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Method Integrating One System of Nonlinear Differential Equations

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Abstract. The Cauchy problem for one system of nonlinear evolution equations, which is a generalization of the Langmuir chain, is considered. The global solvability of the problem is established. By the inverse spectral method, an algorithm for constructing the solution is obtained.

Key Words and Phrases: nonlinear evolution equation, Langmuir chain, spectral data, method of the inverse spectral problem .

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1. Introduction

For positive functions $c_n = c_n(t), c_n(t) \in C^{(1)}[0,\infty)$, we consider the Cauchy problem for the following system of nonlinear evolution equations:

$$\begin{array}{c} \dot{c}_n = c_n \left(\alpha \left(c_{n+1} - c_{n-1} \right) - \beta \left(\left(c_{n+1} - c_{n-1} \right) \sum_{k=0}^{2} c_{n+k} \right) \right), \\ c_n = c_n(t),\ n = 0, 1, ..N-1, t \in (0, \infty],\ \cdot = \frac{d}{dt}, c_{-1} = c_N = 0, \end{array} \tag{1.1}$$

$$c_n(0) = \hat{c}_n > 0,\ n = 0, 1, ..., N-1 \tag{1.2}$$

where α, $\beta$are real numbers and $N \geq 2$ is a given positive integer. This system was first studied in [1], where it was also found there that a system of equations (1.1) can be integrated using the method of the inverse spectral problem (see, also[2]). With $\alpha = 1$, $\beta = 0$, the system of equations (1.1) represents the well-known Volterra model, which was studied by the method of the inverse spectral problem by many authors $[3]-[8]$. In this paper, the global solvability of the problem (1.1) and (1.2)) is established. An algorithm for finding the solution of problem (1.1) and (1.2) by the method of the inverse spectral problem is constructed. Similar issues for various nonlinear evolutionary equations were studied in $[4], [5], [7], [8]$.

2. Preliminary information

Below, we formulate some well-known facts concerning inverse eigenvalue problems for finite Jacobian matrices. Many of these results with proofs can be found in[4] , [6] , [8].

Consider the $(N+1)$dimensional Jacobian matrix

$$L=\begin{pmatrix} 0 & \sqrt{c_0} & 0 & ... & 0 & 0 \\ \sqrt{c_0} & 0 & \sqrt{c_1} & ... & 0 & 0 \\ ... & ... & ... & .... & ... & ... \\ 0 & 0 & 0 & ... & 0 & \sqrt{c_{N-1}} \\ ... & ... & ... & ... & \sqrt{c_{N-1}} & 0 \end{pmatrix}$$

which has zeros on the main diagonal; positive numbers $\sqrt{c_0}, \ldots, \sqrt{c_{N-1}}$ on two adjacent diagonals; and zeros everywhere else. Consider the difference equation

$$\sqrt{c_{n-1}}y_{n-1}+\sqrt{c_n}y_{n+1}=\lambda y_n, n=0,1,...,N,\ c_{-1}=c_N=1, \tag{2.1}$$

where λ is the spectral parameter. Denote by $P_n(\lambda)$the solution to Eq. (2.1) with the initial conditions $P_{-1}(\lambda)=0$ and $P_0(\lambda)=1$. It is well known that the eigenvalues of L are real, simple, and coincide with the zeros of the polynomial $P_{N+1}(\lambda)$ Moreover, these eigenvalues are symmetric about the point $\lambda=0$.

Let $\lambda_0,...,\lambda_N$be the eigenvalues of L. The normalizing coefficients $\alpha_k$are defined as

$$\alpha_k=\sum_{n=0}^{N}P_n^2(\lambda_k), k=0,1,...,N.$$

Moreover, symmetric eigenvalues correspond to identical normalizing coefficients and

$$\sum_{k=0}^{N}\alpha_k^{-1}=1. \tag{2.2}$$

The collection $\{\lambda_k;\alpha_k>0\}_{k=0}^{N}$is called the spectral data of the matrix L. The inverse eigenvalue problem for the matrix L consists in determining the elements $c_n, n=0,...,N-1$ from the spectral data $\{\lambda_k;\alpha_k>0\}_{k=0}^{N}$ satisfying (2.2). The matrix L is uniquely determined by the spectral data and can be found by applying the following algorithm: Let the spectral data $\{\lambda_k;\alpha_k>0\}_{k=0}^{N}$ be given. Construct the moments s_n, $0\le n\le 2N$ by the formula

$$s_n=\sum_{k=0}^{N}\lambda_k^n\alpha_k^{-1} \tag{2.3}$$

and the Hankel determinants $D_n, -1\le n\le N$ by the formulas

$$D_{-1}=1, D_n=\begin{vmatrix} s_0 & s_1 & ... & s_n \\ s_1 & s_2 & ... & s_{n+1} \\ ... & ... & ... & ... \\ s_n & s_{n+1} & ... & s_{2n} \end{vmatrix}, 0\le n\le N. \tag{2.4}$$

Calculate $c_n, n = 0, ..., N-1$ by the formula

$$c_n = D_{n-1} D_{n+1} D_n^{-2}. \quad (2.5)$$

It should be noted that s_n with odd n vanish. This circumstance considerably simplifies the computations of D_n. Additionally, D_n are positive.

