for all complex $\lambda\, (\lambda \in C)$ and are entire functions representable in the form

$$\begin{array}{ll} y_j (x, \lambda, a) = x^{j-1} e^{i\lambda x} + \int_x^\infty K_j^+ (x,t)\, e^{i\lambda t} dt, & Im\lambda \geq 0 \\ y_4 (x, \lambda) = e^{-i\lambda x} + \int_x^\infty K^- (x,t)\, e^{-i\lambda t} dt, & Im\lambda \leq 0 \end{array} \tag{3}$$

whose kernels $K_j^\pm (x, t)$ satisfy the conditions (6), (7) of [19].

Here, according to the results of [16] the kernels $K_j^{\pm}(x,t) \equiv 0,\ j=\overline{1,3},\ K_j^{-}(x,t) \equiv 0$, for $x+t>2a$.

Thus, the system $\left\{y_j^{+}(x,\lambda,a),\ j=\overline{1,3};\ y_4^{-}(x,\lambda,a)\right\}$ forms the solution of the equation (1) with finite coefficients for all λ and by their means as in [19] we can construct the kernel

$$K_a(x,\xi,\lambda)=\begin{cases} K_a^{+}(x,\xi,\lambda), & Im\lambda>0 \\ K_a^{-}(x,\xi,\lambda), & Im\lambda<0 \end{cases} \tag{4}$$

where

$$K_a^{+}(x,\xi,\lambda)=\begin{cases} \sum_{i=1}^{3}\left[h_i^{+}(\xi,\lambda,a)+\omega_i^{+}(\xi,\lambda,a)\right]y_{i-1}^{+}(x,\lambda,a) & for\ \ \xi<x \\ \sum_{i=1}^{3}\left[h_i^{+}(\xi,\lambda,a)\,y_{i-1}^{+}(x,\lambda,a)-\omega_4^{+}(\xi,\lambda,a)\right]y_0^{+}(x,\lambda,a) & for\ \ \xi>x \end{cases} \tag{5}$$

here $y_j^{+}(x,\xi,\lambda),\ j=\overline{1,3};\ y_4^{-}(x,\lambda)$ were renumbered as $y_0^{+}(x,\xi,\lambda),\ y_1^{+}(x,\xi,\lambda),\ y_2^{+}(x,\xi,\lambda),\ y_0^{-}(x,\xi,\lambda)$, $h_i^{+}(x,\lambda,a)=\frac{A_{ia}(\lambda)}{A_a(\lambda)}$, $A_a(\lambda)=\det\left\|U_i(y_k)\right\|_{i,k=1}^{3}\neq 0$; $A_{ia}(\lambda)$ is a determinant obtained from $A_a(\lambda)$ changing $U_v(y_1)$ by $U_v(y_4)$; $\omega_i^{+}(\xi,\lambda,a)=z_{5-i}^{+}(\xi,\lambda,a),\ i=\overline{1,4}$; $z_i^{+}(x,\lambda,a),\ i=\overline{1,4}$ are the solutions of the conjugated equation $l_\lambda^{*}\left(x,\frac{d}{dx},\lambda\right)z(x,\lambda)=0$

$$K_a^{-}(x,\xi,\lambda)=\begin{cases} \left[h^{-}(\xi,\lambda,a)+\omega_1^{-}(\xi,\lambda,a)\right]y_0^{-}(x,\lambda,a), & for\ \ \xi<x \\ h^{-}(\xi,\lambda,a)\,y_0^{-}(x,\lambda,a)-\sum_{i=2}^{4}\omega_i^{-}(\xi,\lambda,a)\,y_{i-2}^{+}(x,\lambda,a), & for\ \ \xi>x \end{cases} \tag{6}$$

$$h^{-}(x,\lambda,a)=\frac{1}{U_v\left(y_0^{-}\right)}\sum_{i=2}^{4}U_v\left(y_{i-2}^{+}\right)\omega_i^{-}(\xi,\lambda,a),\ z_{5-i}^{-}(\xi,\lambda,a)=\omega_i^{-}(\xi,\lambda,a),\ i=\overline{1,4}.$$

