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On Polynomial Estimations with Transcendental Numbers

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Abstract. In the theory of Diophantine Approximations questions on estimation of linear forms with transcendental numbers are considered. Among intensively studied questions of this theory the special place is occupied with metric aspects. Here such questions of the Theory of approximations are considered which take place for almost all real numbers from the set interval. In this paper we study such questions applying the results on estimation of convergence exponent for the special integral of Tarry's problem.

Key Words and Phrases: Diophantine approximation, extremal variety, convergence exponent, special integral, Tarry's problem.

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1. Introduction

In 1932 Mahler K. ([10-11]) entering new classification of transcendental numbers has stated a hypothesis about S-numbers. In the fundamental works [12-13] Sprindzhuk V. G has proved this hypothesis using the method of essential and non-essential domains developed by him. He has formulated new hypotheses generalising Mahler's hypothesis. He has assumed that these hypotheses, and also some other problems, is possible solve entering essential changes into a method of essential and non-essential domains. Now this hypothesis is a special case of the general theorem established by D.J.Klejnbok and G.A.Margulis ([9]).

Let's designate by Π following set of polynomials with integral coefficients of degree not exceeding n:.

$$\Pi = \left\{ f(x) = \sum_{i=0}^{n} a_i x^i \neq 0 | a_i \in Z \right\}.$$

The number

$$h(f) = \max(|a_0|, |a_1|, ..., |a_n|)$$

is called as the height of polynomial $f(x)$. If now to take any transcendental number α it will not be a root for polynomials from the set Π. We take some real number $h > 0$.

For the first time Mahler has proved, that there is a positive constant $\kappa > 0$ such that an inequality

$$|f(\alpha)| > h^{-n\kappa};\ h = h(f)$$

is satisfied for all polynomials $f(x)$ with the height $\leq h$ for almost all real numbers. The initial value of the constant κ found by Mahler was $\kappa = 4 + \varepsilon$, with any positive small $\varepsilon > 0$. The hypothesis stated by Mahler asserts that it is possible to take $\kappa = 1 + \varepsilon$. Justice of this statement has been proved in 1965 by Sprindzhuk by a method of essential and non- essential domains. Simultaneously Sprindzhuk V. G has put some hypotheses generalising Mahler's result.

Let $m_1 < m_2 < \cdots < m_n$ be natural numbers. Consider a polynomial

$$P(x) = a_0 + a_1 x^{m_1} + a_2 x^{m_2} + \cdots + a_n m^{m_n}.$$

Let, further, $v(\bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_{2K})$ means for real vector $\bar{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_k)$ supremum of those $v > 0$ for which there are infinite number of polynomials P, satisfying an inequality

$$|P(\bar{\alpha}_1) + \cdots + P(\bar{\alpha}_K) - P(\bar{\alpha}_{K+1}) - \cdots - P(\bar{\alpha}_{2K})| < H^{-v},$$

$$H = \max\left(|a_0|, ..., |a_n|\right).$$

In the present article we prove the following theorem.

Theorem. There exist a natural number K such that for almost all $(\bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_{2K}) \in [0,1]^{2K}$

$$v(\bar{\alpha}_1, \bar{\alpha}_2, ..., \bar{\alpha}_{2K}) = n.$$

Here we without using Khintschine's transference principle prove this theorem applying the results on convergence exponent of Tarry's problem.

For given real number $h > 0$ we will consider such polynomials which heights does not exceed h (clearly, that number of such polynomials are finite). According to said above for any positive number $\varepsilon > 0$ there will be found unbounded sequence $h_1, h_1, ...$ such, that (**??**) holds for all h_m with $\gamma = \omega_n(\alpha) + \varepsilon$.

2. Auxiliary results

The following result is known as Borel-Kantelli's lemma and plays an important role in a question on extremality of algebraic varieties (see [2, 3, 6-8, 12]).

Lemma 1. Let A_q $(q = 1, 2, ...)$ designate sequence of measurable sets in R^n, and

$$\sum_{q=1}^{\infty} \mathrm{m}esA_q < \infty.$$

Then the measure of a set of such $\bar{x} \in R^n$ which belong into infinite family of sets A_q equals to zero.