3. Solvability of problem (1.1), (1.2)

Let us study the global solvability of problem (1.1) and (1.2).

Theorem. *For any initial data* $c_n(0) = \hat{c}_n, n = 0, ..., N-1$, *problem* (1.1),(1.2) *has a unique solution* $c_n(t)$ *defined on the entire half line* $(0, \infty)$.

Proof. Note that the right hand sides of system (1.1), (1.2) are continuously differentiable functions of $c_0, ..., c_{N-1}$. Then, passing to an integral equation in the standard manner and applying the method of successive approximations, we find that problem (1.1), (1.2) on some interval $[0, \delta]$ has a unique solution$c_n(t), n = 0, ..., N-1$. Let us show that this solution can be extended to the entire positive half line. Assume the opposite. Then there exists a point $t^* \in (0, \infty)$ such that problem (1.1), (1.2) has a solution $c_n(t), n = 0, ..., N-1$ on the interval $[0, t^*)$, but n for some index $\overline{\lim\limits_{t \to t^*}} c_n(t) = \infty$.

On the other hand, the equality (4.1), which will be proved below, imply that the family of matrices $L = L(t)$ are unitarily equivalent; i.e., there exists an $(N+1)$dimensional unitary matrix $U(t)$ $\left(U^*(t) = U^{-1}(t)\right)$ such that

$$U(0) = E, \; L(t) = U^*(t) L(0) U(t)$$

where E is the $(N+1)$dimensional identity matrix. These relations imply

$$\|L(t)\| = \|L(0)\|, t \in [0, t*), \quad (3.1)$$

where $\|\cdot\|$ is the norm of $L = L(t)$ in the$(N+1)$dimensional space of vectors$y = (y_0, y_1, ..., y_N)$ with the norm $\|y\| = \left(\sum_{k=0}^{N} y_k^2\right)^{\frac{1}{2}}$. Combining formula (3.1) with the obvious inequality

$$|c_n(t)| \le \|L(t)\|^2, n = 0, ..., N-1,$$

we see that our assumption that $\overline{\lim\limits_{t \to t^*}} c_n(t) = \infty$is not correct. Thus, the theorem is proved.

4. Solution algorithm for problem (1.1), (1.2)

Assume now that the elements L depend on t: $L = L(t)$ and $c_n(t)$ satisfies Cauchy problem (1.1), (1.2). Consider a $(N+1)$ dimensional matrix $A = A(t)$ that acts on a vector $y = (y_0, y_1, ..., y_N)$ according to the formula

$$(Ay)_n = \frac{1}{2}\left(-\beta \prod_{k-0}^{3} \sqrt{c_{n-1-k}} y_{n-4} - \right.$$

$$-\sqrt{c_{n-1}c_{n-2}}\left(-\alpha+\beta\sum_{k=-1}^{2}c_{n-1-k}\right)y_{n-2}+$$
$$+\sqrt{c_nc_{n+1}}\left(-\alpha+\beta\sum_{k=-1}^{2}c_{n+k}\right)y_{n+2}+\beta\prod_{k=0}^{3}\sqrt{c_{n+k}}y_{n+4}\Big)$$

where $(Ay)_j\,(j=0,1,...,N)$ are calculated taking into account $y_k=0,\,k=-4,-3,-2,-1,N+1,N+2,N+3,N+4$. Note that A is a skew-symmetric matrix: $A^*=-A$. Moreover, it is easy to see that $L$and A form a Lax pair; i.e., system (1.1), (1.2) is equivalent to the matrix equation

$$\dot{L}=[L,\,A]=LA-AL. \qquad (4.1)$$

Since (4.1) implies that the family of matrices $L=L(t)$ are unitarily equivalent (see[8]), the eigenvalues $\lambda_k(t)\,,k=0,1,...,N$ of $L=L(t)$ are independent of t:

$$\lambda_k(t)=\lambda_k(0)=\lambda_k,k=0,1,...,N.$$

Consider the equation

$$Ly=\lambda y \qquad (4.2)$$

where λ is a parameter independent of t. Combining (4.1) with the equality

$$\dot{L}y+L\dot{y}=\lambda\dot{y},$$

we see that the matrix B defined as

$$B=\frac{d}{dt}+A,$$

maps the solutions of Eq. (4.2) to solutions of the same equation.

