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On a boundary value problem with nonlocal integral condition for a parabolic-hyperbolic type equation

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Abstract. In the study, an initial-boundary value problem for a parabolic-hyperbolic type equation considered. The non-local boundary condition is expressed by an integral. A uniqueness theorem for the solvability of this problem is shown and the solution of problem constructed as the sum of Fourier series. The stability of the solution with respect to initial function is proved.

Key Words and Phrases: parabolic-hyperbolic type equation, non-local condition, uniqueness theorem, solvability.

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1. Introduction

Let us define operator L by

$$Lu(x,t) = \begin{cases} u_t - u_{xx}, & \text{for t} > 0, \\ u_{tt} - u_{xx}, & \text{for t} < 0, \end{cases}$$

and $\Omega\left\{(x,t)|0 < x < 1, -\alpha < t < \beta\right\}, \Omega_- = \Omega \cap \{t < 0\}, \Omega_+ = \Omega \cap \{t > 0\}$, where α, β are positive real numbers.

The goal of this paper is to find a function $u(x,t)$ in the domain Ω that satisfy the following conditions:

$$u(x,t) \in C(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega_-) \cap C_x^2(\Omega_+) \tag{1.1}$$

$$Lu(x,t) \equiv 0, \quad (x,t) \in \Omega_- \cap \Omega_+ \tag{1.2}$$

$$u(x,-\alpha) = \psi(x), \quad 0 \le x \le 1, \tag{1.3}$$

$$u_x(0,t) = 0, \quad -\alpha \le t \le \beta, \tag{1.4}$$

$$\int_0^1 xu(x,t)dx = 0, \quad -\alpha \le t \le \beta. \tag{1.5}$$

Here $\psi(x)$ is defined in the interval [0,1] sufficiently smooth function such that

$$\psi'(0) = 0, \quad \int_0^1 x\psi(x)dx = 0.$$

These mixed type equations have many interesting applications in electromagnetic, gas dynamics and similar non-homogeneous processes. For example, in I. M. Gelfand [1], it is shown that the movement of a gas in the channel with wave equation $u_{tt} = a\Delta u$, the movement of agas out of the channel with diffusion equation $u_t = b\Delta u$, where Δ is the Laplacian operator, a, b positive physical parameters. The propagation of electric waves in semi-infinite line is examined by Uflyand Y. S. [2] . It is known that there exists applications of Tricomi type equations in gas dynamics. Ladizhenckaya O. A. and Stupyalis L. [3] investigated the movement of the fluid in the electromagnetic field and this process is expressed with boundary problem for the parabolic-hyperbolic type equation in multi-dimensional space. The boundary problems for mixed type equations was studied in Dzhuraev T. D. [5], Dzhuraev T. D.,Sopuev A. and Mamajanov M. [6], Kapustin N. Yu. and Moiseev E. I. [13], Sabitov K. B. [14], [15], [20], Sabitov K. B. and Sidorov S. N. [16], Sabitova Yu. K. [17], Nachushev A. M. [7], Mamedov K.R [8],Yuldashev T.K [9],[10], Yuldashev T.K and Kadirkulov B.J. [11],[12] and references therein.

The nonlocal integral condition show that physical process is not only at the point but also at the whole object. This type boundary conditions are examined in Cannon J. R. [21], [22], Ionkin N. I. [18], Kamynin L. I. [23], Pulkina L.S. [25], Pulkina L.S. and Savenkova A.E. [27]. Ionkin N. I. [18] studied this boundary condition for diffusion equation in plasma problem. Some references for mixed type equations with nonlocal boundary conditions and integral boundary conditions are given in Dzhuraev T. D.,Sopuev A. and Mamajanov M. [6], Sabitov K. B. [14], Pulkina L.S. and Savenkova A.E. [27].

