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Analytic Functions Expressed with Poisson Distribution Series and their Some Properties

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Abstract. In this paper, we introduce and investigate two new subclasses of analytic functions on the open unit disk in the complex plane. The object of the present paper is to derive characteristic properties of the functions belonging to these classes. In the study, applications of a Poisson distribution series on the analytic functions are also given.

Key Words and Phrases: Analytic function; Starlike function; Convex function; Poisson distribution.

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1. Introduction and preliminaries

Let A be the class of analytic functions f on the disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, normalized by $f(0) = 0 = f'(0) - 1$ of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + ... + a_n z^n + ... = z + \sum_{n=2}^{\infty} a_n z^n, a_n \in \mathbb{C}. \tag{1}$$

By S we will denote the family of all univalent functions in A.

Let T denote the subclass of A in the following form

$$f(z) = z - a_2 z^2 - a_3 z^3 - ... - a_n z^n - ... = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{2}$$

Some of the important and well-investigated subclasses of the univalent functions class S include the classes $\mathrm{S}^*(\alpha)$ and $\mathrm{C}(\alpha)$, respectively, starlike and convex of order $\alpha\,(\alpha \in [0,1))$ on the open unit disk U. By definition, we have (see for details, [5, 6], also [12])

$$\mathrm{S}^*(\alpha) = \left\{ f \in \mathrm{S} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{U} \right\}, \alpha \in [0,1),$$

and

$$\mathrm{C}(\alpha)=\left\{f\in \mathrm{S}:\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha, z\in U\right\}, \alpha\in[0,1).$$

Note that, we will use $TS^*(\alpha)=S^*(\alpha)\cap T$ and $TC(\alpha)=C(\alpha)\cap T$.

For $\beta\in[0,1)$, interesting generalization of the functions classes $S^*(\alpha)$ and $C(\alpha)$, are classes $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$, which defined by

$$\mathrm{S}^*(\alpha,\beta)=\left\{f\in\ \mathrm{S}:\mathrm{Re}\left(\frac{zf'(z)}{\beta zf'(z)+(1-\beta)f(z)}\right)>\alpha, z\in\mathbb{U}\right\}$$

and

$$\mathrm{C}(\alpha,\beta)=\left\{f\in \mathrm{S}:\mathrm{Re}\left(\frac{f'(z)+zf''(z)}{f'(z)+\beta zf''(z)}\right)>\alpha, z\in U\right\},$$

respectively.

We will denote $TS^*(\alpha,\beta)=S^*(\alpha,\beta)\cap T$ and $TC(\alpha,\beta)=C(\alpha,\beta)\cap T$.

The classes $TS^*(\alpha,\beta)$ and $TC(\alpha,\beta)$ were extensively studied by Altintaş and Owa [3] and certain conditions for hypergeometric functions and generalized Bessel functions for these classes were studied Moustafa [7] and Porwal and Dixit [9].

Inspired by the studies mentioned above, we define a unification of the functions classes $\mathrm{S}^*(\alpha,\beta)$ and $\mathrm{C}(\alpha,\beta)$ as follows.

Definition 1.1. *A function f given by (1) is said to be in the class $S^*C(\alpha,\beta;\gamma)$, $\alpha,\beta\in[0,1), \gamma\in[0,1]$ if the following condition is saatisfied*

$$\mathrm{Re}\left(\frac{f'(z)+\gamma z^2f''(z)}{\gamma z\left(f'(z)+\beta zf''(z)\right)+(1-\gamma)\left(\beta zf'(z)+(1-\beta)f(z)\right)}\right)>\alpha, z\in U$$

for $\gamma\in[0,1]$.

We will denote $TS^*C(\alpha,\beta;\gamma)=S^*C(\alpha,\beta;\gamma)\cap T$.