Proof. We shall designate $E \subset R^n$ a subset of such points $\bar{x} \in R^n$ which belong into infinite family of sets A_q. Then for each natural number q and point $\bar{x} \in E$ we have:

$$\bar{x} \in \bigcup_{m=q+1}^{\infty} A_m.$$

Thus,

$$E \subset \bigcup_{m=q+1}^{\infty} A_m,$$

from which the inequality

$$\mu(E) \leq \sum_{m=q+1}^{\infty} \mu(A_m)$$

follows. Since the series on the right side converges, then the set $E \subset R^n$ really has a zero measure. The lemma 1 is proved.

Our basic auxiliary tool is E.Kovalevskaya's lemma (see [2-3, 6, 11]).

Lemma 2. Let q be a natural number, $f_j(\bar{x}), j = 1, ..., N$ be a family of real measurable functions defined in the cube $\Omega = [0,1]^m$. We shall designate by $\mu(q)$ a measure of set of those $\bar{x} \in \Omega = [0,1]^m$ for which

$$\|f_j(\bar{x})\| < q^{-r_j}(1 \leq j \leq N).$$

Then,

$$\mu(q) << q^{-r} \sum_{|c_1|<q^{r_1}} \cdots \sum_{|c_N|<q^{r_N}} \left| \int_{\Omega} e^{2\pi i(c_1 f_1(\bar{x})+\cdots+c_N f_N(\bar{x}))} d\bar{x} \right|,$$

where $r = r_1 + \cdots + r_N$ and the constant under the symbol $<<$ depends only from N.

The proof of this lemma 2 can be found in [6, 11].

Our next auxiliary result is on convergence exponent of special integral of Tarry's problem (see [1, 5]). The number γ is called a convergence exponent for improper integral

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} 1 \times$$

$$\times \left| \int_0^1 e^{2\pi i(\alpha_n x^{m_n} + \alpha_{n-1} x^{m_{n-1}} \cdots + \alpha_1 x^{m_1})} dx \right|^{2k} d\alpha_n d\alpha_{n-1} \cdots d\alpha_1, \quad (1)$$

if it converges for $2k > \gamma$ and disperses for $2k < \gamma$.

Lemma 3. If $n < m_n$, then $\gamma = m_1 + \cdots + m_n$ for integral (1).

The proof of this lemma is given in [1].

3. Proof of the theorem

Notice, that if the inequality (**??**) is executed for some α and

$$f(x) = a_k x^k + \cdots + a_1 x + a_0$$

is a polynomial with the height $h(f) \le q$, then for the polynomial $\varphi(x) = f(x) - a_0$ at the point α following inequality is satisfied:

$$\|\varphi(\alpha)\| < q^{-\gamma}.$$

It is enough to consider only transcendental numbers from the interval [0,1].

In the lemma 2 we put

$$f_1(\bar{x}) = f(x_1) + f(x_2) + \cdots + f(x_K) - f(x_{K+1}) - f(x_{K+2}) - \cdots - f(x_{2K}),$$

where $\bar{x} = (x_1, x_2 ..., x_{2K}) \in [0,1]^{2K}$, $f(x) = \sum_{i=0}^{n} a_i x^i$, $r = n + \delta$, (δ it is small enough), $N = 1$. Let q-natural number. We shall designate $\mu(q)$ a measure of set of those $\alpha \in [0,1]$ for which

$$\|f(\alpha)\| < (2K)^{-1} q^{-n-\delta}. \tag{2}$$

Further, we designate $\mu_0(q)$ a measure of set such $\bar{\alpha} = (\alpha_1, \alpha_2 ..., \alpha_{2K}) \in [0,1]^{2K}$, for which

$$\|f_1(\bar{\alpha})\| < q^{-n-\delta}. \tag{3}$$

It is easy to notice that when (2) satisfied for $\alpha_1, \alpha_2,\ ..., \alpha_{2K} then$ (3) is satisfied for $\bar{\alpha} = (\alpha_1, \alpha_2 ..., \alpha_{2K})$ also, according to the inequality

$$\|f(x_1) + f(x_2) + \cdots + f(x_K) - f(x_{K+1}) - f(x_{K+2}) - \cdots - f(x_{2K})\| \le$$

$$\le \|f(x_1)\| + \|f(x_2)\| + \cdots + \|f(x_K)\| +$$
$$+ \|f(x_{K+1})\| + \|f(x_{K+2})\| + \cdots + \|f(x_{2K})\| \le q^{-n-\delta}.$$

Therefore, if we put

$$A_q = \{\alpha \in [0,1] |\, \|f(\alpha)\| < (2K)^{-1} q^{-n-\delta}\}$$

then

$$A_q^{2K} \subset B_q = \{\bar{\alpha} \in [0,1]^{2K} |\, \|f_1(\bar{\alpha})\| < q^{-n-\delta}\}.$$

But inverse of this statement is not true. Now we denote by B the set of that

$$\bar{\alpha} \in [0,1]^{2K}$$

which falls into the sets B_q for infinite number of natural numbers. For the proof of the theorem we must establish that $mes B = 0$.

