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On Extremality of Affine Image of Some General Class of Algebraic Varieties

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Abstract. In this paper one studies extremality of affine image of some general class of algebraic varieties given by all multivariate monomials the degree of every variable of which does not exceeding given natural numbers. One of conjectures of V. G. Sprindzhuk states that such varieties are extremal. We prove that an affine image of this variety is extremal using theorem on convergence exponent for the special integral of Terry's problem. As a consequence we show that this result is so strong as the Sprindzhuk's hypothesis, that is the initial algebraic variety is extremal. is

Key Words and Phrases: Algebraic variety, Terry's problem, extremality, Diophantine Approximation

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1. Introduction

Let's consider set of all polynomials of variable x with integral coefficients which degrees don't exceed k:

$$\Pi = \left\{ f(x) = \sum_{i=1}^{k} a_i x^i | a_i \in R \right\}.$$

The number

$$H(f) = \max(|a_0|, |a_1|, ..., |a_k|)$$

is called the height of polynomial $f(x)$. Let a real transcendental number α (hence, α is not a root of considered family Π any of polynomials) be given. Let $H > 0$ be a real number. For given $H > 0$, consider such polynomials height of which does not exceeding H. We shall designate by $\gamma_0 = \gamma_0(\alpha)$ the supremum of those positive numbers $\gamma > 0$, for which the inequality

$$|f(\alpha)| < H^{-\gamma};\ H = H(f)$$

is satisfied for infinite number of polynomials from Π, when $H \to \infty$; in other words for any $\varepsilon > 0$ there will be found an increasing sequence $H_1, ..., H_m, ...$ of real numbers not bounded from above such that for all such H_m we have:

$$\gamma = \gamma_0 + \varepsilon.$$

This number is defined for every natural k, and, therefore, it is possible to define number (final or infinite)

$$\rho = \overline{\lim_{k\to\infty}}\frac{\gamma_0}{k}.$$

Let's notice that for transcendental numbers by Dirichlet's principle always $\gamma_0 \geq k$, and, consequently, $\rho \geq 1$. Entering classification ([9]) Mahler ([10]) has come out with the assumption, that $\gamma_0 = n$. This assumption has been proved in [12].

In the works [11], [12, p. 433] Sprindzhuk V. G has formulated a number of problems for the theory Diophantine Approximations concerning extremal varieties. To our days these hypotheses are proved (most general of them is established by D.Klejnbokom and G.Margulis [13]). In the present work we shall prove that an affine image of this variety is extremal (we will notice, that we consider only the real variant), by a new method, using the lemma 2 and the theorem from [1, p. 50].

Let $n_1, n_2, ..., n_k$–be any natural numbers, $P(x_1, x_2, ..., x_k)$ - be polynomial of degrees n_i with respect to the variables $x_i (i = 1, 2, ..., k)$, without a monomial of zero degree, i.e.

$$P(x_1, x_2, ..., x_k) = \sum_{\substack{0\leq i_1\leq n_1\\ \cdots\ \cdots\ \cdots\\ 0\leq i_k\leq n_k}} a_{i_1 i_2\cdots i_k} x_1^{i_1} x_2^{i_2}\cdots x_k^{i_k},$$

with $a_{0,0,...,0} = 0$. Consider the variety defined in $\mathbb{R}^N$ by all monomials of this polynomial:

$$\Gamma = (x_1^{i_1} x_2^{i_2}\cdots x_k^{i_k}); 0\leq i_1\leq n_1, ..., 0\leq i_k\leq n_k.$$

Let, further, $v(\alpha_1, \alpha_2, ..., \alpha_k)$ means, for real vector $\bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_k)$, the supremum of those $v > 0$ for which there is an infinite number of polynomials P, satisfying an inequality

$$|P(\alpha_1, \alpha_2, ..., \alpha_k)| < H^{-v},$$

$$H = \max|a_{i_1 i_2\cdots i_k}| \begin{pmatrix} 0\leq i_1\leq n_1\\ \cdots\\ 0\leq i_k\leq n_k \end{pmatrix}.$$

In agree with Sprindzuk's conjecture, for almost all $\bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_k)$

$$v(\alpha_1, \alpha_2, ..., \alpha_k) = N = (n_1 + 1)(n_2 + 1)\cdots(n_k + 1) - 1$$

for all $n_1, n_2, ..., n_k$.