In special case, we have

$S^*C(\alpha,\beta;0)=S^*(\alpha,\beta); S^*C(\alpha,\beta;1)=C(\alpha,\beta); S^*C(\alpha,0;0)=S^*(\alpha);$
$S^*C(\alpha,0;1)=C(\alpha); TS^*C(\alpha,\beta;0)=TS^*(\alpha,\beta); TS^*C(\alpha,\beta;1)=TC(\alpha,\beta);$
$ST^*C(\alpha,0;0)=TS^*(\alpha); TS^*C(\alpha,0;1)=TC(\alpha).$

Suitably specializing the parameters we note that

1) $S^*C(\alpha,0;0)=S^*(\alpha)$ [11]; 2) $S^*C(\alpha,0;1)=C(\alpha)$ [11]; 3)$TS^*C(\alpha,\beta;0)=TS^*(\alpha,\beta)$ [1],[2], [4] and [10]; 4)$TS^*C(\alpha,0;0)=TS^*(\alpha)$ [11]; 5) $TS^*C(\alpha,\beta;1)=TC(\alpha,\beta)$[3], [10]; 6) $TS^*C(\alpha,0;1)=TC(\alpha,\beta)$ [11].

The object of the present paper is to examine characteristic properties of the classes $S^*C(\alpha,\beta;\gamma)$ and $TS^*C(\alpha,\beta;\gamma)$. In this study, applications of a Poisson distribution series on the analytic functions are also given.

2. Conditions for the classes $S^*C(\alpha,\beta;\gamma)$ and $TS^*C(\alpha,\beta;\gamma)$

In this section, we will examine some characteristic properties of the classes $S^*C(\alpha,\beta;\gamma)$ and $TS^*C(\alpha,\beta;\gamma)$.

A sufficient condition for the functions in the class $S^*C(\alpha,\beta;\gamma)$ is given by the following theorem.

Theorem 2.1. *Let $f \in A$. Then, the function f belongs to the class $S^*C(\alpha,\beta;\gamma)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty}(1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta)|a_n| \leq 1-\alpha. \tag{3}$$

The result is sharp for the functions

$$f_n(z) = z + \frac{1-\alpha}{(1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta)}z^n, z \in U, n = 2,3,\ldots . \tag{4}$$

Proof. From the definition, a function f belong to the class $S^*C(\alpha,\beta;\gamma)$, $\alpha,\beta \in [0,1)$, $\gamma \in [0,1]$ if and only if

$$\operatorname{Re}\left(\frac{f'(z)+\gamma z^2 f''(z)}{\gamma z(f'(z)+\beta z f''(z))+(1-\gamma)(\beta z f'(z)+(1-\beta)f(z))}\right) > \alpha. \tag{5}$$

Also, it can be easily shown that condition (5) holds true if

$$\left|\frac{f'(z)+\gamma z^2 f''(z)}{\gamma z(f'(z)+\beta z f''(z))+(1-\gamma)(\beta z f'(z)+(1-\beta)f(z))}-1\right| \leq 1-\alpha. \tag{6}$$

Now, let us show that this condition is satisfied under hypothesis of the theorem.

By simple computation, we write

$$\begin{aligned}
&\left|\frac{f'(z)+\gamma z^2 f''(z)}{\gamma z(f'(z)+\beta z f''(z))+(1-\gamma)(\beta z f'(z)+(1-\beta)f(z))}-1\right| \\
= &\left|\frac{\sum_{n=2}^{\infty}(1+(n-1)\gamma)(n-1)(1-\beta)a_n z^n}{z+\sum_{n=2}^{\infty}(1+(n-1)\gamma)(1+(n-1)\beta)a_n z^n}\right| \\
\leq &\frac{\sum_{n=2}^{\infty}(1+(n-1)\gamma)(n-1)(1-\beta)|a_n|}{1-\sum_{n=2}^{\infty}(1+(n-1)\gamma)(1+(n-1)\beta)|a_n|}.
\end{aligned}$$

Last expression of the above inequality is bounded by $1-\alpha$ if and only if

$$\sum_{n=2}^{\infty}(1+(n-1)\gamma)(n-1)(1-\beta)|a_n|$$

$$\leq \quad (1-\alpha)\left\{1-\sum_{n=2}^{\infty}(1+(n-1)\gamma)(1+(n-1)\beta)|a_n|\right\},$$

which is equivalent to

$$\sum_{n=2}^{\infty}(1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta)|a_n| \leq 1-\alpha.$$

Thus, the inequality (6) is satisfied if satisfied last inequality.

