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A Family of Theta-Function İdentities İnvolving R_α, R_β and R_m-functions Related to Jacobi's Triple-Product Identity

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Abstract. The author establish a set of three new relationships involving R_α, R_β and R_m-functions, which are based upon q-product identities and Jacobi's celebrated triple-product identity; and also provide answer for a open problem of Srivastava *et al* [32].

Key Words and Phrases: Theta-function identities; R_m-functions; Jacobi's triple-product identity; Ramanujan's theta functions; q-Product identities; Euler's Pentagonal Number Theorem; Rogers-Ramanujan continued fraction; Rogers-Ramanujan identities.
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1. Introduction and Definitions

Throughout this article, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers, respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}$$

and recall the following q-notations (see, for example, [5], [7, Chapter 3, Section 3.2.1], [33, Chapter 6] and [34, pp. 346 *et seq.*]). The q-shifted factorial $(a; q)_n$ is defined (for $|q| < 1$) by

$$(a;q)_n := \begin{cases} 1 & (n = 0) \\ \prod_{k=0}^{n-1}(1 - aq^k) & (n \in \mathbb{N}), \end{cases} \tag{1.1}$$

where $a, q \in \mathbb{C}$ and it is assumed *tacitly* that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$. We also write

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k) = \prod_{k=1}^{\infty}(1 - aq^{k-1}) \qquad (a, q \in \mathbb{C};\ |q| < 1). \tag{1.2}$$

It should be noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in the equation (1.2) diverges. So, whenever $(a; q)_\infty$ is involved in a given formula, the constraint $|q| < 1$

will be *tacitly* assumed to be satisfied.

The following notations are also frequently used in our investigation:

$$(a_1, a_2, \cdots, a_m; q)_n := (a_1; q)_n \ (a_2; q)_n \ \cdots \ (a_m; q)_n \tag{1.3}$$

and

$$(a_1, a_2, \cdots, a_m; q)_\infty := (a_1; q)_\infty \ (a_2; q)_\infty \ \cdots \ (a_m; q)_\infty. \tag{1.4}$$

Ramanujan (see [26] and [27]) defined the general theta function $\mathfrak{f}(a, b)$ as follows (see, for details, [8, p. 31, Eq. (18.1)] and [29]):

$$\begin{aligned} \mathfrak{f}(a, b) &= 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} \ (a^n + b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} \ b^{\frac{n(n-1)}{2}} = \mathfrak{f}(b, a) \qquad (|ab| < 1). \end{aligned} \tag{1.5}$$

We find from this last equation (1.5) that

$$\mathfrak{f}(a, b) = a^{\frac{n(n+1)}{2}} \ b^{\frac{n(n-1)}{2}} \ \mathfrak{f}\big(a(ab)^n, b(ab)^{-n}\big) = \mathfrak{f}(b, a) \qquad (n \in \mathbb{Z}). \tag{1.6}$$

In fact, Ramanujan (see [26] and [27]) also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, is given by (see [8, p. 35, Entry 19]):

$$\mathfrak{f}(a, b) = (-a; ab)_\infty \ (-b; ab)_\infty \ (ab; ab)_\infty \tag{1.7}$$

or, equivalently, by (see [24])

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{n^2} \ z^n &= \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 + zq^{2n-1}\right) \left(1 + \frac{1}{z} \ q^{2n-1}\right) \\ &= \left(q^2; q^2\right)_\infty \ \left(-zq; q^2\right)_\infty \ \left(-\frac{q}{z}; q^2\right)_\infty \qquad (|q| < 1; \ z \neq 0). \end{aligned}$$

Note 1. Equation (1.6) holds true as stated only if n is any integer. Moreover, in case n is not an integer, this result (1.6) is only approximately true (see, for details, [26, Vol. **2**, Chapter XVI, p. 193, Entry 18 (iv)]).

Note 2. Historically speaking, the q-series identity (1.7) or its above-mentioned equivalent form was first proved by Carl Friedrich Gauss (1777–1855).