Let us reduce the (1.5) integral condition into the classic boundary condition. For this, in the equation (1.1) fixing parameter t and multiplying by x, then integrating according to x from ϵ to $1-\epsilon$ such as $\epsilon > 0$ sufficiently small number, we get

$$\int_\epsilon^{1-\epsilon} xu_t dx - \int_\epsilon^{1-\epsilon} xu_{xx} dx, \quad \text{for t} > 0,$$

$$\int_\epsilon^{1-\epsilon} xu_{tt} dx - \int_\epsilon^{1-\epsilon} xu_{xx} dx, \quad \text{for t} < 0.$$

Here when $\epsilon \to 0$ we obtained

$$\frac{d}{dt}\left(\int_0^1 xudx\right) = u_x(1,t) - u(1,t) + u(0,t), \quad t \geq 0,$$

$$\frac{d^2}{dt^2}\left(\int_0^1 xudx\right) = u_x(1,t) - u(1,t) + u(0,t), \quad t \leq 0,$$

and as a result (1.4) integral boundary condition is reduced to the

$$u(0,t) - u(1,t) + u_x(1,t) = 0 \tag{1.6}$$

nonlocal condition.

Therefore, in this paper we study the problem in the domain Ω to find a function $u(x,t)$ satisfying the conditions (1.1)-(1.4) and (1.6).

2. Spectral Problem

Let us find the trivial solution of the boundary problem (1.1)-(1.6) by using method of separating of variables in the form $u(x,t) = X(x)T(t)$. By writing this expression in equation (1.1) and (1.3),(1.6) boundary conditions, we encounter

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < l, \tag{2.1}$$

$$X'(0) = 0, \tag{2.2}$$

$$X'(1) + X(0) - X(1) = 0, \tag{2.3}$$

spectral problem for $X(x)$ and

$$T'(t) + \lambda T(t) = 0, \quad 0 < t < \beta, \tag{2.4}$$

$$T''(t) + \lambda T(t) = 0, \quad -\alpha < t < 0, \tag{2.5}$$

ordinary differential equation for $T(t)$, where λ is a complex parameter.

Boundary conditions (2.2),(2.3) are regular in terms of Birkhof and also strong regular ([Nai], p.56). If the boundary conditions are strongly regular, the root functions of the boundary value problem form Riesz basis on space $L_2(0,1)$ (see [Nai], p.90). Let's examine the spectral properties of the boundary value problem (2.1)-(2.3).

The spectral problem (2.1)-(2.3) has two sequences $\lambda_{1,k}, \lambda_{2,k}$ eigenvalues:

$$\lambda_{1,k} = (\mu_{1k})^2 = (2k\pi)^2, \quad k = 0, 1, 2, ...,$$

$$\lambda_{2,k} = (\mu_{2k})^2 = [(2k+1)\pi]^2 \left[1 + O\left(\frac{1}{k}\right)\right], \quad k \to \infty.$$

Corresponding eigenfunctions are of the form

$$X_{2k} \equiv X_{1,k}(x) = cos 2k\pi x, \quad k = 0, 1, 2, \dots,$$

$$X_{2k-1} \equiv X_{2,k}(x) = cos(2k+1)\pi x + O\left(\frac{1}{k}\right), \quad k \to \infty.$$

Let the sequence of eigenfunctions of the boundary value problem (2.1)-(2.3) be $\{X_k(x)\} = \{X_{1,k}(x), X_{2k}(x)\}$.

We get the form of the adjoint boundary value problem

$$Y''(x) + \lambda Y(x) = 0, \quad 0 < x < 1 \tag{2.6}$$

$$Y'(0) - Y(1) = 0, \tag{2.7}$$

$$Y'(1) - Y(1) = 0. \tag{2.8}$$

Boundary conditions (2.6)-(2.8) is regular and also strongly regular. The eigenfunctions of the adjoint boundary problem form a Riesz basis on the space $L_2(0,1)$. The conditions (2.7),(2.8) are not equivalent to the boundary conditions (2.2),(2.3). Therefore (2.6)-(2.8) boundary value problem are not self adjoint. The sequence of eigenfunctions of the adjoint boundary value problem form a biorthogonal-adjoint sequence with the eigenfunctions of the (2.1)-(2.3) spectral problem. These eigenvalues of the boundary value problem coincide with the eigenvalues of the spectral problem (2.1) - (2.3) . Eigenfunctions of the adjoint problem are of the form

$$Y_{0,1} \equiv Y_0(x) = x$$

$$Y_{k,1}(x) = 2k\pi cos2k\pi x + sin2k\pi x, \quad k = 1, 2, \dots$$

$$Y_{k,2}(x) = (2k+1)\pi cos(2k+1)\pi x - sin(2k+1)\pi x + O\left(\frac{1}{k}\right), k \to \infty.$$

Let eigenfunctions of the (2.6)-(2.8) adjoint problem be $\{Y_k(x)\} = \{Y_{0,1}, Y_{k,1}, Y_{k,2}\}$.