Further, we easily see that the inequality (3) provided as equality for the functions given by (4).

Thus, the proof of Theorem 2.1 is completed. ∎

By setting $\gamma = 0$ and $\gamma = 1$ in Theorem 2.1, we can readily deduce the following results, respectively.

Corollary 2.2. *The function f given by (1) belongs to the class $S^*(\alpha,\beta)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty}(n-\alpha-(n-1)\alpha\beta)|a_n| \leq 1-\alpha.$$

The result is sharp for the functions

$$f_n(z) = z + \frac{1-\alpha}{n-\alpha-(n-1)\alpha\beta}z^n, z \in U, n = 2, 3, \dots .$$

Corollary 2.3. *The function f given by (1) belongs to the class $C(\alpha,\beta)$ if the following condition is satisfied*

$$\sum_{n=2}^{\infty}n(n-\alpha-(n-1)\alpha\beta)|a_n| \leq 1-\alpha.$$

The result is sharp for the functions

$$f_n(z) = z + \frac{1-\alpha}{n(n-\alpha-(n-1)\alpha\beta)}z^n, z \in U, n = 2, 3, \dots .$$

Remark 2.4. *Consequences of the properties given by Corollary 2.2 and Corollary 2.3 can be obtained for each of the classes studied by earlier researches, by specializing the various parameters involved. Many of these consequences were proved by earlier researches on the subject (cf., e.g., [11]).*

For the function $f \in TS^*C(\alpha,\beta;\gamma)$, the converse of Theorem 2.1 is also true.

Theorem 2.5. *Let $f \in T$. Then, the function f belongs to the class $TS^*C(\alpha,\beta;\gamma)$ if and only if*

$$\sum_{n=2}^{\infty}(1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta)|a_n| \leq 1-\alpha. \tag{7}$$

The result is sharp for the functions

$$f_n(z) = z - \frac{1-\alpha}{(1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta)} z^n, z \in U, n = 2, 3, \dots . \tag{8}$$

Proof. The proof of the sufficiency of theorem can be proved similarly to the proof of Theorem 2.1.

We will prove only the necessity of theorem.

Assume that $f \in TS^*C(\alpha, \beta; \gamma), \alpha, \beta \in [0,1), \gamma \in [0,1]$. That is,

$$\operatorname{Re}\left\{ \frac{zf'(z) + \gamma z^2 f''(z)}{\gamma z (f'(z) + \beta z f''(z)) + (1-\gamma)(\beta z f'(z) + (1-\beta) f(z))} \right\} > \alpha, z \in U.$$

Then, by simple computation, we write

$$\operatorname{Re}\left\{ \frac{zf'(z) + \gamma z^2 f''(z)}{\gamma z (f'(z) + \beta z f''(z)) + (1-\gamma)(\beta z f'(z) + (1-\beta) f(z))} \right\}$$
$$= \left\{ \frac{z - \sum\limits_{n=2}^{\infty} n(1+(n-1)\gamma) a_n z^n}{z - \sum\limits_{n=2}^{\infty} (1+(n-1)\gamma)(1+(n-1)\beta) a_n z^n} \right\} > \alpha.$$

The last expression in the brackets of the above inequality is real if choose z real. Hence, from the previous inequality letting $z \to 1$ through real values, we obtain

$$1 - \sum_{n=2}^{\infty} n(1+(n-1)\gamma) a_n \geq \alpha \left\{ 1 - \sum_{n=2}^{\infty} (1+(n-1)\gamma)(1+(n-1)\beta) a_n \right\}.$$

This follows

$$\sum_{n=2}^{\infty} (1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta) |a_n| \leq 1-\alpha,$$

which is the same as (7).

