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On the New Families of $k-$ Pell Numbers and $k-$ Pell-Lucas Numbers and Their Polynomials

Engin Özkan, Merve Taştan, Aysun Aktaş

Abstract. We introduce advances in the study of $k-$Pell and $k-$Pell-Lucas numbers. Then we give some properties of these families and find the generating functions of the families for some k. In addition, we identify the relationships between the family of $k-$Pell numbers and the classical Pell numbers as well as the family of $k-$Pell-Lucas numbers and the classical Pell-Lucas numbers. Further, we define generalized polynomials for these numbers. We also obtain some useful properties of these polynomials. Additionally, we give the relationships between the generalized $k-$Pell polynomials and the classical Pell polynomials as well as the generalized $k-$Pell-Lucas polynomials and the classical Pell-Lucas polynomials. These generalizations are given in matrix representation. Results in this paper include Cassini identities for these families and polynomials.

Key Words and Phrases: Pell Number, Pell-Lucas Numbers, Pell polynomials, Pell-Lucas polynomials, Cassini's Identity, Generating Function.

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1. Introduction

Fibonacci sequences have many important properties as well as a broad range of applications in many areas of sciences [15, 17, 19, 20, 21]. Some authors have worked on a new family of k-Fibonacci numbers [1, 3, 7, 16, 18].

Many authors have been interested in Pell numbers and their applications [2, 4, 5, 6, 8, 9, 10, 11, 12, 13, 14, 22].

In [7], they showed that general $k-$Fibonacci numbers were related to a matrix.

In [4], they studied the generalized Binet's formula, the generating function and some identities for $r-$Pell numbers.

One of the latest works in this area is [21] where it was defined new families of Gauss $k-$Lucas numbers and the generalized polynomials for these numbers. They gave some properties of these numbers and their polynomials. They proved some theorems related to these numbers and their polynomials.

In this paper, we shall be interested in new families of $k-$Pell numbers, $k-$Pell-Lucas numbers and their polynomials.

2. Preliminaries

Now, we introduce Pell numbers P_n, Pell-Lucas numbers Q_n, Pell $M-$matrix, Pell-Lucas $FM-$matrix and Pell polynomials $P_n(x)$, Pell-Lucas polynomials $Q_n(x)$, the Pell polynomials Q^n $P-$matrix and Pell -Lucas polynomials Q^nR-matrix.

Definition 2.1. *The Pell numbers P_n are defined by the recurrence relations*

$$P_n = 2P_{n-1} + P_{n-2}$$

for $n \geq 2$ with initial values $P_0 = 0$ and $P_1 = 1$.[12]

Definition 2.2. *The Pell-Lucas numbers Q_n are defined by the recurrence relations*

$$Q_n = 2Q_{n-1} + Q_{n-2}$$

for $n \geq 2$ with initial values $Q_0 = 2 and$ $Q_1 = 2$.[12]

Binet formulas for P_n and Q_n are given by as follows

$$P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1}$$

and

$$Q_n = \alpha^n + \beta^n \tag{2}$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.[12]

One can also be obtained the members of these integer sequences in different ways. In [4],[11], they showed:

$$M^n = \begin{bmatrix} P_{n+1} & P_n \\ P_n & P_{n-1} \end{bmatrix}$$

and

$$FM^n = \begin{bmatrix} Q_{n+1} & Q_n \\ Q_n & Q_{n-1} \end{bmatrix}$$

where $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ and $F = \begin{bmatrix} 2 & 2 \\ 2 & -2 \end{bmatrix}$respectively. The matrices M and FM are called the Pell $F-$matrix and Pell-Lucas $FM-$matrix, respectively.[5]

Definition 2.3. *The Pell and the Pell-Lucas polynomial $P_n(x)$ and $Q_n(x)$ are defined by the following recurrence relations*

$$P_n(x) = 2P_{n-1}(x) + P_{n-2}(x)$$

for $n \geq 2$ with initial values $P_0(x) = 0$ and $P_1(x) = 1$ and

$$Q_n(x) = 2Q_{n-1}(x) + Q_{n-2}(x)$$

for $n \geq 2$ with initial values $Q_0(x) = 2$ and $Q_1 = 2x$.[13]