Further, separating the term corresponding to the value $c = 0$, we shall have, according to the lemma 2:

$$\mu^{2K}(A_q) \le \mu_0(B_q) << q^{-n-\delta} + q^{-n-\delta} \times$$

$$\times \sum_{0<|c|<q^{n+\delta}} \left| \int_{[0,1]^{2K}} e^{2\pi ic(f(\alpha_1)+\cdots+f(\alpha_K)-f(\alpha_{K+1})+\cdots+f(\alpha_{2K}))} d\alpha_1 \cdots d\alpha_{2K} \right|, \tag{4}$$

Thus the constant in the symbol $<<$ depends only from δ. We can present integral on the last chain of the previous inequality as follows (when $c \neq 0$):

$$\int_{[0,1]^{2K}} e^{2\pi ic(f(\alpha_1)+\cdots+f(\alpha_K)-f(\alpha_{K+1})-\cdots-f(\alpha_{2K}))} d\alpha_1 \cdots d\alpha_{2K} =$$

$$= \left| \int_{[0,1]^K} e^{2\pi ic(\varphi(\alpha_1)+\cdots+\varphi(\alpha_K))} d\alpha_1 \cdots d\alpha_{2K} \right|^2 .$$

If we use the polynomial representation, then we can write:

$$\varphi(\alpha_1) + \cdots + \varphi(\alpha_K) = a_n\alpha_1^{m_n} + \cdots + a_1\alpha_1^{m_1} + \cdots +$$

$$+a_n\alpha_K^{m_n} + \cdots + a_1\alpha_K^{m_1} = a_n(\alpha_1^{m_n} + \cdots + \alpha_K^{m_n}) + \cdots +$$

$$+a_1(\alpha_1^{m_1} + \cdots + \alpha_K^{m_1}).$$

Therefore,

$$\int_{[0,1]^K} e^{2\pi ic(\varphi(\alpha_1)+\cdots+\varphi(\alpha_K))} d\alpha_1 \cdots d\alpha_{2K} =$$

$$= \int_0^1 \cdots \int_0^1 e^{2\pi i(ca_1u_1+\cdots+ca_nu_n)} d\alpha_1 \cdots d\alpha_K;$$

Here we used designations $u_j = \alpha_1^{m_j} + \alpha_2^{m_j} + \cdots + \alpha_K^{m_j}$, $j = 1, ..., n$. Applying consequence of the lemma 1 of the work [5], we can rewrite the last integral as follows:

$$\int_0^1 \cdots \int_0^1 e^{2\pi i(ca_1u_1+\cdots+ca_nu_n)} d\alpha_1 \cdots d\alpha_K =$$

$$\int_0^K \cdots \int_0^K \left(\int_\Pi \frac{ds}{\sqrt{G}} \right) e^{2\pi i(ca_1u_1+\cdots+ca_nu_n)} du_1 \cdots du_n, \tag{5}$$

where Π here designates a surface defined by system of the equations

$$u_j = \alpha_1^{m_j} + \alpha_2^{m_j} + \cdots + \alpha_K^{m_j}, j = 1, ..., n,$$

and G is Gram's determinant of gradients of functions u_j, i.e.

$$G = \det(AA^t);$$

here A designates Jacoby matrix of system of functions u_j:

$$A = \begin{pmatrix} m_n\alpha_1^{m_n-1} & m_n\alpha_2^{m_n-1} & \cdots & m_n\alpha_K^{m_n-1} \\ m_{n-1}\alpha_1^{m_{n-1}-1} & m_{n-1}\alpha_2^{m_{n-1}-1} & \cdots & m_{n-1}\alpha_K^{m_{n-1}-1} \\ \vdots & \vdots & \ddots & \vdots \\ m_1\alpha_1^{m_1-1} & m_1\alpha_2^{m_1-1} & \cdots & m_1\alpha_K^{m_1-1} \end{pmatrix}.$$

According to the problem 48, sec. 5, chap.1 of [14], each minor of this matrix of a maximal order is distinct from zero, if numbers $\alpha_1, \alpha_2, ..., \alpha_K$ are in pairs various. Hence, the set of those points $(\alpha_1, \alpha_2, ..., \alpha_K) \in [0, 1]^K$ for which $G = 0$ represents a Jordan set of a zero measure.