Further, it is clear that the inequality (7) provided as equality for the functions given by (8).

Thus, the proof of Theorem 2.5 is completed. ∎

By taking $\gamma = 0$ and $\gamma = 1$ in Theorem 2.5, we can readily deduce the following results, respectively.

Corollary 2.6. *The function f given by (2) belongs to the class $TS^*(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} (n-\alpha-(n-1)\alpha\beta) a_n \leq 1-\alpha.$$

The result is sharp for the functions

$$f_n(z) = z - \frac{1-\alpha}{n-\alpha-(n-1)\alpha\beta} z^n, z \in U, n = 2, 3, \dots\ .$$

Corollary 2.7. *The function f given by (2) belongs to the class $TC(\alpha, \beta)$ if and only if*

$$\sum_{n=2}^{\infty} n(n-\alpha-(n-1)\alpha\beta)|a_n| \leq 1-\alpha.$$

The result is sharp for the functions

$$f_n(z) = z - \frac{1-\alpha}{n(n-\alpha-(n-1)\alpha\beta)} z^n, z \in U, n = 2, 3, \dots\ .$$

Remark 2.8. *The results obtained by Corollary 2.6 and Corollary 2.7 would reduce to known results in [3].*

Remark 2.9. *Consequences of the properties given by Corollary 2.6 and Corollary 2.7 can be obtained for each of the classes studied by earlier researches, by specializing the various parameters involved. Many of these consequences were proved by earlier researches on the subject (cf., e.g., [11]).*

3. Applications of a Poisson distribution series on the analytic functions

In this section, we will investigate certain class of analytic functions associated with Poisson distribution series.

A variable x is said to have Poisson distribution if it takes the values $0, 1, 2, 3, \dots$ with probabilities e^{-p}, $\frac{p}{1!}e^{-p}$, $\frac{p^2}{2!}e^{-p}$, $\frac{p^3}{3!}e^{-p}$, ..., respectively, where p is a parameter.

Thus,

$$P(x = n) = \frac{p^n}{n!} e^{-p}, n = 0, 1, 2, 3, \dots\ . \quad (9)$$

Now, we introduce a Poisson distribution series as follows

$$z + \sum_{n=2}^{\infty} \frac{p^{n-1}}{(n-1)!} e^{-p} z^n, z \in U. \quad (10)$$

We can easily show that series (10) is convergent and the radius of convergence is infinity.

We define the function $F : \mathbb{C} \to \mathbb{C}$ by

$$F(z) = z + \sum_{n=2}^{\infty} \frac{p^{n-1}}{(n-1)!} e^{-p} z^n, z \in U. \quad (11)$$

Also, let

$$G(z) = 2z - F(z) = z - \sum_{n=2}^{\infty} \frac{p^{n-1}}{(n-1)!} e^{-p} z^n, z \in U. \quad (12)$$

It is clear that $F \in A$ and $G \in T$.

Now, we will give sufficient condition for the function F defined by (11), belonging to the class $S^{*}C(\alpha, \beta; \gamma)$, and necessary and sufficient condition for the function G defined by (12), belonging to the class $TS^{*}C(\alpha, \beta; \gamma)$, respectively.

Theorem 3.1. *Let $p > 0$ and the following condition is provided*

$$\{(1-\alpha\beta)\gamma p + (1-\alpha\beta + (2-(1+\beta)\alpha)\gamma)\} pe^{p} \leq 1-\alpha. \tag{13}$$

Then, the function F defined by (11) belongs to the class $S^{*}C(\alpha, \beta; \gamma)$.