Binet formulas for $P_n(x)$ and $Q_n(x)$ are given by as follows

$$P_n(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)} \tag{3}$$

and

$$Q_n = \alpha^n(x) + \beta^n(x) \tag{4}$$

where $\alpha = x + \sqrt{x^2+1}$ and $\beta = x - \sqrt{x^2+1}$.[13]

One can also be obtained the members of these integer sequences in different ways. In [5], [13], they showed:

$$M^n = \begin{bmatrix} P_{n+1}(x) & P_n(x) \\ P_n(x) & P_{n-1}(x) \end{bmatrix}$$

where $M = \begin{bmatrix} 2x & 1 \\ 1 & 0 \end{bmatrix}$.[5]

Then $n \geq 1$

$$\begin{bmatrix} Q_{n+1}(x) \\ Q_n(x) \end{bmatrix} = M^n \begin{bmatrix} 2x \\ 2 \end{bmatrix}.$$

3. Main results

Definition 3.1. *For $n, k \in N(k \neq 0)$, there are only numbers m and r such that $n = mk + r$ and $0 \leq r < k$. The generalized $k-$Pell numbers $P_n^{(k)}$ are defined by*

$$P_n^{(k)} = \frac{1}{(2\sqrt{2})^k} \left(\alpha^m - \beta^m\right)^{k-r} \left(\alpha^{m+1} - \beta^{m+1}\right)^r, n = mk + r$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

Also, we can find the generalized $k-$Pell numbers by matrix methods. Indeed, it is clear that

$$P_n^{k-1} M^n = \begin{bmatrix} P_{kn+1}^{(k)} & P_{kn}^{(k)} \\ P_{kn}^{(k)} & P_{kn-1}^{(k)} \end{bmatrix}$$

where $M = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$.

Let's give some values for the generalized $k-$Pell numbers in the Table 1.

From (5), we get the following relation

$$P_n^{(k)} = (P_m)^{k-r}(P_{m+1})^r, n = mk + r \tag{5}$$

If $k = 1$ in last equation, we have that $m = n$ and r=0 so $P_n^{(1)} = P_n$.

Throughout this paper, let $k, m \in \{1, 2, 3, \ldots\}$.

Table 1:

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$P_0^{(k)}$	0	0	0	0	0	0
$P_1^{(k)}$	1	0	0	0	0	0
$P_2^{(k)}$	2	1	0	0	0	0
$P_3^{(k)}$	5	2	1	0	0	0
$P_4^{(k)}$	12	4	2	1	0	0
$P_5^{(k)}$	29	10	4	2	1	0

Theorem 3.2. *For k and m, we have*

$$P_{m+1}^k - P_m^k = [P_{mk+k}^{(k)} - P_{mk}^{(k)}].$$

Proof. From (5), we have

$$\begin{aligned} P_{mk+k}^{(k)} - P_{mk}^{(k)} &= [(P_m)^{k-k}(P_{m+1})^k] - [(P_m)^{k-0}(P_{m+1})^0] \\ &= (P_{m+1})^k - (P_m)^k. \end{aligned}$$

□

For n, we have the following interesting properties related to these numbers.

$$\begin{aligned} P_{2n}^{(2)} &= P_n^2, \\ P_{2n+1}^{(2)} &= P_n P_{n+1}, \\ P_{2n+1}^{(2)} &= 2P_n^{(2)} + P_{n-1}^{(2)}. \end{aligned}$$

Theorem 3.3. *We have that the generating function for $P_n^{(2)}$ is*

$$G_n^{(2)}(z) = \frac{1}{-z^4 - 2z^3 - 2z + 1}.$$

Proof. By using definition 4. and the following formal power series for$P_n^{(2)}(z)$, $P_n^{(2)}(z) = \sum_{i=1}^{n} P_i^{(2)} z^i$, we have

$$\begin{aligned} P_n^{(2)}(z) &= z + z^2 + z^3 + 3z^4 + 9z^5 + 15z^6 + 25z^7 + ... \\ 2zP_n^{(2)}(z) &= 2z^2 + 2z^3 + 2z^4 + 6z^5 + 18z^6 + 30z^7 + 50z^8 + ... \\ 2z^3 P_n^{(2)}(z) &= 2z^4 + 2z^5 + 2z^6 + 6z^7 + 18z^8 + 30z^9 + 50z^{10} + ... \\ z^4 P_n^{(2)}(z) &= z^5 + z^6 + z^7 + 3z^8 + 9z^9 + 15z^{10} + 25z^{11} + ... \end{aligned}$$