Now let $P(\lambda_k,t)=(P_n(\lambda_k,t))_{n=0}^{N}$ be an eigenvector of $L=L(t)$ corresponding to the eigenvalue $\lambda_k,k=0,1,...,N$, and let $P_0(\lambda_k,t)=1$. Then $BP(\lambda_k,t)$ is also an eigenvector of $L=L(t)$ corresponding to λ_k. After simple transformations, you can show that

$$(BP(\lambda_k,t))_0=\frac{1}{2}\left[\beta\lambda_k^4-\alpha\lambda_k^2+(\alpha-\beta(c_0+c_1))\,c_0\right].$$

However, since the eigenvalues $\lambda_k,k=0,1,...,N$ are simple, we have

$$BP(\lambda_k,t)=\frac{1}{2}\left[\beta\lambda_k^4-\alpha\lambda_k^2+(\alpha-\beta(c_0+c_1))\,c_0\right]P(\lambda_k,t) \qquad (4.3)$$

Let us determine the dynamics of the normalizing coefficient

$$\alpha_k(t)=\sum_{n=0}^{N}P_n^2(\lambda_k,t)\,,k=0,1,...,N.$$

Taking into account

$$\dot{\alpha}_k(t)=2\sum_{n=0}^{N}P_n(\lambda_k,t)\,\dot{P}_n(\lambda_k,t)\,,k=0,1,...,N.$$

and applying formula (4.3), we find that

$$\dot{\alpha}_k(t) = \left[\beta\lambda_k^4 - \alpha\lambda_k^2 + (\alpha - \beta(c_0 + c_1))\, c_0\right] \alpha_k(t) + \sum_{n=0}^{N} (AP(\lambda_k, t))_n P_n(\lambda_k, t), k = 0, 1, ..., N.$$

Since A is skew-symmetric, the sum

$$\sum_{n=0}^{N} (AP(\lambda_k, t))_n P_n(\lambda_k, t), k = 0, 1, ..., N$$

vanishes. Therefore, we have

$$\dot{\alpha}_k(t) = \left[\beta\lambda_k^4 - \alpha\lambda_k^2 + (\alpha - \beta(c_0 + c_1))\, c_0\right] \alpha_k(t), k = 0, 1, ..., N.$$

whence

$$\alpha_k^{-1}(t) = \alpha_k^{-1}(0) \exp\left[\left(\alpha\lambda_k^2 - \beta\lambda_k^4\right) t\right] \exp\left[\int_0^t \left[\beta(c_0(\tau) + c_1(\tau)) - \alpha\right] c_0(\tau)\, d\tau\right]. \quad (4.4)$$

Here, $c_0(t)$ and $c_1(t)$cannot be arbitrary but must satisfy

$$\sum_{k=0}^{N} \alpha_k^{-1}(t) = 1.$$

Taking this into account in (4.4) yields

$$\exp\left[\int_0^t \left[\beta(c_0(\tau) + c_1(\tau)) - \alpha\right] c_0(\tau)\, d\tau\right] = \left(\sum_{k=0}^{N} \alpha_k^{-1}(0) \exp\left[\left(\alpha\lambda_k^2 - \beta\lambda_k^4\right) t\right]\right)^{-1}.$$

Substituting this equality into (4.4), we finally obtain

$$\alpha_k^{-1}(t) = \alpha_k^{-1}(0) \exp\left[\left(\alpha\lambda_k^2 - \beta\lambda_k^4\right) t\right] \left(\sum_{k=0}^{N} \alpha_k^{-1}(0) \exp\left[\left(\alpha\lambda_k^2 - \beta\lambda_k^4\right) t\right]\right)^{-1}, k = 0, ..., N. \quad (4.5)$$

Using relation (4.5), we obtain the following algorithm for solving the problem (1.1), (1.2) by the inverse spectral method: Let the initial conditions $c_n(0) = \hat{c}_n > 0$, $n = 0, 1, ..., N-1$ be given. Construct the spectral data $\{\lambda_k; \alpha(t)_k > 0\}_{k=0}^{N}$. Compute $\alpha_k(t)$, $k = 0, 1, ..., N$ by formulas (4.5). Given the collection $\{\lambda_k; \alpha(t)_k > 0\}_{k=0}^{N}$ solve the inverse eigenvalue problem by applying (2.3)–(2.5).

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