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On the Internal Compactness of the Space of Regular Solutions of one Class of Second Order Operator-Differential Equations with a Discontinuous Coefficient

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Abstract. In paper given definition of the internal compactness of the space of regular solutions of one class of homogeneous operator-differential equations of second order with a discontinuous coefficient and found sufficient conditions for the internal compactness of these solutions.

Key Words and Phrases: Hilbert space, operator-differential equation, regular solution, internal compactness.

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1. Introduction

Let H be a separable Hilbert space, A a normal operator with inverse in H.

Obviously, $D(A) = D(A^*)$, $A^*A = AA^*$ and the operator A is representable in the form $A = UC = CU$, here U is a unitary operator, C is a self-adjoint positive operator in H, so that, $D(C) = D(A)$ and

$$\|Ax\| = \|A^*x\| = \|Cx\|, x \in D(A),$$

$$\|A^\gamma x\| = \|C^\gamma x\|, \ x \in D(A^\gamma) = D(C^\gamma).$$

It is known that, the domain of the operator C^γ $(\gamma \geq 0)$ will be H Hilbert space with scalar product:

$$H_\gamma = D(C^\gamma), \ (x, y)_\gamma = (C^\gamma x, C^\gamma y), \ x, y \in H_\gamma \ .$$

If $\gamma = 0$ then we assume that, $H_0 = H$ and

$$(x, y)_0 = (x, y), \ x, y \in H.$$

Let $-\infty \leq a < b \leq \infty$. Denote by $L_2((a, b); H)$ set of all vector functions with the values in H, strongly measured and having finite integral:

$$\int_a^b \|f(t)\|_H^2 dt < \infty.$$

Note that $L_2((a,b)\,;H)$ be a Hilbert space with the scalar product and norm as follows:

$$(f,g)_{L_2((a,b);H)} = \int_a^b (f(t)\,,g(t))dt$$

$$\|f\|_{L_2((a,b);H)} = \left(\int_a^b \|f(t)\|_H^2 dt\right)^{1/2}.$$

Consider $n \geq 1$, $C^n u(t) \in L_2((a,b)\,;H)$ vector functions and $u^{(n)}(t) \in L_2((a,b)\,;H)$ are their generalized derivatives. Then the following linear set can be defined:

$$W_2^n\left((a,b)\,;H\right) =$$

$$= \left\{u(t):\ C^n u(t) \in L_2\left((a,b)\,;H\right), u^{(n)}(t) \in L_2((a,b)\,;H)\right\}.$$

Let us determine the scalar product in the $W_2^n\left((a,b)\,;H\right)$ linear set as follows:

$$(u,\vartheta)_{W_2^n((a,b);H)} = (u^{(n)},\vartheta^{(n)})_{L_2((a,b);H)} + (C^n u, C^n \vartheta)_{L_2((a,b);H)},$$

here $u,\vartheta \in W_2^n\left((a,b)\,;H\right)$. We obtain $W_2^n\left((a,b)\,;H\right)$ (see [6]) a complete Hilbert space. It is clear that

$$\|u\|_{W_2^n((a,b);H)} = \left(\left\|u^{(n)}\right\|^2_{L_2((a,b);H)} + \|C^n u\|^2_{L_2((a,b);H)}\right)^{1/2}.$$

This norm can also be written as follows

$$\|u\|_{W_2^n((a,b);H)} = \left(\left\|u^{(n)}\right\|^2_{L_2((a,b);H)} + \|A^n u\|^2_{L_2((a,b);H)}\right)^{1/2}.$$

It is known (see [6]) that $W_2^n\left((a,b)\,;H\right)$ space is included in $W_2^{n-1}\left((a,b)\,;H\right)$ space as a density set and this input is compact.

It is obvious that the norm of space $W_2^{n-1}\left((a,b)\,;H\right)$ is weaker than the norm of space $W_2^n\left((a,b)\,;H\right)$.

Let us consider a homogeneous operator-differential equation of the second order with the discontinuous coefficients in H separabel Hilbert space:

$$P\left(\frac{d}{dt}\right)u(t) \equiv -\frac{d^2u(t)}{dt^2} + \rho(t)\,A^2u(t) + A_1\frac{du(t)}{dt} +$$

$$+A_2u(t) = 0, \quad t \in R_+ = [0,+\infty)\ , \tag{1.1}$$

here $u(t)$ vector function with the values H defined in R_+, the operator coefficients of the equation satisfy the following conditions:

1) A is a normal operator with completely inverse A^{-1} and its spectrum is contained in the angular sector

$$S_\varepsilon = \left\{\lambda:\ |arg\lambda| \leq \varepsilon, \quad 0 \leq \varepsilon < \frac{\pi}{2}\right\};$$

2) $\rho(t) = \begin{cases} \alpha^2, & t \in (0,1), \\ \beta^2, & t \in (1, +\infty), \end{cases}$ $\alpha,\ \beta > 0$ (we assume that for simplicity $\alpha \leq \beta$);

3) A_1 and A_2 are linear operators such that, $B_1 = A_1 A^{-1}$, $B_2 = A_2 A^{-2}$ are bounded operators in H.

Definition 1. *If exists a vector-function $u(t) \in W_2^2(R_+; H)$ satisfying equation* (1.1) *almost everywhere in R_+ , then we say that it is a regular solution of equation* (1.1).