Several q-series identities, which emerge naturally from Jacobi's triple-product identity (1.7), are worthy of note here (see, for details, [8, pp. 36–37, Entry 22]):

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}$$

$$= \left\{(-q;q^2)_\infty\right\}^2 (q^2;q^2)_\infty = \frac{(-q;q^2)_\infty\,(q^2;q^2)_\infty}{(q;q^2)_\infty\,(-q^2;q^2)_\infty}; \tag{1.8}$$

$$\psi(q) := \mathfrak{f}(q,q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}; \tag{1.9}$$

$$f(-q) := \mathfrak{f}(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n\, q^{\frac{n(3n-1)}{2}}$$
$$= \sum_{n=0}^{\infty}(-1)^n\, q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty}(-1)^n\, q^{\frac{n(3n+1)}{2}} = (q;q)_\infty. \tag{1.10}$$

Equation (1.10) is known as Euler's *Pentagonal Number Theorem.* Remarkably, the following q-series identity:

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty} = \frac{1}{\chi(-q)} \tag{1.11}$$

provides the analytic equivalent form of Euler's famous theorem (see, for details, [5] and [7]).

Theorem 1.1. (Euler's Pentagonal Number Theorem) The number of partitions of a given positive integer n into distinct parts is equal to the number of partitions of n into odd parts.

We also recall the Rogers-Ramanujan continued fraction $R(q)$ given by

$$R(q) := q^{\frac{1}{5}}\frac{H(q)}{G(q)} = q^{\frac{1}{5}}\frac{\mathfrak{f}(-q,-q^4)}{\mathfrak{f}(-q^2,-q^3)} = q^{\frac{1}{5}}\frac{(q;q^5)_\infty\,(q^4;q^5)_\infty}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty}$$
$$= \frac{q^{\frac{1}{5}}}{1+}\;\frac{q}{1+}\;\frac{q^2}{1+}\;\frac{q^3}{1+} \qquad (|q|<1). \tag{1.12}$$

Here $G(q)$ and $H(q)$, which are associated with the widely-investigated Roger-Ramanujan identities, are defined as follows:

$$G(q) := \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q,-q^4)}$$
$$= \frac{1}{(q;q^5)_\infty\,(q4;q^5)_\infty} = \frac{(q^2;q^5)_\infty\,(q^3;q^5)_\infty\,(q^5;q^5)_\infty}{(q;q)_\infty} \tag{1.13}$$

and

$$H(q) := \sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q^2,-q^3)} = \frac{1}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty}$$
$$= \frac{(q;q^5)_\infty\,(q^4;q^5)_\infty\,(q^5;q^5)_\infty}{(q;q)_\infty}, \tag{1.14}$$

and the functions $\mathfrak{f}(a,b)$ and $f(-q)$ are given by the equations (1.5) and (1.10), respectively.

For a detailed historical account of (and for various related developments stemming from) the Rogers-Ramanujan continued fraction (1.12) as well as the Rogers-Ramanujan identities (1.13) and (1.14), the interested reader may refer to the monumental work [8, p. 77 *et seq.*] (see also [29] and [33]).

The following continued-fraction results may be recalled now (see, for example, [10, p. 5, Eq. (2.8)]).

Theorem 1.2. Suppose that $|q|<1$. Then

$$(q^2;q^2)_\infty(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{1}{1-}\,\frac{q}{1+}\,\frac{q(1-q)}{1-}\,\frac{q^3}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^5}{1+}\,\frac{q^3(1-q^3)}{1-\cdots}$$

$$= \cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1-\vdots}}}}}}}, \tag{1.15}$$

$$\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots$$

$$= \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\cfrac{q^6}{1+\vdots}}}}}}}, \tag{1.16}$$

and

$$C(q) = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1+\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots$$

$$= 1 + \cfrac{q}{1 + \cfrac{q^2}{1 + \cfrac{q^3}{1 + \cfrac{q^4}{1 + \cfrac{q^5}{1 + \cfrac{q^6}{1 + \vdots}}}}}} \tag{1.17}$$