Operations $[-z^4 P_n^{(2)}(z) - 2z^3 P_n^{(2)}(z) - 2zP_n^{(2)}(z) + P_n^{(2)}(z)]$ give

$$(-z^4 - 2z^3 - 2z + 1)P_n^{(2)}(z) = 1$$

Hence, the generating function $G_n^{(2)}(z)$ for $P_n^{(2)}(z)$ is

$$G_n^{(2)}(z) = \frac{1}{-z^4 - 2z^3 - 2z + 1}$$

which concludes the proof. □

For n, the generalized 3−Pell numbers and the known Pell numbers satisfy

$$\begin{aligned} P_{3n}^{(3)} &= P_n^3, \\ P_{3n+1}^{(3)} &= P_n^2 P_{n+1} \\ P_{3n+2}^{(3)} &= P_n P_{n+1}^2, \\ P_{3n+1}^{(3)} &= 2P_{3n}^{(3)} + P_{3n-1}^{(3)}. \end{aligned}$$

Here we note that $P_{-1}^{(3)} = 0$. More generally, we have $P_{-n}^{(k)} = 0$. for$k = 1, 2, \ldots$.

For fixed numbers n, the generalized 4−Pell numbers and the known Pell numbers satisfy

Theorem 3.4. *For n, we have the relation*

$$P_{kn+1}^{(k)} = 2P_{kn}^{(k)} + P_{kn-1}^{(k)}.$$

Proof.

$$\begin{aligned} 2P_{kn}^{(k)} + P_{kn-1}^{(k)} &= 2P_n^k + P_{n-1}P_n^{k-1} \\ &= P_n^{k-1}(2P_n + P_{n-1}) \\ &= P_n^{k-1}P_{n+1} \\ &= P_{kn+1}^{(k)}. \end{aligned}$$

□

Theorem 3.5. *For $n, s \geq 0$ and $n + s > 1$, we have*

$$P_{2(n+s-1)}^{(2)} - P_{n+s}P_{n+s-2} = (-1)^{n+s}$$

Proof. For $n + s = a$, we get

$$P_{2(a-1)}^{(2)} - P_a P_{a-2} = (-1)^a.$$

We will prove the theorem by the induction method on a. For $a = 2$,we have

$$P_{2(2-1)}^{(2)} - P_2 P_{2-2} = 1 - 2.0 = (-1)^{2-2}.$$

Now, assume that the theorem holds for $a = k$, that is

$$P_{2(k-1)}^{(2)} - P_k P_{k-2} = (-1)^k.$$

Then, for $a = k+1$, we have

$$\begin{aligned}
P^{(2)}_{2(k+1-1)} - P_{k+1}P_{k+1-2} &= (-1)^{k+1}, \\
P^{(2)}_{2(k+1-1)} &= P_k^2, \\
P_k^2 - P_{k+1}P_{k-1} &= (2P_{k-1}+P_{k-2})^2 - (2P_k + P_{k-1})P_{k-1} \\
&= 4P_{k-1}^2 + 4P_{k-1}P_{k-2} + P_{k-2}^2 - [2(2P_{k-1}+P_{k-2}) + 2P_{k-1}]P_{k-1} \\
&= 4P_{k-1}^2 + 4P_{k-1}P_{k-2} + P_{k-2}^2 - 5P_{k-1}^2 - 2P_{k-1}P_{k-2} \\
&= -P_{k-1}^2 + 2P_{k-1}P_{k-2} + P_{k-2}^2 \\
&= -P_{k-1}^2 + P_{k-2}(2P_{k-1}+P_{k-2}) \\
&= -P_{k-1}^2 + P_kP_{k-2} \\
&= -(P_{k-1}^2 - P_kP_{k-2}) \\
&= (-1)(-1)^k \\
&= (-1)^{k+1}.
\end{aligned}$$

□

Theorem 3.6. *Let $P_n^{(k)}$ be generalized k-Pell numbers. For $n, k \geq 2$, Cassini's Identity for $P_n^{(k)}$ is as follows:*

$$P^{(k)}_{kn+t}P^{(k)}_{kn+t-2} - (P^{(k)}_{kn+t-1})^2 = \left\{ \begin{array}{c} P_n^{2k-2}(-1)^n, t = 1 \\ 0, t \neq 1 \end{array} \right\}.$$