Proof. Since $F \in A$ and

$$F(z) = z + \sum_{n=2}^{\infty} \frac{p^{n-1}}{(n-1)!} e^{-p} z^{n}, z \in U,$$

according to Theorem 2.1, we must show that

$$\sum_{n=2}^{\infty} (1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta) \frac{p^{n-1}}{(n-1)!} e^{-p} \leq 1-\alpha. \tag{14}$$

Let

$$L_1(\alpha, \beta; \gamma) = \sum_{n=2}^{\infty} (1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta) \frac{p^{n-1}}{(n-1)!} e^{-p}.$$

Setting

$$\begin{aligned} & (1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta) \\ = \ & (n-2)(n-1)(1-\alpha\beta)\gamma + (n-1)(1-\alpha\beta+(2-(1+\beta)\alpha)\gamma) + 1-\alpha \end{aligned}$$

and by simple computation, we obtain

$$\begin{aligned} & L_1(p; \alpha, \beta; \gamma) \\ = \ & e^{-p} \left\{ \sum_{n=2}^{\infty} \begin{pmatrix} (n-2)(n-1)(1-\alpha\beta)\gamma + \\ (n-1)(1-\alpha\beta+(2-(1+\beta)\alpha)\gamma) + 1-\alpha \end{pmatrix} \frac{p^{n-1}}{(n-1)!} \right\} \\ = \ & (1-\alpha\beta)\gamma p^{2} + (1-\alpha\beta+(2-(1+\beta)\alpha)\gamma) p + (1-\alpha)\left(1-e^{-p}\right). \end{aligned}$$

Teherfore, inequality (14) holds true if

$$(1-\alpha\beta)\gamma p^{2} + (1-\alpha\beta+(2-(1+\beta)\alpha)\gamma) p + (1-\alpha)\left(1-e^{-p}\right) \leq 1-\alpha,$$

which is equivalent to (13).

Thus, the proof of Theorem 3.1 is completed. ∎

By taking $\gamma = 0$ and $\gamma = 1$ in Theorem 3.1, we arrive at the following results.

Corollary 3.2. *If $p > 0$ and satisfied the following condition*

$$(1-\alpha\beta)\, pe^{p} \leq 1-\alpha,$$

then the function F defined by (11) belongs to the class $S^{*}(\alpha,\beta)$.

Corollary 3.3. *If $p > 0$ and satisfied the following condition*

$$\{(1-\alpha\beta)\, p + (3-2\alpha\beta-\alpha)\}\, pe^{p} \leq 1-\alpha,$$

then the function F defined by (11) belongs to the class $C(\alpha,\beta)$.

Remark 3.4. *Consequences of the results given by Corollary 3.2 and Corollary 3.3 can be obtained for each of the classes, by specializing the various parameters involved.*

Theorem 3.5. *Let $p > 0$. Then, the function G defined by (12) belongs to the class $TS^{*}C(\alpha,\beta;\gamma)$ if and only if*

$$\{(1-\alpha\beta)\,\gamma p + (1-\alpha\beta+(2-(1+\beta)\,\alpha)\,\gamma)\}\, pe^{p} \leq 1-\alpha. \tag{15}$$

Proof. Firstly, let us provide sufficiency of theorem.

From (12), we have

$$G(p,z) = z - \sum_{n=2}^{\infty} \frac{p^{n-1}}{(n-1)!} e^{-p} z^{n},\, z \in U.$$

Also, since

$$\begin{aligned} & (1+(n-1)\,\gamma)\,(n-\alpha-(n-1)\,\alpha\beta) \\ = \; & (n-2)\,(n-1)\,(1-\alpha\beta)\,\gamma + (n-1)\,(1-\alpha\beta+(2-(1+\beta)\,\alpha)\,\gamma) + 1-\alpha, \end{aligned}$$

we write

$$\begin{aligned} & \sum_{n=2}^{\infty} (1+(n-1)\,\gamma)\,(n-\alpha-(n-1)\,\alpha\beta)\, \frac{p^{n-1}}{(n-1)!} e^{-p} \\ = \; & (1-\alpha\beta)\,\gamma p^{2} + (1-\alpha\beta+(2-(1+\beta)\,\alpha)\,\gamma)\, p + (1-\alpha)\left(1-e^{-p}\right). \end{aligned} \tag{16}$$