The kernel $K_a(x,\xi,\lambda)$ is a holomorphic function in the upper and lower half-plane with the exception of finitely many eigenvalues

$$\lambda_1^{+}(a),\ldots,\lambda_l^{+}(a), Im\lambda_j^{+}(a)>0, \quad j=\overline{1,l}$$

$$\lambda_1^{-}(a),\ldots,\lambda_m^{-}(a), Im\lambda_j^{-}(a)<0, \quad j=\overline{1,m}$$

and $K_a^{+}(x,\xi,\lambda)$ allows analytic continuation from the upper halfplane to the lower one ($K_a^{-}(x,\xi,\lambda)$ from the lower halfplane to the upper one) and is a meromorphic function. This time $K_a^{+}(x,\xi,\lambda)$ may have finitely many poles $\mu_1^{+},\ldots,\mu_s^{+}$ (for $K_a^{-}(x,\xi,\lambda)$ such numbers may be $\mu_1^{-},\ldots,\mu_n^{-}$) in the form of spectral properties.

Let $f(x)$ be a finite function up to the 7-th order whose support does not contain the point $x=0$. Using the equality

$$\begin{aligned}\delta_a(x-\xi)=&\frac{\partial^4 K_a(x,\xi,\lambda)}{\partial x^4}-2i\lambda\frac{\partial^3 K_a(x,\xi,\lambda)}{\partial x^3}+2i\lambda^3\frac{\partial K_a(x,\xi,\lambda)}{\partial x}-\\ &-\lambda^4 K_a(x,\xi,\lambda)+r(x)\frac{\partial_0 K_a(x,\xi,\lambda)}{\partial x}+(\lambda p(x)+q(x))\,K_a(x,\xi,\lambda)\end{aligned} \tag{7}$$

and integration by parts, we get

$$\int_0^{\infty}K_a(x,\xi,\lambda)\,f(\xi)\,d\xi=\frac{f(x)}{\lambda^4}+O\left(\frac{1}{\lambda^5}\right). \tag{8}$$

Assume

$$y_{\pm}(x,\lambda)=\int_0^{\infty}K_a^{\pm}(x,\xi,\lambda)\,f(\xi)\,d\xi \tag{9}$$

and

$$y_a(x,\lambda)=\begin{cases} y_a^{+}(x,\lambda), & for \quad Im\lambda>0\\ y_a^{-}(x,\lambda), & for \quad Im\lambda<0.\end{cases}$$

Then

$$y_a(x,\lambda)=\frac{f(x)}{\lambda^4}+O\left(\frac{1}{\lambda^5}\right). \tag{10}$$

Assume that the operator $L_a^{\alpha}(\lambda)$ has no spectral properties and it has only finitely many eigenvalues in the above introduced domains. Let Γ_N be a contour of a circle of radius N centered at the zero. Then from (10) it follows that as $N\to\infty$ the following equality is valid:

$$\frac{1}{2\pi i}\int_{\Gamma_N}y_a(x,\lambda)\,d\lambda=0(1),\ \frac{1}{2\pi i}\int_{\Gamma_N}\lambda y_a(x,\lambda)\,d\lambda=0(1),\ \frac{1}{2\pi i}\int_{\Gamma_N}\lambda^2 y_a(x,\lambda)\,d\lambda=0(1) \tag{11}$$

$$\frac{1}{2\pi i}\int_{\Gamma_N}\lambda^3 y_a(x,\lambda)\,d\lambda=f(x)+0(1) \tag{12}$$

and by the Cauchy theorem

$$\frac{1}{2\pi i}\int_{\Gamma_N}\lambda^j y_a(x,\lambda)\,d\lambda=\sum_{j=1}^{l}Res\lambda^j y_a^{+}(x,\lambda)\Big|_{\lambda=\lambda_j^{+}(a)}+\sum_{j=1}^{m}Res\lambda^j y_a^{-}(x,\lambda)\Big|_{\lambda=\lambda_j^{-}(a)}-$$

$$-\frac{1}{2\pi i}\int_{-N}^{N}\lambda^j y_a^{+}(x,\lambda)+\int_{-N}^{N}\lambda^j y_a^{-}(x,\lambda)+d\lambda. \tag{13}$$