Let's notice also that the surface integral

$$g(\bar{u}) = \int_{\Pi} \frac{ds}{\sqrt{G}},$$

for $\bar{u} = (u_1, ..., u_K)$ is defined in improper sense, as a limit

$$\lim_{\lambda \to 0} \int_{\Pi, G \geq \sqrt{\lambda}} \frac{ds}{\sqrt{G}},$$

which defines a summable function of $\bar{u}$. The integral on the right part of the equality (4) can be broken into the sum of integrals through unite cubes:

$$\int_0^K \cdots \int_0^K g(\bar{u}) e^{2\pi i(ca_1 u_1 + \cdots + ca_n u_n)} du_1 \cdots du_n =$$

$$= \sum_{0 \leq j_1, ..., j_n \leq K-1} \int_{j_1}^{j_1+1} \cdots \int_{j_n}^{j_n+1} g(\bar{u}) e^{2\pi i(ca_1 u_1 + \cdots + ca_n u_n)} du_1 \cdots du_n.$$

Each integral on the right part of this equality is a Fourier coefficient

$$g_{ca_1, ..., ca_n}(j_1, ..., j_n) = \int_{j_1}^{j_1+1} \cdots \int_{j_n}^{j_n+1} g(\bar{u}) e^{2\pi i(ca_1 u_1 + \cdots + ca_n u_n)} du_1 \cdots du_n$$

for the restriction of the function $g(\bar{u})$ into the cube $[j_1, j_1 + 1] \times \cdots \times [j_n, j_n + 1]$.

From (4) and (5) we receive

$$\mu^{2K}(A_q) \leq \mu_0(B_q) \leq q^{-n-\delta} +$$

$$+ q^{-n-\delta} \sum_{0 < |c| < q^{n+\delta}} \left| \sum_{0 \leq j_1, ..., j_n \leq K-1} g_{ca_1, ..., ca_n}(j_1, ..., j_n) \right|^2 \leq$$

$$\leq q^{-n-\delta} + q^{-n-\delta} K^n \sum_{0 < |c| < q^{n+\delta}} \sum_{0 \leq j_1, ..., j_n \leq K-1} \left| g_{ca_1, ..., ca_n}(j_1, ..., j_n) \right|^2. \quad (6)$$

Designate by $b_m = ca_m$ in the last sum on the right part of inequalities (6). Note that according to the restrictions imposed on indexes we have $|b_m| \leq q^{n+1+\delta}$. Therefore, the number of solutions of an equation $xy = b_m$ in integers don't exceed $4\tau(b_m) <<_\varepsilon q^\varepsilon$ for all positive $\varepsilon > 0$, and a constant under the sign of $<<$ depends only on ε. We have:

$$\mu^{2K}(A_q) \leq \mu_0(B_q) \leq q^{-n-\delta} +$$

$$+q^{-n-\delta}K^n \sum_{0\le j_1,\dots,j_n\le K-1} \sum_{0<|c|<q^{n+\delta}} |g_{ca_1,\dots,ca_n}(j_1,\dots,j_n)|^2 \le$$

$$<< q^{-n-\delta} + q^{-n-\delta}K^n \sum_{0\le j_1,\dots,j_n\le K-1} \sum_{0<|b_j|<q^{n+1+\delta}, j=1,\dots,n} |g_{b_1,\dots,b_n}(j_1,\dots,j_n)|^2 .$$

This inequality is fair for given polynomial f with height $h(f) \le q$ and given transcendental number α. Using indicated above estimation for the number of divisors, we only increase the sum. Therefore, we have:

$$\mu_0(B_q) \le q^{-n-\delta+\varepsilon} + q^{-n-\delta+\varepsilon}K^n \sum_{0\le j_1,\dots,j_n\le K-1} 1\times$$

$$\times \sum_{|b_1|\le q^{n+2}} \cdots \sum_{|b_n|\le q^{n+2}} |g_{b_1,\dots,b_n}(j_1,\dots,j_n)|^2 . \qquad (7)$$