By introducing the general family $R(s, t, l, u, v, w)$, Andrews *et al.* [6] investigated a number of interesting double-summation hypergeometric q-series representations for several families of partitions and further explored the rôle of double series in combinatorial-partition identities:

$$R(s, t, l, u, v, w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn}\, r(l, u, v, w; n), \tag{1.18}$$

where

$$r(l, u, v, w : n) := \sum_{j=0}^{[\frac{n}{u}]} (-1)^j \, \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}\, (q^{uv};q^{uv})_j}. \tag{1.19}$$

We also recall the following interesting special cases of (1.18) (see, for details, [6, p. 106, Theorem 3]; see also [29]):

$$R(2, 1, 1, 1, 2, 2) = (-q; q^2)_\infty, \tag{1.20}$$

$$R(2, 2, 1, 1, 2, 2) = (-q^2; q^2)_\infty \tag{1.21}$$

and

$$R(m, m, 1, 1, 1, 2) = \frac{(q^{2m}; q^{2m})_\infty}{(q^m; q^{2m})_\infty}. \tag{1.22}$$

Recently, Srivastva *et al* [32] has introduced following three notations:

$$R_\alpha = R(2, 1, 1, 1, 2, 2); R_\beta = R(2, 2, 1, 1, 2, 2); R_m = R(m, m, 1, 1, 1, 2) \qquad (m \in \mathbb{N}). \tag{1.23}$$

for multivaraite R-functions, which we shall use for computation of our main results in section 2.

Ever since the year 2015, several new advancements and generalizations of the existing results were made in regard to combinatorial partition-theoretic identities (see, for example, [11] to [17], [20] to [21] and [29] to [31]). In particular, Chaudhary *et al.* generalized several known results on character formulas (see [19]), Roger-Ramanujan type identities (see [16]), Eisenstein series, the Ramanujan-Gollnitz-Gordon continued fraction (see [17]), the 3-dissection property (see [14]), Ramanujan's modular equations of degrees 3, 7 and 9 (see [12] and [13]), and so on, by using combinatorial partition-theoretic identities. An interesting recent investigation on the subject of combinatorial partition-theoretic identities by Hahn *et al.* [23] is also worth mentioning in this connection.

Each of the following preliminary results will be needed for the demonstration of our main results in this paper (see [4, Theorem 5.1]; [8, Entry 51, p.204]; [25, Theorem 3.1]):

I. If

$$U = \frac{\phi^4(-q)}{\phi^4(-q^3)} \quad \text{and} \quad V = \frac{\psi^4(q)}{\psi^4(q^3)},$$

then

$$U - UV = 9 - V. \tag{1.24}$$

II. If

$$M = \frac{f^2(-q)}{q^{\frac{1}{6}} f^2(-q^3)} \quad \text{and} \quad N = \frac{f^2(-q^2)}{q^{\frac{1}{3}} f^2(-q^6)},$$

then

$$MN + \frac{9}{MN} = \left(\frac{N}{M}\right)^3 + \left(\frac{M}{N}\right)^3. \tag{1.25}$$

III. If

$$U = \frac{\phi^4(q)}{\phi^4(q^3)},$$

then

$$\frac{f^4(q) f^4(-q^2)}{q f^4(q^3) f^4(-q^6)} = \frac{U(U-9)}{1-U}, \quad U \neq 1. \tag{1.26}$$

2. A Set of Main Results

In this section, we establish a set of three new relationships involving R_α, R_β and R_m-functions, which are based upon q-product identities and Jacobi's celebrated triple-product identity.

Theorem 2.1. *Each of the following relationships holds true:*

$$\left(9 - \frac{1}{q}\left\{\frac{R_1}{R_3}\right\}^4\right) = \left(1 - \frac{1}{q}\left\{\frac{R_1}{R_3}\right\}^4\right) \cdot \frac{(q,q,q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{(q^3,q^3,q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}$$
$$\cdot \frac{(-q^3,-q^3,-q^3,-q^3,-q^6,-q^6,-q^6,-q^6;q^6)_\infty}{\{R_\alpha R_\beta\}^4} \tag{2.1}$$

Equation (2.1) gives inter-relationships between R_1, R_3, R_α and R_β.