Here as $N\to\infty$ we can pass to the limit. Then we find

$$0=\frac{1}{2\pi i}\sum_{j=1}^{l}\int_{|\lambda-\lambda_j^{+}(a)|=\delta}\lambda^j y_a^{+}(x,\lambda)\,d\lambda+\frac{1}{2\pi i}\sum_{j=1}^{m}\int_{|\lambda-\lambda_j^{-}(a)|=\delta}\lambda^j y_a^{-}(x,\lambda)\,d\lambda+$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\lambda^j\left\{y_a^{-}(x,\lambda)-y_a^{+}(x,\lambda)\right\}d\lambda,\quad j=0,1,2 \tag{14}$$

and

$$f(x)=\frac{1}{2\pi i}\sum_{j=1}^{l}\int_{|\lambda-\lambda_j^{+}(a)|=\delta}\lambda^3 y_a^{+}(x,\lambda)\,d\lambda+\frac{1}{2\pi i}\sum_{j=1}^{m}\int_{|\lambda-\lambda_j^{-}(a)|=\delta}\lambda^3 y_a^{-}(x,\lambda)\,d\lambda+$$

$$+\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\lambda^3\left\{y_a^{-}(x,\lambda)-y_a^{+}(x,\lambda)\right\}d\lambda\,. \tag{15}$$

Thus, taking into account (15) we get the proof of the following theorem.

Theorem. Assume that the complex functions $r(x), p(x), q(x)$ are determined and continuous on the finite interval [0,a], has continuous derivatives to the third, fourth fifth order, inclusively and vanish outside the interval [0,a], while the pencil $L^{\alpha}_{\lambda a}$ has no spectral properties. Then for any finite function $f(x)$ smooth to the 7-th order and whose support does not contain the point $x = 0$ we have a uniformly convergent formula of spectral expansion (15).

Case b) When the coefficients of the equation (1) are exponentially decreasing functions on the semi-axis $[0, \infty)$.

Let

$0 \le \eta_a(x) \le 1, \eta_a(x) = \begin{cases} 1, & 0 \le x \le a \\ 0, & x \ge a \end{cases}$ $r_a(x) = r(x)\eta_a(x)$, $q_a(x) = q(x)\eta_a(x)$, $p_a(x) = p(x)\eta_a(x)$ and $y_a(x, \lambda)$ be the solution of the boundary value problem

$$l_a\left(x, \frac{d}{dx}, \lambda\right) y_a(x, \lambda) = \left(\frac{d}{dx} - i\lambda\right)^3 \left(\frac{d}{dx} + i\lambda\right) y_a(x, \lambda) + r_a(x) y'_a(x, \lambda) +$$

$$(\lambda p_a(x) + q_a(x))\, y_a(x, \lambda) = f(x), \tag{16}$$

$U_v\left(y_a(x, \lambda)\right) = 0, \quad 0 \le x < \infty, \quad y_a(x, \lambda) \in L_2(0, \infty), \quad f(x)$-is a finite function.

Then

$$y(x, \lambda) = y_a(x, \lambda) + \int_0^{\infty} K(x, \xi, \lambda)\left[r(\xi)\left(1 - \eta_a(\xi)\right) + \left(\lambda p(\xi) + q(\xi)\eta(\xi)\right)\right] y_a(\xi, \lambda) d\xi, \tag{17}$$

and it is evident that

$$y(x, \lambda) = \int_0^{\infty} K(x, \xi, \lambda) f(\xi) d\xi, \quad y_a(x, \lambda) = \int_0^{\infty} K_a(x, \xi, \lambda) f(\xi) d\xi.$$

From determination of the system of solutions $y_i(x, \lambda)$, $i = 1, 4$ and $y_{ja}(x, \lambda, a)$, $j = \overline{1, 4}$ it follows that $\lim\limits_{a \to \infty} y_{ja}(x, \lambda, a) = y_j(x, \lambda)$ uniformly in the domain $0 \le x < \infty$ and $Im\lambda \ge 0$. Their derivatives to the third order have the same property. As $a \to \infty$ we have $A_a(\lambda) \to A(\lambda)$, $B_a(\lambda) \to B(\lambda)$, (where $B_a(\lambda) \equiv U_v(y_{4a})$, V is one of the numbers 1,2,3, $B(\lambda) \equiv U_v(y_4)$ uniformly in the domain $\pm Im\lambda \ge 0$.