Now, suppose that condition (15) is satisfied. It follows that

$$(1-\alpha\beta)\,\gamma p^{2} + (1-\alpha\beta+(2-(1+\beta)\,\alpha)\,\gamma)\, p \leq (1-\alpha)\, e^{-p},$$

which is equivalent to

$$(1-\alpha\beta)\,\gamma p^{2} + (1-\alpha\beta+(2-(1+\beta)\,\alpha)\,\gamma)\, p + (1-\alpha)\left(1-e^{-p}\right) \leq 1-\alpha.$$

Therefore, considering (16), we obtain

$$\sum_{n=2}^{\infty} (1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta)\frac{p^{n-1}}{(n-1)!}e^{-p} \leq 1-\alpha.$$

Thus, according to Theorem 2.5, the function G belongs to the class $TS^*C(\alpha,\beta;\gamma)$.

Now, let we prove of the necessity.

Assume that $G \in TS^*C(\alpha,\beta;\gamma)$. Then, according to Theorem 2.5, the following condition is satisfied

$$\sum_{n=2}^{\infty} (1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta)\frac{p^{n-1}}{(n-1)!}e^{-p} \leq 1-\alpha.$$

Therefore, from (16), we have

$$(1-\alpha\beta)\gamma p^2 + (1-\alpha\beta+(2-(1+\beta)\alpha)\gamma)p + (1-\alpha)\left(1-e^{-p}\right) \leq 1-\alpha,$$

which readily yields

$$(1-\alpha\beta)\gamma p^2 + (1-\alpha\beta+(2-(1+\beta)\alpha)\gamma)p \leq (1-\alpha)e^{-p}.$$

Last condition is equivalent to condition (15).

This completes the proof of necessity, and the proof of Theorem 3.5. ∎

By taking $\gamma = 0$ and $\gamma = 1$ in Theorem 3.5, we can readily deduce the following results, respectively.

Corollary 3.6. *Let $p > 0$. Then, the function G defined by (12) belongs to the class $TS^*(\alpha,\beta)$ if and only if*

$$(1-\alpha\beta)pe^p \leq 1-\alpha.$$

Corollary 3.7. *Let $p > 0$. Then, the function G defined by (12) belongs to the class $TS^*(\alpha,\beta)$ if and only if*

$$\{(1-\alpha\beta)p + (3-2\alpha\beta-\alpha)\}pe^p \leq 1-\alpha.$$

Remark 3.8. *Results obtained in Corollary 3.6 and Corollary 3.7 verifies results given by Theorem 3 and Theorem 4 in [10], respectively.*

4. Some integral operators involving the functions F and G

In this section, we will examine some inclusion properties of integral operators associated with the functions F and G as follows

$$\overset{\Lambda}{F}(p,z) = \int_0^z \frac{F(p,t)}{t}dt \quad \text{and} \quad \overset{\Lambda}{G}(p,z) = \int_0^z \frac{G(p,t)}{t}dt. \tag{17}$$

Theorem 4.1. *Let $p > 0$ and the following condition is provided*

$$\left\{(1-\alpha\beta)\gamma p + (1-\beta)(1-\gamma)\alpha\left[1+\left(e^{-p}-1\right)p^{-1}\right]\right\} e^{p} \leq 1-\alpha. \tag{18}$$

Then, the function $\overset{\Lambda}{F}$ defined by (17) belongs to the class $S^{*}C(\alpha,\beta;\gamma)$.