In this work we show that the affine image Γ^* (see [13, p. 67]) of Γ defined as below:

$$\Gamma^* = \left(\sum_{j=1}^{2h}(-1)^j x_{j1}^{i_1} x_{j2}^{i_2}\cdots x_{jk}^{i_k}; j = 1, ..., N\right)$$

is extremal for some h. The value of h is defined by the exponent of convergence of special integral of Tarry's problem. The aim of this paper is proving of the following theorem and its consequence ([6]).

Theorem. Let $m = max(n_1, ...n_k) \geq 2$ and $2h > Nm$. Then the variety Γ^* is extremal.

Consequence. The variety Γ is extremal.

2. Auxiliary results

For an establishment of our results it is necessarily to pass to an equivalent variant of a problem formulating it in the form of extremality of varieties. At first we will notice, that received above relation it is possible to interpret in another way. Namely, let we are given with the system of real numbers $\beta_1, \beta_2, ..., \beta_n$ is set. We shall consider an inequality

$$\|b_1\beta_1 + b_2\beta_2 + \cdots + b_n\beta_n\| \leq b^{-v},$$

where $\|x\|$ means distance from x to the nearest integer, $b = \max(|b_1|, ..., |b_n|)$, and $b > 0, v > 0$. We will designate through $v(\beta_1, \beta_2, ..., \beta_n)$ a supremum of all of those $v > 0$ for which it is executed for infinite number of integral vectors $(b_1, ..., b_n)$. Simultaneously, we shall consider the following inequality:

$$\max\left(\|q\beta_1\|, \|q\beta_2\|, ..., \|q\beta_n\|\right) \leq q^{-u}.$$

Now we shall designate through $u(\beta_1, \beta_2, ..., \beta_n)$ a supremum of all that $u > 0$ for which this inequality is executed for infinite number positive integers $q > 0$. In the theory of Diophantine Approximations there is a principle named as the Khintschin transference principle (see [3, 4, 12]), asserting, that equalities $v(\beta_1, \beta_2, ..., \beta_n) = n$ and $u(\beta_1, \beta_2, ..., \beta_n) = 1/n$ are equivalent. Again from the Dirichlet's principle the relation below follows

$$u(\beta_1, \beta_2, ..., \beta_n) \geq 1/n.$$

As numbers $\beta_1, \beta_2, ..., \beta_n$ we take monomials $\alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_k^{i_k}$ of considered polynomial P, with $n = N$. Then the inequality will coincide with the inequality considered above for the polynomial P, thus b will be replaced by H. Then, taking into account Khintschin transference principle, we should show, that

$$u(\beta_1, \beta_2, ..., \beta_N) = 1/N,$$

where $u(\beta_1, \beta_2, ..., \beta_N)$ designates the supremum of those $u > 0$ for which it is executed for infinite number whole positive $q > 0$.

Lemma 1. Let A_q, $(q = 1, ..., n)$ some sequence of measurable subsets of the space R^n such, that the series

$$\sum_{q=1}^{\infty} mesA_q < \infty.$$

converges. Then, the measure of subset A of those points $\bar{x} \in R^n$ which belongs into infinite family of subsets A_q is equal to zero.

This lemma well-known in the literature and known as the Borel-Kantelli's lemma (see [12]).

The following lemma belongs to E.I.Kovalevskaya (see [2, 7, 11]).

Lemma 2. Let $m \leq N$ q be natural numbers, $f_j(\bar{x})$, $j = 1, ..., N$ be real measurable functions defined in the cube $\Omega = [0,1]^m$. We shall designate by $\mu(q)$ a measure of a set of such points $\bar{x} \in \Omega = [0,1]^m$, for which

$$\|f_j(\bar{x})\| < q^{-r_j}(1 \leq j \leq N).$$

Then,

$$\mu(q) << q^{-r} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \int_{\Omega} e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} d\bar{x} \right|,$$

where $r = r_1 + \cdots + r_N$, and the constant hidden under the symbol $<<$ depends only on N.

The proof of this lemma can be found in [2, 13].