3. Uniqueness of the solution

We obtain

$$T_k(t) = \begin{cases} a_k e^{-\mu_k^2 t}, & t > 0, \\ b_k cos\mu_k t + c_k sin\mu_k t, & t < 0, \end{cases} \tag{3.1}$$

by writing $\lambda = \mu_k^2$ in the equations (2.4),(2.5),where a_k, b_k, c_k are arbitrary constants. Since $u(x,t) \in \Omega$, the solution $u_k(x,t) = X_k(x)T_k(t) \in \Omega$ also has this property, and let's choose a_k, b_k, c_k constants such that conditions

$$T_k(0_+) = T_k(0_-), \quad T_k'(0_+) = T_k'(0_-) \tag{3.2}$$

hold. The function (3.1) satisfies conditions (3.2) only when $a_k = c_k, b_k = -c_k\mu_k$. Then function (3.1) is in the form of

$$T_k(t) = \begin{cases} c_k e^{-\mu_k^2 t}, & t > 0, \\ c_k cos\mu_k t - c_k\mu_k sin\mu_k t, & t < 0. \end{cases} \tag{3.3}$$

Let us consider the

$$u_k(t) = \int_0^1 u(x,t) Y_k(x) dx \tag{3.4}$$

function and find the differential equation which $u_k(t)$ satisfies. Firsly,let us integrate the equation (3.4) from ϵ to $1-\epsilon$

$$u_{k,\epsilon}(t) = \int_\epsilon^{1-\epsilon} u(x,t) Y_k(x) dx, \tag{3.5}$$

where $\epsilon > 0$ is a small number. Then differentiating the equation (3.5) with respect to t, once when $t > 0$ is, twice when $t < 0$, following equations are obtained

$$u'_{k,\epsilon}(t) = \int_{\epsilon}^{1-\epsilon} u_t(x,t)Y_k(x)dx = \int_{\epsilon}^{1-\epsilon} u_{xx}(x,t)Y_k(x)dx, \tag{3.6}$$

$$u''_{k,\epsilon}(t) = \int_{\epsilon}^{1-\epsilon} u_{tt}(x,t)Y_k(x)dx = \int_{\epsilon}^{1-\epsilon} u_{xx}(x,t)Y_k(x)dx. \tag{3.7}$$

By partial integration twice on the right side of (3.6) and (3.7) and as taking the limit when $\epsilon \to 0$, we have

$$u'_k(t) + \mu_k^2 u_k(t) = 0, \quad t > 0, \tag{3.8}$$

$$u''_k(t) + \mu_k^2 u_k(t) = 0, \quad t < 0. \tag{3.9}$$

Differential equations of (3.8),(3.9) coincide with equations of (2.4),(2.5) for $\lambda = \mu_k^2$ and thus $u_k(t) \equiv T_k(t)$ is obtained for $-\alpha \leq t \leq \beta$, that is the functions $u_k(t)$ are defined by (3.3). By using the initial condition (1.2)

$$u_k(-\alpha) = \int_0^1 u(x,-\alpha)Y_k(x)dx = \int_0^1 \psi(x)Y_k(x)dx = \psi_k, \tag{3.10}$$

and from (3.3) and (3.10) the coefficients c_k in (3.3) satisfy

$$c_k[cos\mu_k\alpha + \mu_k sin\mu_k\alpha] = \psi_k. \tag{3.11}$$

Hence, for the condition

$$d(k) = cos\mu_k\alpha + \mu_k sin\mu_k\alpha \neq 0, \tag{3.12}$$

the coefficients c_k obtained as

$$c_k = \frac{\psi_k}{cos\mu_k\alpha + \mu_k sin\mu_k\alpha} = \frac{\psi_k}{d(k)}. \tag{3.13}$$