Proof. Since

$$\overset{\Lambda}{F}(p,z) = z + \sum_{n=2}^{\infty} \frac{p^{n-1}}{n!} e^{-p} z^{n}, z \in U.$$

according to Theorem 2.1, the function $\overset{\Lambda}{F}$ belongs to the class $S^{*}C(\alpha,\beta;\gamma)$ if the following condition is satisfied

$$\sum_{n=2}^{\infty} (1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta) \frac{p^{n-1}}{n!} e^{-p} \leq 1-\alpha. \tag{19}$$

Let

$$L_2(\alpha,\beta;\gamma) = \sum_{n=2}^{\infty} (1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta) \frac{p^{n-1}}{n!} e^{-p}.$$

Setting

$$\begin{aligned} & (1+(n-1)\gamma)(n-\alpha-(n-1)\alpha\beta) \\ = \; & (n-1)n(1-\alpha\beta)\gamma + n((1-\alpha\beta)(1-\gamma)+(1-\alpha)\gamma) - (1-\beta)(1-\gamma)\alpha \end{aligned}$$

and by simple computation, we obtain

$$\begin{aligned} L_2(\alpha,\beta;\gamma) &= \sum_{n=2}^{\infty} \left\{ \begin{array}{c} (n-1)n(1-\alpha\beta)\gamma + n(1-\beta)(1-\gamma)\alpha \\ -(1-\beta)(1-\gamma)\alpha + n(1-\alpha) \end{array} \right\} \frac{p^{n-1}}{e^{p} n!} \\ &= (1-\alpha\beta)\gamma p + (1-\beta)(1-\gamma)\alpha\left(1-e^{-p}\right) \\ &\quad - \frac{(1-\beta)(1-\gamma)\alpha}{p}\left(1-(1+p)e^{-p}\right) + (1-\alpha)\left(1-e^{-p}\right) \\ &= (1-\alpha\beta)\gamma p + (1-\beta)(1-\gamma)\alpha\left(1-\left(1-e^{-p}\right)p^{-1}\right) \\ &\quad + (1-\alpha)\left(1-e^{-p}\right) . \end{aligned}$$

Therefore, inequality (19) holds true if

$$(1-\alpha\beta)\gamma p + (1-\beta)(1-\gamma)\alpha\left(1-\left(1-e^{-p}\right)p^{-1}\right) + (1-\alpha)\left(1-e^{-p}\right) \leq 1-\alpha,$$

which is equivalent to (18).
Thus, the proof of Theorem 4.1 is completed. ∎

By taking $\gamma = 0$ and $\gamma = 1$ in Theorem 4.1, we arrive at the following results.

Corollary 4.2. *If $p > 0$ and the following condition is satisfied*

$$(1-\beta)\,\alpha\left(1-\left(1-e^{-p}\right)p^{-1}\right)e^{p} \leq 1-\alpha,$$

then the function $\overset{\Lambda}{F}$ defined by (17) belongs to the class $S^{*}(\alpha,\beta)$.

Corollary 4.3. *If $p > 0$ and the following condition is satisfied*

$$(1-\alpha\beta)\,pe^{p} \leq 1-\alpha,$$

then the function $\overset{\Lambda}{F}$ defined by (17) belongs to the class $C(\alpha,\beta)$.

Remark 4.4. *Consequences of the results given by Corollary 4.2 and Corollary 4.3 can be obtained for each of the classes, by specializing the various parameters involved.*

Theorem 4.5. *If $p > 0$, then the function $\overset{\Lambda}{G}$ defined by (17) is in the class $TS^{*}C(\alpha,\beta;\gamma)$ if and only if*

$$\left\{(1-\alpha\beta)\,\gamma p+(1-\beta)(1-\gamma)\,\alpha\left[1+\left(e^{-p}-1\right)p^{-1}\right]\right\}e^{p} \leq 1-\alpha. \tag{20}$$

Proof. Firstly, let us provide the sufficiency of theorem.