Let's designate U_q set of all those transcendental numbers for which the inequality (**??**) is satisfied for some polynomial with the height $h(f) \le q$. Then, summarising both parts of received above inequalities over all polynomials of specified kind, from (7) we deduce:

$$\mu(U_q) << q^{-\delta+\varepsilon} + q^{-n-\delta+\varepsilon}K^n \sum_{|a_1|\le q} \cdots \sum_{|a_n|\le q} \sum_{0\le j_1,\dots,j_n\le K-1} 1\times$$

$$\times \sum_{|b_1|\le q^{n+2}} \cdots \sum_{|b_n|\le q^{n+2}} |g_{b_1,\dots,b_n}(j_1,\dots,j_n)|^2 \le$$

$$\le q^{-\delta+\varepsilon} + q^{-\delta+\varepsilon}K^n \sum_{0\le j_1,\dots,j_n\le K-1} \sum_{|b_1|\le q^{n+2}} \cdots \sum_{|b_n|\le q^{n+2}} |g_{b_1,\dots,b_n}(j_1,\dots,j_n)|^2 .$$

Applying Parseval's equality, we can write:

$$\sum_{b_1=-\infty}^{\infty} \cdots \sum_{b_n=-\infty}^{\infty} |g_{b_1,\dots,b_n}(j_1,\dots,j_n)|^2 = \int_{j_1}^{j_1+1} \cdots \int_{j_n}^{j_n+1} (g(\bar u))^2\, du_1 \cdots du_n.$$

Hence, writing $\varepsilon = \delta/2$, we receive:

$$\mu(U_q) << q^{-\delta/2} +$$

$$+q^{-\delta/2}K^n \sum_{0\le j_1,\dots,j_n\le K-1} \int_{j_1}^{j_1+1} \cdots \int_{j_n}^{j_n+1} (g(\bar u))^2\, du_1 \cdots du_n \le$$

$$\le q^{-\delta/2} + q^{-\delta/2}K^n \int_0^K \cdots \int_0^K \left(\int_\Pi \frac{ds}{\sqrt{G}} \right)^2 du_1 \cdots du_n. \qquad (8)$$

As it has been shown in [1], last integral is equal to the special integral (1) of Tarry's problem:

$$(2\pi)^{-n} \int_0^\infty \cdots \int_0^\infty \left| \int_0^1 e^{2\pi i(\alpha_1 x^{m_1}+\cdots+\alpha_n x^{m_n})} dx \right|^{2K} d\alpha_1 \cdots d\alpha_n.$$

According to the lemma 3, this special integral has an exponent of convergence equal to $m_1 + m_2 + \cdots + m_n$. So, when $2K > m_1 + m_2 + \cdots + m_n$ last integral a constant. So, we have

$$\mu(U_q) << q^{-\delta/2}, \tag{9}$$

According to (8). Taking $q = 2^k$ we will consider the union

$$\bigcup_{k=1}^{\infty} U_{2^k}.$$

Recall that the set U_q is a set of those transcendental numbers $\alpha \in [0,1]$ for which the inequality (**??**) is satisfied for $f \in \Pi$ with heights not exceeding q. Then,

$$U_q = \bigcup_{f \in \Pi, h(f) \leq q} W_f,$$

where

$$W_f = \{\alpha | \, |f(\alpha)| < q^{-n-\delta}\}.$$

Since

$$\sum_{k=1}^{\infty} \mu\left(U_{2^k}\right) \leq \sum_{m=0}^{\infty} 2^{-m\delta/2},$$

then by the lemma 1 the set of those α which belong into infinite set of subsets W_f has a measure 0. We, hence, have proved that the number of polynomials with heights $\leq q$ satisfying (3) is unbounded when $q \to \infty$, only for the set of transcendental numbers of a zero measure.

Taking sequence of numbers $\delta_1, \delta_2, ..., \delta_n \to \infty$, we find sequence of sets $M_1, M_2, ...$ of zero measure for points of each of which above relation (3) is executed for infinite number of polynomials $f_1(\bar{x})$, with corresponding value δ_n. Designate now $M = \bigcup_{i=1}^{\infty} M_i$. We have $mesM = 0$. For points $(\alpha_1, ..., \alpha_{2K}) \in [0,1]^{2K} \backslash M$ considered above inequality for any positive $\delta > 0$ can be satisfied only for finite number of polynomials. It means, that for all $(\alpha_1, ..., \alpha_{2K}) \in [0,1]^{2K} \backslash M$ supremum of those numbers $u > 0$, for which the system (3) is executed for infinite set of polynomials, does not exceed $1/N$. Further, from Dirichlet's principle it follows that this supremum is not less than $1/N$. Hence, in designations entered above, the equality $u(\beta_1, ..., \beta_N) = 1/N$, as was to be shown, holds. The theorem is proved.

The theorem is proved.

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