$$\frac{(q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{q^{\frac{1}{2}}(q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty} + \frac{9q^{\frac{1}{2}}(q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}{(q,q,q^2,q^2,q^2,q^2;q^2)_\infty}$$
$$= \frac{1}{q^{\frac{1}{2}}}\left\{\frac{R_1 R_{12}(q^3;q^6)_\infty(q^6;q^{12})_\infty(q^{12};q^{24})_\infty}{(q^2;q^2)_\infty(q^6;q^{12})_\infty(q^{24};q^{24})_\infty}\right\}^6 + q^{\frac{1}{2}}\left\{\frac{R_2 R_3 (q;q^2)_\infty (q^2;q^4)_\infty}{(q^4;q^4)_\infty (q^6;q^6)_\infty}\right\}^6. \tag{2.2}$$

Equation (2.2) gives inter-relationships between R_1, R_2, R_3 and R_{12}.

$$\frac{(-q,-q,-q,-q;-q)_\infty(q^2,q^2,q^2,q^2;q^2)_\infty}{q.(-q^3,-q^3,-q^3,-q^3;-q^3)_\infty(q^6,q^6,q^6,q^6;q^6)_\infty} = \frac{\{R_\alpha R_1(q^3,-q^6;q^6)_\infty\}^4}{\{(-q^3,q^6;q^6)_\infty\}^4}$$
$$\cdot\frac{\{R_1R_\alpha(q^3,-q^6;q^6)_\infty\}^4 - 9\{R_2(-q^3,q^6;q^6)\}^4}{\{R_\beta(-q^3,q^6;q^6)_\infty\}^4}$$
$$\cdot\frac{\{(-q^3,q^6;q^6)_\infty\}^4}{\{R_2(-q^3,q^6;q^6)_\infty\}^4 - \{R_1R_\alpha(q^3,q^6;q^6)\}^4} \tag{2.3}$$

Equation (2.3) gives inter-relationships between R_1, R_2, R_α and R_β.
It is assumed that each member of the assertions (2.1) to (2.3) exists.

Proof: First of all, in order to prove the assertion (2.1) of Theorem 2.1, apply the identities (1.8) and (1.9), under the given precondition of result (1.24); and further using(1.23), we obtain.

$$U = \frac{\phi^4(-q)}{\phi^4(-q^3)} = \frac{(q,q,q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{\{R_\alpha R_\beta\}^4}$$
$$\cdot\frac{(-q^3,-q^3,-q^3,-q^3,-q^6,-q^6,-q^6,-q^6;-q^6)_\infty}{(q^3,q^3,q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}, \tag{2.4}$$

$$V = \frac{\psi^4(q)}{q\ \psi^4(q^3)} = \frac{1}{q}\left\{\frac{R_1}{R_3}\right\}^4, \tag{2.5}$$

and

$$U - UV = \left(1 - \frac{1}{q}\left\{\frac{R_1}{R_3}\right\}^4\right)\cdot\frac{(q,q,q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{\{R_\alpha R_\beta\}^4}$$
$$\cdot\frac{(-q^3,-q^3,-q^3,-q^3,-q^6,-q^6,-q^6,-q^6;-q^6)_\infty}{(q^3,q^3,q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}. \tag{2.6}$$

Now, upon substituting from these last results (2.4)–(2.6) into (1.24), if we rearrange the terms and use some algebraic manipulations, we are led to the first assertion (2.1) of Theorem 2.1.
Secondly, we prove the second relationship (2.2) of Theorem 2.1. Indeed, if we first apply the identity (1.10), under the given precondition of the assertion (1.25), ater some simplifications, the following values for M and N would follow:

$$M = \frac{f^2(-q)}{q^{\frac{1}{6}}f^2(-q^3)} = \frac{\{(q;q)_\infty\}^2}{q^{\frac{1}{6}}\{(q^3;q^3)_\infty\}^2}, \tag{2.7}$$

and

$$N = \frac{f^2(-q^2)}{q^{\frac{1}{3}}f^2(-q^6)} = \frac{\{(q^2;q^2)_\infty\}^2}{q^{\frac{1}{3}}\{(q^6;q^6)_\infty\}^2}. \tag{2.8}$$