From (17), we obtain

$$\overset{\Lambda}{G}(p,z) = z-\sum_{n=2}^{\infty}\frac{p^{n-1}}{n!}e^{-p}z^{n},\ z\in U.$$

Also, since

$$\begin{aligned} & (1+(n-1)\,\gamma)\,(n-\alpha-(n-1)\,\alpha\beta) \\ = \ & (n-1)\,n\,(1-\alpha\beta)\,\gamma+n\,((1-\alpha\beta)(1-\gamma)+(1-\alpha)\,\gamma)-(1-\beta)(1-\gamma)\,\alpha, \end{aligned}$$

we write

$$\begin{aligned} & \sum_{n=2}^{\infty}(1+(n-1)\,\gamma)\,(n-\alpha-(n-1)\,\alpha\beta)\,\frac{p^{n-1}}{n!}e^{-p} \qquad (21)\\ = \ & (1-\alpha\beta)\,\gamma p+(1-\beta)(1-\gamma)\,\alpha\left(1-e^{-p}\right)-\frac{(1-\beta)(1-\gamma)\,\alpha}{p}\left(1-(1+p)\,e^{-p}\right) \\ & +(1-\alpha)\left(1-e^{-p}\right) \\ = \ & (1-\alpha\beta)\,\gamma p+(1-\beta)(1-\gamma)\,\alpha\left(1-\left(1-e^{-p}\right)p^{-1}\right)+(1-\alpha)\left(1-e^{-p}\right) \end{aligned}$$

Now, suppose that (20) is satisfied. It follows that

$$(1-\alpha\beta)\,\gamma p+(1-\beta)\,(1-\gamma)\,\alpha\left(1-\left(1-e^{-p}\right)p^{-1}\right)\leq(1-\alpha)\,e^{-p},$$

which is equivalent to

$$(1-\alpha\beta)\,\gamma p+(1-\beta)\,(1-\gamma)\,\alpha\left(1-\left(1-e^{-p}\right)p^{-1}\right)+(1-\alpha)\left(1-e^{-p}\right)\leq 1-\alpha.$$

Hence, from (21), we have

$$\sum_{n=2}^{\infty}(1+(n-1)\,\gamma)\,(n-\alpha-(n-1)\,\alpha\beta)\,\frac{p^{n-1}}{n!}e^{-p}\leq 1-\alpha.$$

Thus, according to Theorem 2.5, the function $\overset{\Lambda}{G}$ belongs to the class $TS^{*}C\,(\alpha,\beta;\gamma)$.

Now, let we prove the necessity of theorem.

Assume that $\overset{\Lambda}{G}\in TS^{*}C\,(\alpha,\beta;\gamma)$. Then, according to Theorem 2.5, the following condition is satisfied

$$\sum_{n=2}^{\infty}(1+(n-1)\,\gamma)\,(n-\alpha-(n-1)\,\alpha\beta)\,\frac{p^{n-1}}{n!}e^{-p}\leq 1-\alpha.$$

This follows from (21)

$$(1-\alpha\beta)\,\gamma p+(1-\beta)\,(1-\gamma)\,\alpha\left(1-\left(1-e^{-p}\right)p^{-1}\right)+(1-\alpha)\left(1-e^{-p}\right)\leq 1-\alpha,$$

which readily yields

$$(1-\alpha\beta)\,\gamma p+(1-\beta)\,(1-\gamma)\,\alpha\left(1-\left(1-e^{-p}\right)p^{-1}\right)\leq(1-\alpha)\,e^{-p}.$$

Last condition is equivalent to (20). This completes the proof of necessity.

Thus, the proof of Theorem 4.5 is completed. ∎

By taking $\gamma=0$ and $\gamma=1$ in Theorem 4.5, we can readily deduce the following results, respectively.

Corollary 4.6. *If $p>0$, then the function $\overset{\Lambda}{G}$ defined by (17) belongs to the class $TS^{*}\,(\alpha,\beta)$ if and only if*

$$(1-\beta)\,\alpha\left(1+\left(e^{-p}-1\right)p^{-1}\right)e^{p}\leq 1-\alpha.$$

Corollary 4.7. *If $p>0$, then the function $\overset{\Lambda}{G}$ defined by (17) belongs to the class $TC\,(\alpha,\beta)$ if and only if*

$$(1-\alpha\beta)\,pe^{p}\leq 1-\alpha.$$

Remark 4.8. *Result obtained in Corollary 4.7 verifies result given by Theorem 5 in [10].*

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