Lemma 3. There is a natural number $h \geq N$ satisfying such that the Gram determinant of gradients of functions

$$\sum_{j=1}^{h} x_{1j}^{i_1} x_{2j}^{i_2} \cdots x_{kj}^{i_k}; 0 \leq i_1 \leq n_1, \cdots, 0 \leq i_k \leq n_k, i_1 + \cdots + i_k > 0,$$

is distinct from zero almost everywhere in $\mathbb{R}^{kh}$.

Proof. Let $h \geq N$ where

$$N = (n_1 + 1)(n_2 + 1) \cdots (n_k + 1) - 1$$

expresses the number of monomials. If the polynomial P has a nonzero monomial of first degree then we suppose that it is x_1. In this case the Jacoby matrix contains N columns formed by partial derivatives of

$$\sum_{j=1}^{h} x_{1j}^{i_1} x_{2j}^{i_2} \cdots x_{kj}^{i_k}; 0 \leq i_1 \leq n_1, \cdots, 0 \leq i_k \leq n_k, i_1 + \cdots + i_k > 0$$

taken with respect to the variables $x_{11}, x_{12}, \cdots x_{1N}$. These columns contain monomials of Nk variables being pairwisely different. Therefore, the minor with such columns contains $N!$ in pairs different products in its expansion which cannot vanished by reduction of like terms. Then this minor represents a nonzero polynomial which vanishes only on the closed subset of a zero measure. As Gram determinant is equal to the sum of squares of all possible minors of a Jacoby matrix (see [14, p. 245]), then the lemma's statement is proved.

3. Proof of the theorem

Let's order monomials of the given polynomial someway, and accept designations $f_j(\bar{x}) = qx_1^{i_1}x_2^{i_2}\cdots x_k^{i_k}$ for $j = 1, ..., N$. Then we can present the variety Γ as follows

$$\Gamma = (f_j(\bar{x}); j = 1, ..., N),$$

where $\bar{x} = (x_1, ..., x_k) \in \Omega = [0,1]^k$.

Let's designate

$$g_j(\bar{x}_1, ..., \bar{x}_{2h}) = f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h) - f_j(\bar{x}_{h+1}) - \cdots - f_j(\bar{x}_{2h}),$$

for $(\bar{x}_1, ..., \bar{x}_h, \bar{x}_{h+1}, ..., \bar{x}_{2h}) \in \Omega^{2h}$. Now we put

$$B_q = \{(\bar{x}_1, ..., \bar{x}_h, \bar{x}_{h+1}, ..., \bar{x}_{2h}) \in \Omega^{2h} : \|g_j(\bar{x}_1, ..., \bar{x}_{2h})\| \le q^{-r_j}, j = 1, ..., N, \}$$

Denote by B the set of those vectors $(\bar{x}_1, ..., \bar{x}_h, \bar{y}_1, ..., \bar{y}_h)$ which fall into infinite set of subsets B_q. To prove the theorem's statement it is enough to show that $mesB = 0$. For this purpose, according to the Borel-Kantelly's lemma, we must prove convergence of the series $\sum_{q=1}^{\infty} \mu_q$, where $\mu_q = mes(B_q)$.

By the lemma 2, for the measure $\mu_q = mes(B_q)$ of a subset of points, we receive:

$$\mu_q << q^{-1-N\delta} \times$$

$$\times \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \int_\Omega \cdots \int_\Omega e^{2\pi i(c_1 g_1(\bar{x}_1,..,\bar{x}_{2h}) + \cdots + c_N g_N(\bar{x}_1,..,\bar{x}_{2h}))} d\bar{x}_1 \cdots d\bar{x}_{2h} \right| =$$

$$= q^{-1-N\delta} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \left(\int_\Omega e^{2\pi i(c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} d\bar{x} \right)^h \right|^2.$$

Further, we have:

$$\left(\int_\Omega e^{2\pi i(c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} d\bar{x} \right)^h =$$

$$= \int_\Omega \cdots \int_\Omega e^{2\pi i(c_1(f_1(\bar{x}_1) + \cdots + f_1(\bar{x}_h)) + \cdots + c_N(f_N(\bar{x}_1) + \cdots + f_N(\bar{x}_h)))} d\bar{x}_1 \cdots d\bar{x}_h.$$