By substituting (3.13) in (3.3) we get

$$u_k(t) = \begin{cases} \frac{\psi_k}{d(k)} e^{-\mu_k^2 t}, & t > 0, \\ \frac{cos\mu_k t - \mu_k sin\mu_k t}{d(k)} \psi_k, & t < 0. \end{cases} \tag{3.14}$$

Now assume that $\psi(x) \equiv 0$ and condition (3.12) hold for each $k = 1, 2, \cdots$. Then $\psi_k \equiv 0$ and according to the formulas for (3.14),(3.4)

$$\int_0^1 u(x,t)Y_k(x)dx = 0 \quad k = 1, 2, \ldots$$

for each $t \in [-\alpha, \beta]$. The sequence $\{y_k(x)\}$ is the complete sequence as well, since it forms the basis on $L_2(0,1)$. Using this property, $u(x,t) \equiv 0$ almost everywhere for each $t \in [-\alpha, \beta]$. Since the function $u(x,t)$ is continuous in the closed $\bar{\mathcal{D}}$ region, $u(x,t) \equiv 0$ in this region.

Then we prove the following theorem

Theorem 3.1. *If there is a solution of the boundary problem (1.1)-(1.6), this solution is unique if and only if the (3.12) condition is satisfied.*

Proof. Suppose that the condition(3.12) is not satisfied for an α and $k = p$, so $\sigma(p) = cos\mu_p\alpha + \mu_p sin\mu_p\alpha = 0$. The problem (where $\psi(x) \equiv 0$) boundary problem (1.1)-(1.6) has a non-zero

$$u_p(x,t) = \begin{cases} c_p e^{-\mu_p^2 t} X_p(x), & t > 0, \\ c_p(cos\mu_p t - \mu_p sin\mu_p t) X_p(x), & t < 0 \end{cases} \tag{3.15}$$

solution, where $c_p \neq 0$ is arbitrary constant. □

4. The existence of the solution

Suppose that $d(k) \neq 0$ then there is a constant c_0 such that $|d(k)| \geq c_0 > 0$. The solution of the (1.1)-(1.6) boundary value problem can be represented in the form of series

$$u(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k(x). \tag{4.1}$$

It is clear that the product of $u_k(x,t) = u_k(t)X_k(x)$ satisfies the equation (1.1). To show that the series (3.1) is the solution of the boundary problem (1.1)-(1.6), it is required that this series is uniformly convergent in the closed $\bar{\mathcal{D}}$ region and when $t < 0$ this series can be differentiated once with respect to t, twice with respect to x term by term ; and when $t > 0$, this series can be differentiated twice with respect to t and x term by term. For this, let's evaluate the $u_k(t)$.

Lemma 4.1. *The following evaluations are provided for each $k \in \mathbb{Z}_+$: for $t \in [-\alpha, \beta]$*

$$|u_k(t)| \leq A_1 k |\psi_k|, \quad |u_k'(t)| \leq A_2 k^2 |\psi_k| \tag{4.2}$$

and for $t \in [-\alpha, 0]$

$$|u_k''(t)| \leq A_3 k^3 |\psi_k|. \tag{4.3}$$

Proof. From the formula (3.14) for $t \in [0, \beta]$, we have

$$|u_k(t)| = \left| \frac{e^{-\mu_k^2 t}}{d(k)} \psi_k \right| \leq \frac{1}{c_0} |\psi_k| \leq \tilde{A}_1 k |\psi_k|,$$

$$|u_k'(t)| = \left| \frac{-\mu_k^2 e^{-\mu_k^2 t}}{d(k)} \psi_k \right| \leq \frac{|\mu_k|^2}{c_0} |\psi_k| \leq \tilde{A}_2 k^2 |\psi_k|.$$

Similarly, for $t \in [-\alpha, 0]$, we get

$$|u_k(t)| = \left| \frac{\psi_k}{d(k)} (cos\mu_k t - \mu_k sin\mu_k t) \right| \leq \frac{|\psi_k|}{c_0} \sqrt{1 + \mu_k^2} \leq \tilde{A}_3 k |\psi_k|$$

$$|u_k'(t)| = \left|\frac{\psi_k\mu_k}{d(k)}(sin\mu_k t + \mu_k cos\mu_k t)\right| \leq \frac{|\mu_k||\psi_k|}{c_0}\sqrt{1+\mu_k^2} \leq \tilde{A}_4 k^2|\psi_k|.$$