Hence it follows that for any $\delta > 0$ there will be found such a number $a_0 > 0$ that for all $a > a_0$ outside the circles of radius $\delta > 0$ centered at $\lambda_1^+, \ldots, \lambda_{m+}^+$ and the functions $A_a(\lambda)$, $B_a(\lambda)$ do not vanish in the upper and lower half-planes, respectively.

Equation (18) in certain denotations can be writhen in the form:

$$y(\cdot, \lambda) = y_a(\cdot, \lambda) - R_\lambda\left(B_a + Q_a\right) y_a(\cdot, \lambda),$$

where B_a and Q_a determine appropriate operators with respect to $(\lambda r(x) + q(x))\, y$ and Ay'.

On the circle $\Gamma_j^{\pm} = \left\{\lambda : \left|\lambda - \lambda_j^{\pm}\right| = \delta\right\}$ the operator R_λ is uniformly bounded, while $\|B_a\| \le c \cdot e^{-\varepsilon a}$. Therefore $\lim\limits_{a \to \infty} \|R_\lambda B_a\| = 0$ uniformly on $\Gamma_j^{\pm}$.

We now consider the operator $R_\lambda Q_a$ that will be determined in the following way:

$$R_\lambda Q_a y = \int_0^\infty K(x,\xi,\lambda)\, r(\xi)\,(1-\eta_a(\xi))\, y'(\xi)\, d\xi =$$

$$= -\int_0^\infty K_+(x,\xi,\lambda)\left\{[1-\eta_a(\xi)]\, r(\xi) + [1-\eta_a(\xi)\, r(\xi)]' \cdot K(x,\xi,\lambda)\right\} y(\xi,\lambda)\, d\xi.$$

Hence it follows that the norm of the operators generated by the kernel $[1-\eta_a(\xi)\, r(\xi)]'$ $K(x,\xi,\lambda)$ uniformly with respect to $\lambda \in \Gamma_j$ tends to zero as $a \to \infty$. Using directly $K(x,\xi,\lambda)$ and $K_0(x,\xi,\lambda)$ (where $K_0(x,\xi,\lambda)$ is a kernel of the operator only with principle terms) it is verified that the operator generated by the kernel $K(x,\xi,\lambda)$ in $L_2(0,\infty)$ is a bounded operator. Therefore the norm of the operator generated by the kernel $K_+(x,\xi,\lambda)$ $[1-\eta_a(\xi)\, r(\xi)]'$ also uniformly with respect to $\lambda \in \Gamma_j$ tends to zero as $a \to \infty$. Hence it follows that as $a \to \infty$ $\lim\limits_{a\to\infty} y_a(x,\lambda) = y(x,\lambda)$, $\lambda \in \Gamma_j^{\pm}$ is valid in the sence of convergence of $L_2(0,\infty)$.

Therefore in (14), (15) one can pass to the limit as $a \to \infty$ locally in $L_2(0,\infty)$:

$$f(x) = \frac{1}{2\pi i}\sum_{j=1}^{m_+}\int_{|\lambda-\lambda_j^+|=\delta} \lambda^3 y^+(x,\lambda)\, d\lambda +$$

$$+\frac{1}{2\pi i}\sum_{j=1}^{m_-}\int_{|\lambda-\lambda_j^-|=\delta} \lambda^3 y^-(x,\lambda) + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \lambda^j \left\{y^-(x,\lambda) - y^+(x,\lambda)\right\} d\lambda \qquad (18)$$

$$f(x) = \frac{1}{2\pi i}\sum_{v=1}^{m_+}\int_{|\lambda-\lambda_j^+|=\delta} \lambda^j y^+(x,\lambda)\, d\lambda +$$

$$+\frac{1}{2\pi i}\sum_{j=1}^{m_-}\int_{|\lambda-\lambda_j^-|=\delta} \lambda^j y^-(x,\lambda)\, d\lambda + \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \lambda^j \left\{y^-(x,\lambda) - y^+(x,\lambda)\right\} d\lambda j = \overline{0,1,2}. \qquad (19)$$

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