According to the formula on exchange of variables from work [5], we have:

$$\int_\Omega \cdots \int_\Omega e^{2\pi i(c_1(f_1(\bar{x}_1) + \cdots + f_1(\bar{x}_h)) + \cdots + c_N(f_N(\bar{x}_1) + \cdots + f_N(\bar{x}_h)))} d\bar{x}_1 \cdots d\bar{x}_h =$$

$$= \int_0^h \cdots \int_0^h du_1 \cdots du_N e^{2\pi iq(c_1 u_1 + \cdots + c_N u_N)} \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}},$$

where $\Pi(\bar{u})$ designates a surface defined by system of the equations

$$(f_j(\bar{x}_1) + \cdots + f_j(\bar{x}_h))/q = u_j, \; j = 1, ..., N,$$

G designates the Gram determinant of gradients of the functions standing on the left parts of the system's equations. Notice that in consent with the lemma 3, the left part of the equality is possible to present in the form of improper integral, preliminarily removing some union of open neighbourhoods (see [14, p. 204]) of points, where $G = 0$, with enough small measure. Then, accordingly, the right part of the equality also should be understood in improper sense. More strictly speaking, we shall enter the set

$$M_\eta = \{(\bar{x}_1, ..., \bar{x}_h) \in R | G \geq \eta > 0\}.$$

Then, we receive:

$$\int_\Omega \cdots \int_\Omega e^{2\pi i(c_1(f_1(\bar{x}_1)+\cdots+f_1(\bar{x}_h))+\cdots+c_N(f_N(\bar{x}_1)+\cdots+f_N(\bar{x}_h)))} d\bar{x}_1 \cdots d\bar{x}_h =$$

$$= \lim_{\eta\to 0} \int_{M_\eta} e^{2\pi i(c_1(f_1(\bar{x}_1)+\cdots+f_1(\bar{x}_h))+\cdots+c_N(f_N(\bar{x}_1)+\cdots+f_N(\bar{x}_h)))} d\bar{x}_1 \cdots d\bar{x}_h.$$

Hence, the right part of equality should be written down in the form

$$\lim_{\eta\to 0} \int_0^h \cdots \int_0^h \left(\int_{\Pi(\bar{u}) \bigcap M_\eta} \frac{ds}{\sqrt{G}} \right) e^{2\pi i q(c_1 u_1+\cdots+c_N u_N)} d\bar{u} =$$

$$= \int_0^h \cdots \int_0^h \left(\lim_{\eta\to 0} \int_{\Pi(\bar{u}) \bigcap M_\eta} \frac{ds}{\sqrt{G}} \right) e^{2\pi i q(c_1 u_1+\cdots+c_N u_N)} d\bar{u}.$$

Expression $\lim_{\eta\to 0} \int_{\Pi(\bar{u}) \bigcap M_\eta} \frac{ds}{\sqrt{G}}$ under integral defines function from $\bar{u}$ which, owing to positivity and monotony with respect to η, has quite certain, final or infinite value. Believing for convenience $\eta = 1/m$, we have:

$$\int_0^h \cdots \int_0^h | \lim_{\lambda\to 0} \int_{\Pi(\bar{u}) \bigcap M_\lambda} \frac{ds}{\sqrt{G}} - \left(\int_{\Pi(\bar{u}) \bigcap M_\eta} \frac{ds}{\sqrt{G}} \right) | d\bar{u} \leq$$

$$\int_0^h \cdots \int_0^h \sum_{n\geq m} \int_{\Pi(\bar{u}) \bigcap \left(M_{1/n} \setminus M_{1/(n+1)} \right)} \frac{ds}{\sqrt{G}} \leq$$

$$\leq \int_{[0,1]^{kh} \setminus M_{1/m}} dx_{11} \cdots dx_{kh} \to 0,$$

as $\eta \to 0$. This proves possibility specified above passing to the limit under the sign of integral.