Proof. By using definition 4, we get

$$\begin{aligned}
P^{(k)}_{kn+t}P^{(k)}_{kn+t-2} - (P^{(k)}_{kn+t-1})^2 &= (P_n^{k-1}P_{n+t})(P_n^{k-1}P_{n+t-2}) - (P_n^{k-1}P_{n+t-1})^2 \\
&= (P_n^{k-1})^2(P_{n+t}P_{n+t-2} - (P_{n+t-1})^2) \\
&= P_n^{2k-2}(P_{n+t}P_{n+t-2} - (P_{n+t-1})^2)
\end{aligned}$$

For $t = 1$;

$$\begin{aligned}
&= P_n^{2k-2}(P_{n+1}P_{n-1} - (P_n)^2) \\
&= P_n^{2k-2}(-1)^n.
\end{aligned}$$

For $t \neq 1$, $t = m, (m \in N)$;

$$\begin{aligned}
&= P_n^{2k-2}(P_{n+m}P_{n+m-2} - (P_{n+m-1})^2) \\
&= P_n^{2k-2}(P^{(2)}_{2n+2m-2} - P^{(2)}_{2n+2m-2}) \\
&= 0.
\end{aligned}$$

□

Definition 3.7. *For $n, k \in N(k \neq 0)$, there are only numbers m and r such that $n = mk + r$ and $0 \leq r < k$. Then we define the generalized $k-$Pell-Lucas numbers $Q_n^{(k)}$ by*

$$Q_n^{(k)} := (\alpha^m + \beta^m)^{k-r}(\alpha^{m+1} + \beta^{m+1})^r$$

where $\alpha = 1 + \sqrt{2}$ and $\beta = 1 - \sqrt{2}$.

We can obtain the generalized k-Pell-Lucas numbers by matrixes as follows:

$$Q_n^{k-1} FM^n = \begin{bmatrix} Q_{kn+1}^{(k)} & Q_{kn}^{(k)} \\ Q_{kn}^{(k)} & Q_{kn-1}^{(k)} \end{bmatrix}$$

Let's give some values for the generalized $k-$Pell-Lucas numbers in the Table 2.

Table 2:

	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$Q_0^{(k)}$	2	4	8	16	32	64
$Q_1^{(k)}$	2	4	8	16	32	64
$Q_2^{(k)}$	6	4	8	16	32	64
$Q_3^{(k)}$	14	12	8	16	32	64
$Q_4^{(k)}$	34	36	24	16	32	64
$Q_5^{(k)}$	82	84	72	48	32	64

From (6) and Definition 5, we have the relation

$$Q_n^{(k)} := (Q_m)^{k-r}(Q_{m+1})^r, n = mk + r \tag{6}$$

When $k = 1$ in last equation, we get that $m = n$ and $r = 0$ so $Q_n^{(1)} = Q_n$.

Theorem 3.8. *For k and m, we have the following relation between the generalized $k-$Pell-Lucas numbers and the known Pell-Lucas numbers*

$$[(Q_{m+1})^k - (Q_m)^k] = [Q_{(m+1)k}^{(k)} - Q_{mk}^{(k)}]. \tag{7}$$

Proof. From (6), we get

$$\begin{aligned} Q_{(m+1)k}^{(k)} - Q_{mk}^{(k)} &= [(Q_m)^{k-k}(Q_{m+1})^k] - [(Q_m)^{k-0}(Q_{m+1})^0] \\ &= (Q_{m+1})^k - (Q_m)^k. \end{aligned}$$

□

For $k = 2, 3, 4$ and n, we have the interesting following properties between these numbers.

$$
\begin{aligned}
Q_{2n}^{(2)} &= Q_n^2, \\
Q_{2n+1}^{(2)} &= Q_n Q_{n+1}, \\
Q_{3n}^{(3)} &= Q_n^3, \\
Q_{3n+1}^{(3)} &= Q_n^2 Q_{n+1}, \\
Q_{3n+2}^{(3)} &= Q_n Q_{n+1}^2, \\
Q_{4n}^{(4)} &= Q_n^4, \\
Q_{4n+1}^{(4)} &= Q_n^3 Q_{n+1}, \\
Q_{4n+2}^{(4)} &= Q_n^2 Q_{n+1}^2, \\
Q_{4n+3}^{(4)} &= Q_n Q_{n+1}^3.
\end{aligned}
$$