Moreover from equation (3.9) it is obtained tat

$$|u_k''(t)| = |\mu_k^2||u_k(t)| \leq A_3 k^3 |\psi_k|$$

where $\tilde{A}_j, (j = 1, 2, 3, 4)$ is positive constants. The proof is over. □

By applying Lemma 4.1, first order derivatives of this series in the closed region $\bar{\mathcal{D}}$ and the second order derivatives of series in the regions $\bar{\mathcal{D}}_+, \bar{\mathcal{D}_-}$ can be bounded above by following numerical series

$$A_4 \sum_{k=1}^{\infty} k^3 |\psi_k|. \tag{4.4}$$

Now let us find evaluations for ψ_k.

Lemma 4.2. *Let* $\psi(x) \in C^4[0, l]$ *and* $\psi^{(i+1)}(1)+\psi^{(i)}(0)-\psi^{(i)}(1) = 0, \psi^{(i+1)}(0) = 0, i = 0, 2$ *is satisfied. Then*

$$\psi_k = \frac{\psi_k^{(4)}}{|\mu_k|^4}, \quad k \in \mathbb{Z}_+, \tag{4.5}$$

$$\sum_{k=1}^{\infty} |\psi_k^{(4)}|^2 \leq \|\psi^{(4)}(x)\|_{L_2(0,1)}. \tag{4.6}$$

Proof. By using the integral (3.10) and equation (3.1) we have

$$\psi_k = -\frac{1}{|\mu_k|^2}\int_0^1 \psi(x) Y_k''(x)dx.$$

Then integrating twice and considering the conditions of the lemma , we get

$$\psi_k = -\frac{1}{|\mu|_k^2}\psi_k^{(2)}. \tag{4.7}$$

In similar way we find

$$\psi_k^{(2)} = -\frac{1}{|\mu|_k^2}\int_0^1 \psi^{(4)}(x) y_k(x)dx = -\frac{1}{|\mu|_k^2}\psi_k^{(4)}. \tag{4.8}$$

We obtain the formula (4.5) by writing the equation (4.8) in (4.7). According to the condition of the lemma $\psi^{(4)}(x) \in C[0, 1]$, the Fourier series theory, the $\sum_{k=1}^{\infty} |\psi_k^{(4)}|^2$ series is convergent and the inequality (4.6) holds. The proof is over. □

Under the conditions of Lemma 4.2, the series (4.4) is bounded above by

$$A_5 \sum_{k=1}^{\infty} \frac{1}{k} |\psi_k^{(4)}|. \tag{4.9}$$

Then, due to the convergence of (4.9) series and according to Weierstrass criterion, series (4.1) is absolutely and uniformly convergent in $\bar{\mathcal{D}}_+$ and $\bar{\mathcal{D}}_-$ closed regions in accordance with series

$$u_t(x,t) = \sum_{k=1}^{\infty} u_k'(t) X_k(x), \quad t > 0, \tag{4.10}$$

$$u_{tt}(x,t) = \sum_{k=1}^{\infty} u_k''(t) X_k(x), \quad t < 0, \tag{4.11}$$

$$u_{xx}(x,t) = \sum_{k=1}^{\infty} u_k(t) X_k''(x), \quad t < 0. \tag{4.12}$$

Hence the sum of the series (4.1), $u(x,t) \in \Omega$, i.e satisfies the condition (1.1)-(1.6). By appliying the series (4.1),(4.10),(4.11),(4.12) in (1.1), we get

$$u_t - u_{xx} = \sum_{k=1}^{\infty} [u_k'(t) X_k(x) - u_k(t) X_k''(x)] \equiv 0, \quad t > 0,$$

.

$$u_{tt} - u_{xx} = \sum_{k=1}^{\infty} [u_k'(t) X_k(x) - u_k(t) X_k''(x)] \equiv 0, \quad t < 0.$$

Hence equation (3.1) provides equation (1.1). As a result of that we give the following theorem.