Let's present integral on the right part as sum of Fourier coefficients of inner surface integral. For this purpose we shall break integral on the right part into the sum of integrals taken over unite cubes:

$$\int_0^h \cdots \int_0^h du_1 \cdots du_N e^{2\pi i(c_1 q u_1+\cdots+c_N q u_N)} \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} =$$

$$= \sum_{h_1=1}^{h} \cdots \sum_{h_N=1}^{h} \int_{h_1-1}^{h_1} \cdots \int_{h_N-1}^{h_N} \phi(u_1,...,u_N) e^{2\pi i (c_1 q u_1 + \cdots + c_N q u_N)} du_1 \cdots du_N,$$

and,

$$\phi(u_1,...,u_N) = \begin{cases} \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}}, \, if \Pi(\bar{u}) \neq \emptyset \\ 0, \quad \text{othervice.} \end{cases}$$

Hence, for every fixed q, according to told above, we receive:

$$\left| \left(\int_{\Omega} e^{2\pi i (c_1 f_1(\bar{x}) + \cdots + c_N f_N(\bar{x}))} d\bar{x} \right)^h \right|^2 \leq h^N \times$$

$$\sum_{h_1=1}^{h} \cdots \sum_{h_N=1}^{h} \left| \int_{h_1-1}^{h_1} \cdots \int_{h_N-1}^{h_N} \phi(u_1,...,u_N) e^{2\pi i (c_1 q u_1 + \cdots + c_N q u_N)} du_1 \cdots du_N \right|^2.$$

Including constants h and N into the Vinogradiv's symbol, we receive:

$$\mu_q << q^{-1-N\delta} \sum_{h_1=1}^{h} \cdots \sum_{h_N=1}^{h} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} 1 \times$$

$$\times \left| \int_{h_1-1}^{h_1} \cdots \int_{h_N-1}^{h_N} \phi(u_1,...,u_N) e^{2\pi i (c_1 q u_1 + \cdots + c_N q u_N)} du_1 \cdots du_N \right|^2$$

Obviously, on the right part of this inequality, the sum over $(c_1,...,c_N)$ doesn't exceed all sum of squares of modules of Fourier coefficients for surface integral. Then, by Parseval's equality one has:

$$\mu_q << q^{-1-N\delta} \times$$

$$\times \sum_{h_1=1}^{h} \cdots \sum_{h_N=1}^{h} \int_{h_1-1}^{h_1} \cdots \int_{h_N-1}^{h_N} du_1 \cdots du_N \left(\int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} \right)^2 =$$

$$= q^{-1-N\delta} \int_0^h \cdots \int_0^h du_1 \cdots du_N \left(\int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G}} \right)^2$$

As it is established in [4, p. 78], [5], the multiple integral on the right part of the last equality represents the special integral of Tarry's problem with the polynomial

$$P(x_1, x_2, ..., x_k) = \sum_{\substack{0 \leq i_1 \leq n_1 \\ \cdots \quad \cdots \quad \cdots \\ 0 \leq i_k \leq n_k}} a_{i_1 i_2 \cdots i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k} = \sum_{j=1}^{N} \alpha_j x^{(j)},$$

where $x^{(j)} = x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}$, i.e. the integral

$$(2\pi)^{-N} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \left| \int_0^1 \cdots \int_0^1 e^{2\pi i (\sum_{j=1}^{N} \alpha_j x^{(j)})} dx_1 \cdots dx_k \right|^{2h} d\alpha_1 d\alpha_2 \cdots d\alpha_N.$$

As

$$h > N \max(n_1, n_2, ..., n_k),$$

then as has been established in [1], the special integral converges. Then the series

$$\sum_{q=1}^{\infty} \mu_q$$

converges. According to Borel-Kantelly's lemma, subset B has zero measure.

We have proved that for any positive $\delta > 0$ a set of those points

$$(\bar{x}_1, ..., \bar{x}_h, \bar{x}_{h+1}, ..., \bar{x}_{2h}) \in \Omega^{2h}$$

for which system of inequalities

$$\|g_j(\bar{x}_1, ..., \bar{y}_h)\| \le q^{-r_j}, j = 1, ..., N,$$

are satisfied for infinite number of natural numbers q, has a measure zero. Taking sequence of numbers $\delta_1, \delta_2, ..., \delta_n \to \infty$, we find sequence of sets $M_1, M_2, ...$ of a measure zero for points of each of which above relation is executed for infinite quantity of natural numbers q, with corresponding value δ_n. Designate now $M = \bigcup_{i=1}^{\infty} M_i$. We have $mesM = 0$. For points $(x_1, ..., x_k) \in [0,1]^k \backslash M$ considered above system of inequalities for any positive $\delta > 0$ can be satisfied only for finite number of natural numbers q. It means, that for all $(x_1, ..., x_k) \in [0,1]^k \backslash M$ supremum of those numbers $u > 0$, for which the system