Theorem 3.9. *For n, we have the relation*

$$Q_{kn+1}^{(k)} = 2Q_{kn}^{(k)} + Q_{kn-1}^{(k)}.$$

Proof. By using (6), we get

$$
\begin{aligned}
2Q_{kn}^{(k)} + Q_{kn-1}^{(k)} &= 2Q_n^k + Q_{n-1}Q_n^{k-1} \\
&= Q_n^{k-1}(2Q_n + Q_{n-1}) \\
&= Q_n^{k-1}Q_{n+1} \\
&= Q_{kn+1}^{(k)}.
\end{aligned}
$$

□

Theorem 3.10. *For $n, k \geq 2$, Cassini's Identity for $Q_n^{(k)}$ is as follows:*

$$Q_{kn+t}^{(k)} Q_{kn+t-2}^{(k)} - (Q_{kn+t-1}^{(k)})^2 = \left\{ \begin{array}{c} Q_n^{2k-2}(-1)^{n+1}8, t = 1 \\ 0, t \neq 1 \end{array} \right\}.$$

Proof. By using (6), we get

$$
\begin{aligned}
Q_{kn+t}^{(k)} Q_{kn+t-2}^{(k)} - (Q_{kn+t-1}^{(k)})^2 &= (Q_n^{k-1}Q_{n+t})(Q_n^{k-1}Q_{n+t-2}) - (Q_n^{k-1}Q_{n+k-1})^2 \\
&= (Q_n^{k-1})^2(Q_{n+t}Q_{n+t-2} - (Q_{n+k-1})^2) \\
&= Q_n^{2k-2}(Q_{n+t}Q_{n+t-2} - (Q_{n+k-1})^2)
\end{aligned}
$$

For $t = 1$;

$$
\begin{aligned}
&= Q_n^{2k-2}(Q_{n+1}Q_{n-1} - (Q_n)^2) \\
&= Q_n^{2k-2}(-1)^{n+1}8.
\end{aligned}
$$

For $t \neq 1$, $t = m$, $(m \in N)$;

$$\begin{aligned} &= Q_n^{2k-2}(Q_{n+t}Q_{n+t-2} - (Q_{n+k-1})^2) \\ &= Q_n^{2k-2}(Q_{2n+2n-2}^{(2)} - Q_{2n+2n-2}^{(2)}) \\ &= 0. \end{aligned}$$

□

For $t = 0, 1, 2, \ldots, k-1$, we obtain the following relations:

$$P_{nk+t}^{(k)} = P_n^{k-t} P_{n+1}^t$$

and

$$Q_{nk+t}^{(k)} = Q_n^{k-t} Q_{n+1}^t.$$

Definition 3.11. *For $n, k \in N (k \neq 0)$, there are only numbers m and r such that $n = mk + r$ and $0 \leq r < k$. Then we define the generalized k-Pell polynomials $P_n^{(k)}(x)$ by*

$$P_n^{(k)}(x) := \left(\frac{\alpha^m(x) - \beta^m(x)}{\alpha(x) - \beta(x)} \right)^{k-r} \left(\frac{\alpha^{m+1} - \beta^{m+1}}{\alpha(x) - \beta(x)} \right)^r, n = mk + r$$

where $\alpha(x) = x + \sqrt{x^2+1}$ and $\beta(x) = x - \sqrt{x^2+1}$.

Moreover, we can obtain the generalized $k-$Pell polynomials by matrixes as follows:

$$P_{n+1}^{k-1} M^n = \begin{bmatrix} P_{kn+k+1}^{(k)} & P_{kn+k}^{(k)} \\ P_{kn+k}^{(k)} & P_{kn+k-1}^{(k)} \end{bmatrix}.$$

Let's give sample values for the generalized $k-$Pell polynomials in Table 3.