Theorem 4.3. *The (1.1) - (1.6) boundary value problem has only one solution and is defined by the series (3.1).*

Note that if $\sigma_\alpha(p) = 0$,α is a rational number, for $k = p = k_1, k_2, \ldots, k_m$, the necessary and sufficient condition for the problem (1.1) to be solvable is to ensure that the following equality hold

$$\psi_k = \int_0^1 \psi(x) Y_k(x) dx = 0, \quad k = k_1, k_2, \ldots, k_m, \tag{4.13}$$

where $1 \leq k_1 < k_2 < \cdots < k_m \leq k_0$, $k_i, i = 1, 2, \ldots, m, m \in \mathbb{N}_+$ is the given number. In this case, the solution is defined as

$$u(x,t) = \left(\sum_{k=1}^{k_1 - 1} + \cdots \sum_{k=k_{m-1}+1}^{k_m - 1} + \sum_{k=k_m+1}^{+\infty} \right) u_k(t) X_k(x) + \sum_p B_p u_p(x,t) \tag{4.14}$$

In the last term, the number p takes the values of $k_1, k_2, \ldots, k_m$, B_p is an arbitrary constant, the function $u(x,t)$ is expressed by the formula (3.15), such that in sum terms on the right side of (4.14 is zero if the upper bound is less than the lower bound.

5. Stability of the solution

Let us obtain evaluations according to the function given at the initial for the stability of the solution of the boundary value problem (1.1)-(1.6).Firstly, let us define the following norms.

$$\|u(x,t)\|_{L_2(0,1)} = \left(\int_0^1 |u(x,t)|^2 dx\right)^{\frac{1}{2}}, \|u(x,t)\|_{C(\mathcal{D})} = \max_{\mathcal{D}}|u(x,t)|.$$

Theorem 5.1.

$$\|u(x,t)\|_{C(\bar{\mathcal{D}})} \leq M\|\psi''(x)\|_{C[0,1]} \tag{5.1}$$

for the $u(x,t) \in \Omega$ solution of the boundary problem (1.1)-(1.6) , where $M > 0$ is constant and does not depend on $\psi(x)$.

Proof. Let $(x,t) \in \bar{\mathcal{D}}$ be any point.By using the expression of $X_k(x)$ we have $|X_k(x)| \leq M_1 > 0, M_1$ is constant. From (4.7) and according to the Cauchy-Schwarz inequality we get

$$|u(x,t)| \leq \sum_{k=1}^{\infty} |u_k(t)||X_k(x)| \leq M_1 \sum_{k=1}^{\infty} k|\psi_k| \leq M_2 \sum_{k=1}^{\infty} \frac{1}{k}|\psi_k^{(2)}| \leq$$

$$\leq M_2 \left(\sum_{k=1}^{\infty} \frac{1}{k^2}\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} |\psi_k^{(2)}|^2\right)^{\frac{1}{2}} \leq M_3\|\psi''(x)\|_{L_2(0,1)},$$

where $M_i, i = 1,2,3$ is positive constants. Thus theorem is proved. □

References

[1] I. M. Gelfand, *Some a questions of analysis and differential equations*, Uspechi Mat Nauk (Russian Math. Syrveys)., 14(3), 3-19 (1959) ,[Am. Math. Soc. Transl., 26, 201-219 (1963)].

[2] Uflyand Y. S., *Propagation of oscillation in composite electric lines*, Insherno-Fizicheski Zhurnal., 7(1), 89-92 (1964).

[3] Ladizhenckaya O. A. and Stupyalis L., *On mixed type equations*, Vestnik LGU, Ser.Mat-Mekh-Astr., 19(4), 36-46 (1965).

[4] Zoline, *On a boundary-value problems for a model equation of hyperbolic-parabolic type*, Zh.Vychisl. Math. i Mat. Fiz., 6(6), 991-1001 (1966).

[5] Dzhuraev T. D.,*Boundary value problems for equation of mixed and mixed composite types*, Fan Tashkent [In Russian]., (1979).

[6] Dzhuraev T. D.,Sopuev A. and Mamajanov M., *Boundary value problems for equations of parabolic-hyperbolic type*, Fan Tashkent, [In Russian]., (1986).