$$\|f_j(\bar{x})\| \le q^{-u} (j = 1, ..., N)$$

takes place for infinite set of natural numbers q, does not exceed $1/N$. Further, from Dirichlet's principle it follows that this supremum is not less than $1/N$. Hence, in designations entered above, the equality $u(\beta_1, ..., \beta_N) = 1/N$, as was to be shown, holds. The theorem is proved.

Proof of the consequence. Suppose the conditions of the theorem be satisfied and $2h > N(m+2)$. Let us at first establish the connection between measures of the set B_q and the set

$$A_q = \{(\bar{x} \in \Omega | \, \|f_j(\bar{x})\| \le q^{-1/N-\delta/2}, j = 1, ..., N\}.$$

Let $\mu_0(q) = mes(B_q)$ and $\mu_q = mes(A_q)$. According to said above

$$\mu_0(q) = \int_{(\bar{x}_1, ..., \bar{x}_{2h}) \in \Omega^{2h}, \|q \sum_{j=1}^{h} (\phi_j(\bar{x}_j) - \phi_j(\bar{x}_{h+j}))\| < q^{-1/N-\delta/2}, 1 \le j \le N} d\bar{x} d\bar{x}',$$

moreover $\phi_j(\bar{x}) = f_j(\bar{x})/q$. To bound this integral from below, note that over variables does not imposed any conditions except constraints of integration. Let's make change of variables of the form:

$$u_j = \phi_j(\bar{x}) - \phi_j(\bar{x}'); j = 1, ..., N; \quad (1)$$

here $\bar{x} = (\bar{x}_1, ..., \bar{x}_h), \bar{x}' = (\bar{x}_{h+1}, ..., \bar{x}_{2h})$, applying Lemma 2,[5]. We find:

$$\mu_0 = \int_{-h, \|u_1\| \leq^{-1/N-\delta}}^{h} \cdots \int_{-h, \|u_N\| \leq^{-1/N-\delta}}^{h} du_1 \cdots du_N \int_{\Pi(\bar{u})} \frac{ds}{\sqrt{G(\bar{x})}}, \quad (2)$$

where $G(\bar{x})$ means Gram's determinant for the system of functions $\phi_j(\bar{x}) - \phi_j(\bar{x}')$, and $\Pi(\bar{u})$ is a surface defined by the system of equations (1). It is known that the equation $G = 0$ can be satisfied in a subset of Ω^{2h} with zero measure. Denote this exceptional set by W. It is known that $G(\bar{x})$ is possible to represent as a sum of all minors of maximal order N. So, the set Ω^{2h} can be dissected into the union of finite number (does not dependent of q) closed one-connected subsets in every of which one of these minors takes maximal values among others ([5]), the surface element there can be represented as follows:

$$ds = \frac{\sqrt{G(\bar{x})}}{|J|},$$

and J denotes the maximal minor. Then the right hand side of the equality (1) is possible write as a repeated integral as below:

$$\mu_0 = \int_{\|u_1\| \leq^{-1/N-\delta}} \cdots \int_{\|u_N\| \leq^{-1/N-\delta}} du_1 \cdots du_N \int_{\Pi(\bar{u})} \frac{\sqrt{G(\bar{x})}}{|J|} d\bar{x},$$

and $d\bar{x}$ means differential over independent variables This minor defines in the corresponding domains the variables as implicit functions of other variables, which are independent. Don't destroying the generality, we can assume that the variable $\bar{x}_1$ is independent. It is clear that set of values of one of differences $\|u_j - \phi_j(\bar{x}_1)$ overlaps the interval of length at least $q^{-1/N}(q^{-\delta/2} - q^{\delta})$ and others overlap intervals with the length $q^{-1/N-\delta}$. So, we have

$$\mu_0 \geq c\mu_q q^{-1-(N-1/2)\delta}.$$

This inequality is sufficient for completing of the proof.

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