Now, with the help these last results (2.7) and (2.8), and using (1.23), we obtain

$$MN+\frac{9}{MN}=\frac{(q,q,q^2,q^2,q^2,q^2;q^2)_\infty}{q^{\frac{1}{2}}(q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}+\frac{9q^{\frac{1}{2}}(q^3,q^3,q^6,q^6,q^6,q^6;q^6)_\infty}{q^{\frac{1}{2}}(q,q,q^2,q^2,q^2,q^2;q^2)_\infty}, \tag{2.9}$$

and

$$\left(\frac{N}{M}\right)^3+\left(\frac{M}{N}\right)^3$$

$$=\frac{1}{q^{\frac{1}{2}}}\left\{\frac{R_1R_{12}(q^3;q^6)_\infty(q^6;q^{12})_\infty(q^{12};q^{24})_\infty}{(q^2;q^2)_\infty(q^6;q^{12})_\infty(q^{24};q^{24})_\infty}\right\}^6+q^{\frac{1}{2}}\left\{\frac{R_2R_3(q;q^2)_\infty(q^2;q^4)_\infty}{(q^4;q^4)_\infty(q^6;q^6)_\infty}\right\}^6. \tag{2.10}$$

equating (3.9) and (3.10), as precondition given in (1.25), we are led to the second assertion (2.2) of Theorem 1.2.

Finally, we have to prove our third identity (2.3) of Theorem 1.2. In order to establish this identity, using(1.8) and (1.10), and further applying (1.23), we get

$$U=\left\{\frac{R_\alpha R_1(q^3,-q^6;q^6)_\infty}{R_\beta(-q^3,q^6;q^6)_\infty}\right\}^4. \tag{2.11}$$

Hence

$$\frac{U^2-9U}{1-U}=\frac{\{R_\alpha R_1(q^3,-q^6;q^6)_\infty\}^4}{\{(-q^3,q^6;q^6)_\infty\}^4}\cdot\frac{\{R_1R_\alpha(q^3,-q^6;q^6)_\infty\}^4-9\{R_2(-q^3,q^6;q^6)\}^4}{\{R_\beta(-q^3,q^6;q^6)_\infty\}^4}$$

$$\cdot\frac{\{(-q^3,q^6;q^6)_\infty\}^4}{\{R_2(-q^3,q^6;q^6)_\infty\}^4-\{R_1R_\beta(q^3,-q^6;q^6)\}^4}. \tag{2.12}$$

and

$$\frac{f^4(q)f^4(-q^2)}{qf^4(q^3)f^4(-q^6)}=\frac{(-q,-q,-q,-q;-q)_\infty(q^2,q^2,q^2,q^2;q^2)_\infty}{q.(-q^3,-q^3,-q^3,-q^3;-q^3)_\infty(q^6,q^6,q^6,q^6;q^6)_\infty} \tag{2.13}$$

equating (2.12) and(2.13), as precondition given in (1.26), we obtain(2.3). We thus have completed our proof of the above Theorem.

3. Concluding Remarks and Observations

The present investigation was motivated by several recent developments dealing essentially with theta-function identities and combinatorial partition-theoretic identities. Here, in this article, we have established a family of three presumably new theta-function identities which depict the inter-relationships that exist among R_α, R_β and derivatives of R_m-functions. We have also considered several closely-related identities such as (for example) q-product identities and Jacobi's triple-product identities. And, with a view to further motivating researches involving theta-function identities and combinatorial partition-theoretic identities,we have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article.

The list of citations, which we have included in this article, is believed to be potentially useful for indicating some of the directions for further researches and related developments on the subject-matter which we have dealt with here. In particular, the recent works by Adiga *et al.* (see [1] – [3]), Cao *et al.* [9], Chaudhary *et al.* (see [10], [18] to [22]), Hahn *et al.* [23], and Srivastava *et al.* (see [30], [35]–[37]) are worth mentioning here.

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