[7] Nachushev A. M., *An a uniqueness condition of the Dirichlet problem for an equation of mixed type in a cylindrical domain*, Differ. Uravn. [In Russian]., 6(1), 190-191 (1970).

[8] Mamedov K.R., *An Initial Boundary Value Problem for a Mixed Type Equation in a Rectangular Domain*, Lobachevskii Journal of Mathematics, 42(3), 572-578 (2021).

[9] Yuldashev T.K., *On solvabilility of a mixed value problem for nonlinear partial differential equation of higher order*,Vestn. Samar. Gos. Tekhn. Univ. Ser. Fiz.-Mat. Nauki, 4(),46–57 2013.

[10] Yuldashev T.K., *Nonlocal mixed-value problem for a Boussinesq-type integrodifferential equation with degenerate kernel*, LUkrainian Mathematical Journal, 68(8), 1278–1296 (2016).

[11] Yuldashev T.K. and Kadirkulov B.J., *Nonlocal problem for a mixed type fourth-order differential equation with Hilfer fractional operator*, Ural Math. J., 6(1), 153–167 (2020).

[12] Yuldashev T.K. and Kadirkulov B.J., *Inverse boundary value problem for a fractional differential equations of mixed type with integral redefinition conditions*, Lobachevskii Journal of Mathematics, 42(3) , 649–662,(2021)

[13] Kapustin N. Yu. and Moiseev E. I., *On a spectral problem from the theory of parabolic-hyperbolic heat equations*, Dokl. Ross. Akad. Nauk., 352(4), 451-454, (1997).

[14] Sabitov K. B., *On the theory of mixed-type equations*, Fizmatlit, Moscow, [In Russian]., (2014).

[15] Sabitov K. B., *Tricomi problem for a mixed parabolic-hyperbolic equation in a rectangular domain*, Mat. Zametki., 86(2), 273-279 (2009) [Math. Notes] 86(2), 249-254 (2009)].

[16] Sabitov K. B. and Sidorov S. N., *Initial-boundary- value problem for inhomogeneous degenerate equations of mixed parabolic-hyperbolic type*, Journal of Mathematics

Sciences., 2019, 236(6), 604-640 (2019).

[17] Sabitova Yu. K., *Dirichlet problem for a mixed type equation with characteristic degeneration*, Vestn. Sam. Gas. Tekhn. Univ., Ser. Fiz-Mat. Nauki., 23(4), 622-645 (2019).

[18] Ionkin N. I., *Solution of boundary value problem in heat conduction with a nonclassical boundary condition*, Differential Equations., 13(2), 294-304, (1977).

[19] Sabitova Yu. K., *Boundary-value problem with nonlocal integral condition for mixed-type equations with degeneracy on the transition line*, Mathematical Notes., 98(3), 454-465, (2015).

[20] Sabitov K.B., *Boundary value problem for a parabolic-hyperbolic equation with a nonlocal integral condition*, Differ. Equations, 46(10), 1472-1481, (2010).

[21] Cannon J. R., *The solution of the heat equation subject to the specification of energy*, Quarterly of Applied Mathematics., 1963, 21(2), 155-160.

[22] Cannon J. R.,*The one-dimensional heat equations*, Cambridge University, Press., 1984.

[23] Kamynin L. I., *A boundary value problem in the theory of heat conduction with a nonclassical boundary condition*, Computational Mathematics and Mathematical Physics., 1964, 4(6), 33-59.

[24] Naimark M. A., *Linear differential operators*, Part1, Ungar, New York., (1967).

[25] Pulkina L.S., *A nonlocal problem with integral conditions for a hyperbolic equation*, Differential equations, 2004, 40(7), 947-953.

[26] Avalishvili G., Avalishvili M. and Gordeziani D., *On integral nonlocal boundary problems for some partical differential equations*, Bull. Georg. Nat. Acad. Sci., 2011, 5(1), 31-37.

[27] Pulkina L.S. and Savenkova A.E., *A problem with nonlocal integral conditions of the second kind for one-dimensional hyperbolic equation*, Y.Samara State Tech.Univ., ser. Phys-Math.Sci., 2016, 20(2), 276-289.

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