Table 3:

	$k = 1$	$k = 2$	$k = 3$	$k = 4$
$P_0^{(k)}(x)$	0	0	0	0
$P_1^{(k)}(x)$	1	0	0	0
$P_2^{(k)}(x)$	$2x$	1	0	0
$P_3^{(k)}(x)$	$1 + 4x^2$	$2x$	1	0
$P_4^{(k)}(x)$	$4x + 8x^3$	$4x^2$	$2x$	16
$P_5^{(k)}(x)$	$1 + 12x^2 + 16x^4$	$2x + 8x^3$	$4x^2$	$2x$

The generalized $k-$Pell polynomials and known Pell polynomials are related by

$$P_n^{(k)}(x) := (P_m(x))^{k-r}(P_{m+1}(x))^r, n = mk + r \tag{8}$$

When $k = 1$in last equation, we get that $m = n$ and $r = 0$ so $P_n^{(1)}(x) = P_n(x)$.

Theorem 3.12. *For k and m, we have the following relationship between the generalized $k-$ Pell polynomials and the classical Pell polynomials*

$$[(P_{m+1}(x))^k - (P_m(x))^k] = [P^{(k)}_{(m+1)k}(x) - P^{(k)}_{mk}(x)].$$

Proof. From (8), we get

$$\begin{aligned}
[P^{(k)}_{(m+1)k}(x) - P^{(k)}_{mk}(x)] &= [(P_m(x))^{k-k}(P_{m+1}(x))^k] - [(P_m(x))^{k-0}(P_{m+1}(x))^0] \\
&= (P_{m+1}(x))^k - (P_m(x))^k.
\end{aligned}$$

□

The following equations are provided.

$$\begin{aligned}
P^{(2)}_{2n}(x) &= P^2_n(x), \\
P^{(2)}_{2n+1} &= P_n(x)P_{n+1}(x), \\
P^{(2)}_{2n+1} &= 2xP^{(2)}_{2n}(x) + P^{(2)}_{2n-1}(x), \\
P^{(3)}_{3n}(x) &= P^3_n(x), \\
P^{(3)}_{3n+1} &= P^2_n(x)P_{n+1}(x), \\
P^{(3)}_{3n+2} &= P_n(x)P^2_{n+1}(x), \\
P^{(3)}_{3n+1} &= 2xP^{(3)}_{3n}(x) + P^{(3)}_{3n-1}(x), \\
P^{(4)}_{4n}(x) &= P^4_n(x), \\
P^{(4)}_{4n+1} &= P^3_n(x)P_{n+1}(x), \\
P^{(4)}_{4n+2} &= P^2_n(x)P^2_{n+1}(x), \\
P^{(4)}_{4n+3} &= P_n(x)P^3_{n+1}(x), \\
P^{(4)}_{4n+1} &= 2xP^{(4)}_{4n}(x) + P^{(4)}_{4n-1}(x).
\end{aligned}$$

Theorem 3.13. *For n, we have the following relation*

$$P^{(k)}_{kn+1} = 2xP^{(k)}_{kn}(x) + P^{(k)}_{kn-1}(x).$$

Proof. From (8), we have

$$\begin{aligned}
2xP^{(k)}_{kn}(x) + P^{(k)}_{kn-1}(x) &= 2xP^k_n(x) + P_{n-1}(x)P^{k-1}_n(x) \\
&= P^{k-1}_n(x)(2xP_n(x) + P_{n-1}(x)) \\
&= P^{k-1}_n(x)P_{n+1}(x) \\
&= P^{(k)}_{kn+1}(x).
\end{aligned}$$

□

Theorem 3.14. *For $n, k \geq 2$, Cassini's Identity for $P_n^{(k)}(x)$ is as follows:*

$$P_{kn+t}^{(k)}(x)P_{kn+t-2}^{(k)}(x) - (P_{kn+t-1}^{(k)}(x))^2 = \left\{ \begin{array}{c} P_n^{2k-2}(x)(-1)^n, t = 1 \\ 0, t \neq 1 \end{array} \right\}.$$

Proof. By using (8) we get

$$\begin{aligned} P_{kn+t}^{(k)}(x)P_{kn+t-2}^{(k)}(x) - (P_{kn+t-1}^{(k)}(x))^2 &= (P_n^{k-1}(x)P_{n+t}(x))(P_n^{k-1}(x)P_{n+t-2}(x)) - \\ &- (P_n^{k-1}(x)P_{n+t-1}(x))^2 \\ &= (P_n^{k-1}(x))^2(P_{n+t}(x)P_{n+t-2}(x) - (P_{n+t-1}(x))^2) \\ &= P_n^{2k-2}(x)(P_{n+t}(x)P_{n+t-2}(x) - (P_{n+t-1}(x))^2) \end{aligned}$$

For $t = 1$;

$$\begin{aligned} &= P_n^{2k-2}(x)(P_{n+1}(x)P_{n-1}(x) - (P_n(x))^2) \\ &= P_n^{2k-2}(x)(-1)^n. \end{aligned}$$

For $t \neq 1$, $t = m$, $(m \in N)$;

$$\begin{aligned} &= P_n^{2k-2}(x)(P_{n+m}(x)P_{n+m-2}(x) - (P_{n+m-1}(x))^2) \\ &= P_n^{2k-2}(x)(P_{2n+2m-2}^{(2)}(x) - P_{2n+2m-2}^{(2)}(x)) \\ &= 0. \end{aligned}$$

□

Definition 3.15. *For $n, k \in N (k \neq 0)$, there are only numbers m and r such that $n = mk + r$ and $0 \leq r < k$. Then we define the generalized $k-$Pell-Lucas polinomials $Q_n^{(k)}(x)$ by*

$$Q_n^{(k)}(x) := (\alpha^m(x) + \beta^m(x))^{k-r}(\alpha^{m+1}(x) + \beta^{m+1}(x))^r$$

where $\alpha = x + \sqrt{x^2+1}$ and $\beta = x - \sqrt{x^2+1}$.

Moreover, we can obtain the generalized k-Pell-Lucas polynomials by matrixes as follows:

$$\begin{bmatrix} Q_{kn+k+1}^{(k)}(x) \\ Q_{kn+1}^{(k)}(x) \end{bmatrix} = Q_{n+1}^{k-1}(x)M^n \begin{bmatrix} 2x \\ 2 \end{bmatrix}.$$

Let's give sample values for the generalized $k-$Pell-Lucas polynomials $Q_n^{(k)}(x)$ in the Table 4.

The generalized k-Pell-Lucas polynomials and the classical Pell-Lucas polynomials are related by

$$Q_n^{(k)}(x) := (Q_m(x))^{k-r}(Q_{m+1}(x))^r, n = mk + r \tag{9}$$

When $k = 1$ in last equation, we get that $m = n$ and $r = 0$ so $Q_n^{(1)}(x) = Q_n(x)$.

Table 4:

	$k=1$	$k=2$	$k=3$	$k=4$
$Q_0^{(k)}$	2	4	8	16
$Q_1^{(k)}$	$2x$	$4x$	$8x$	$16x$
$Q_2^{(k)}$	$2+4x^2$	$4x^2$	$8x^2$	$16x^2$
$Q_3^{(k)}$	$6x+8x^3$	$4x+8x^3$	$8x^3$	$16x^3$
$Q_4^{(k)}$	$2+16x^2+16x^4$	$4+16x+16x^4$	$8x^2+16x^4$	$16x^4$
$Q_5^{(k)}$	$10x+40x^3+32x^5$	$12x+40x^3+32x^5$	$8x+32x^3+32x^5$	$16x^3+32x^5$

Theorem 3.16. *For fixed k and m, we have*

$$[(Q_{m+1}(x))^k - (Q_m(x))^k] = [Q^{(k)}_{(m+1)k}(x) - Q^{(k)}_{mk}(x)].$$

Proof. From (9), we have

$$\begin{aligned} Q^{(k)}_{(m+1)k}(x) - Q^{(k)}_{mk}(x) &= [(Q_m(x))^{k-k}(Q_{m+1}(x))^k] - [(Q_m(x))^{k-0}(Q_{m+1}(x))^0] \\ &= (Q_{m+1}(x))^k - (Q_m(x))^k. \end{aligned}$$

□

The following equations are provided.

$$\begin{aligned} Q^{(2)}_{2n}(x) &= Q^2_n(x), \\ Q^{(2)}_{2n+1}(x) &= Q_n(x)Q_{n+1}(x), \\ Q^{(3)}_{3n}(x) &= Q^3_n(x), \\ Q^{(3)}_{3n+1}(x) &= Q^2_n(x)Q_{n+1}(x), \\ Q^{(3)}_{3n+2}(x) &= Q_n(x)Q^2_{n+1}(x), \\ Q^{(4)}_{4n}(x) &= Q^4_n(x), \\ Q^{(4)}_{4n+1}(x) &= Q^3_n(x)Q_{n+1}(x), \\ Q^{(4)}_{4n+2}(x) &= Q^2_n(x)Q^2_{n+1}(x), \\ Q^{(4)}_{4n+3}(x) &= Q_n(x)Q^3_{n+1}(x). \end{aligned}$$

Theorem 3.17. *For fixed natural numbers n, we have*

$$Q^{(k)}_{kn+1}(x) = 2xQ^{(k)}_{kn}(x) + Q^{(k)}_{kn-1}(x).$$

Proof. By using (9), we get

$$2xQ^{(k)}_{kn}(x) + Q^{(k)}_{kn-1}(x) = 2xQ^k_n(x) + Q_{n-1}(x)Q^k_{n-1}(x)$$

$$\begin{aligned} &= Q_n^{k-1}(x)(2xQ_n(x)+Q_{n-1}(x)) \\ &= Q_n^{k-1}(x)Q_{n+1}(x) \\ &= Q_{kn+1}^{(k)}(x). \end{aligned}$$

□

Theorem 3.18. *For $n, k \geq 2$, Cassini's Identity for $Q_n^{(k)}(x)$ is as follows:*

$$Q_{kn+t}^{(k)}(x)Q_{kn+t-2}^{(k)}(x) - (Q_{kn+t-1}^{(k)}(x))^2 = \left\{ \begin{array}{c} Q_n^{2k-2}(x)(-1)^{n-1}4(1+x^2), t=1 \\ 0, t \neq 1 \end{array} \right\}$$

Proof. From (9), we get

$$\begin{aligned} Q_{kn+t}^{(k)}(x)Q_{kn+t-2}^{(k)}(x) - (Q_{kn+t-1}^{(k)}(x))^2 &= (Q_n^{k-1}(x)Q_{n+t}(x))(Q_n^{k-1}(x)Q_{n+t-2}(x)) - \\ &- (Q_n^{k-1}(x)Q_{n+k-1}(x))^2 \\ &= (Q_n^{k-1}(x))^2(Q_{n+t}(x)Q_{n+t-2}(x) - (Q_{n+k-1}(x))^2) \\ &= Q_n^{2k-2}(x)(Q_{n+t}(x)Q_{n+t-2}(x) - (Q_{n+k-1}(x))^2) \end{aligned}$$

For $t = 1$;

$$\begin{aligned} &= Q_n^{2k-2}(x)(Q_{n+1}(x)Q_{n-1}(x) - (Q_n(x))^2) \\ &= Q_n^{2k-2}(x)(-1)^{n-1}4(1+x^2). \end{aligned}$$

For $t \neq 1$, $t = m$, $(m \in N)$;

$$\begin{aligned} &= Q_n^{2k-2}(x)(Q_{n+m}(x)Q_{n+m-2}(x) - (Q_{n+m-1}(x))^2) \\ &= Q_n^{2k-2}(x)(Q_{2n+2m-2}^{(2)}(x) - Q_{2n+2m-2}^{(2)}(x)) \\ &= 0. \end{aligned}$$

□

For $t = 0, 1, 2, \ldots, k-1$, we obtain the following relations:

$$P_{nk+t}^{(k)}(x) = P_n^{k-t}(x)P_{n+1}^t(x)$$

and

$$Q_{nk+t}^{(k)}(x) = Q_n^{k-t}(x)Q_{n+1}^t(x).$$

Conclusion

In the present paper, we defined new families of $k-$Pell, $k-$ Pell-Lucas numbers and their polynomials. We obtained the generating functions of the families for some k. We gave some relations among these families and the classical Pell numbers and the classical Pell-Lucas numbers. Further, we introduced the polynomials for these families. Then we gave relations among their polynomials and the classical Pell polynomials and the classical Pell-Lucas polynomials. Moreover, we showed these generalizations in matrixes. Also, we proved Cassini identities for these families and polynomials.

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Engin ÖZKAN
Department of Mathematics, Erzincan Binali Yıldırım University, Faculty of Arts and Sciences, Erzincan, Turkey.
E-mail: eozkanmath@gmail.com or eozkan@erzincan.edu.tr

Merve TAŞTAN
Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Erzincan, Turkey
E-mail: mervetastan24@gmail.com

Aysun Aktaş
Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Erzincan, Turkey
E-mail: aysun.cicek36@gmail.com

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