Let denote with the $Ker(P\left(\frac{d}{dt}\right), R_+)$ set of solutions of equation (1.1) from the space $W_2^2(R_+; H)$:

$$Ker\left(P\left(\frac{d}{dt}\right), R_+\right) = \left\{u(t) :\ P\left(\frac{d}{dt}\right) u(t) = 0,\ u(t) \in\ W_2^2(R_+; H)\right\}.$$

In other words, $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ is a space of regular solutions of equation (1.1).

Obviously, if conditions 1), 2) and 3) are fulfilled and the operator

$$P\left(\frac{d}{dt}\right) :\ W_2^2(R_+; H) \to L_2(R_+; H)$$

is a continuous then the set $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ is a closed subspace of the space $W_2^2(R_+; H)$.

Now let's denote with $\mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ closure of the space $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ respect to the norm of the space $W_2^1(R_+; H)$.

Definition 2. *Suppose that*

$$0 \leq a < a' < b' < b < +\infty$$

and $M > 0$ *is arbitrary number. If the set*

$$W_M = \left\{u(t) : u(t) \in\ \mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right),\ \|u\|_{W_2^1((a,\ b); H)} \leq M\right\}$$

is compact set in the norm of space $W_2^1((a',\ b'); H)$, *then we will say that* $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ *space is an internal compact space in the norm of space* $W_2^1(R_+; H)$.

For the first time in P.D. Lax's work [5] is given the definition of internal compactness for some solution spaces at infinite intervals and shown its close connection with the Frackman-Lindelöf principle for solutions of elliptic equations. For example, such theorems for abstract equations are obtained in the works [1], [7].

2. Main results

Let us denote the subspace of space $W_2^2(R_+; H)$ by $\dot{W}_2^2(R_+; H)$ and define it as follows:

$$\dot{W}_2^2(R_+; H) = \left\{u(t) : u(t) \in W_2^2(R_+; H),\ u(0) = u'(0) = 0\right\}.$$

In this section defines the internal compact property of the space $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ with respect norm of $\|u\|_{W_2^1(R_+; H)}$.

The following theorem is valid.

Theorem. *Let conditions* 1), 2), 3) *be fulfilled and for a vector function* $u(t)$ *from space* $\dot{W}_2^2(R_+;H)$ *the following inequality hold:*

$$\left\| P\left(\frac{d}{dt}\right) u \right\|_{L_2(R_+;H)} \geq const \|u\|_{W_2^2(R_+;H)}. \qquad (2.1)$$

Then the space $\mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ *is an internal compact space and there is such a number that* $\varkappa_0 > 0$, *for any* $u(t) \in \mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ *the following inequality hold:*

$$\int_0^{+\infty} e^{-2\varkappa_0 t} \left(\left\| \frac{du(t)}{dt} \right\|_H^2 + \|Cu(t)\|_H^2 \right) dt < +\infty.$$

Proof. Let $u(t) \in Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$. Then it is obvious that, for any $\eta > 0$ $u(t+\eta) \in Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$, that is, space $Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$ is invariant space with respect to sliding. Let $0 \leq a < b < +\infty$ and $a < a' < b' < b$. Let's define a numerical function $\varphi(t) \in C^\infty(a,\ b)$ as follows:

$$\varphi(t) = \begin{cases} 1, & t \in (a',\ b'), \\ 0, & t \notin (a',\ b'). \end{cases}$$

Then for a vector function $u(t) \in Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$

$$v(t) = \varphi(t)\, u(t) \in \dot{W}_2^2(R_+;H).$$

Therefore, we obtain from inequality (2.1) that

$$\left\| P\left(\frac{d}{dt}\right) v \right\|_{L_2(R_+;H)} \geq const \|v\|_{W_2^2(R_+;H)}.$$

Since $v(t) = 0$ in the case $t \notin (a',\ b')$ we can write the last inequality as follows:

$$\left\| P\left(\frac{d}{dt}\right) v \right\|_{L_2(R_+;H)} = \left\| P\left(\frac{d}{dt}\right) (\varphi u) \right\|_{L_2(R_+;H)} \geq$$

$$\geq const \|\varphi u\|_{W_2^2(R_+;H)} \geq const \|\varphi u\|_{W_2^2((a,\ b);H)} =$$

$$= const \left(\int_a^b \left(\left\| \frac{d^2(\varphi u)}{dt^2} \right\|_H^2 + \left\| A^2(\varphi u) \right\|_H^2 \right) dt \right)^{1/2} \geq$$

$$\geq const \left(\int_{a'}^{b'} \left(\left\| \frac{d^2(\varphi u)}{dt^2} \right\|_H^2 + \left\| A^2(\varphi u) \right\|_H^2 \right) dt \right)^{1/2} =$$

$$= const \left(\int_{a'}^{b'} \left(\left\| \frac{d^2 u}{dt^2} \right\|_H^2 + \left\| A^2 u \right\|_H^2 \right) dt \right)^{1/2} =$$

$$= const\|u\|_{W_2^2((a',\ b');H)}. \tag{2.2}$$

On the other hand, it is easy to see that

$$P\left(\frac{d}{dt}\right)v\left(t\right)=P\left(\frac{d}{dt}\right)\left(\varphi\left(t\right)u\left(t\right)\right)=-\frac{d^2\left(\varphi\left(t\right)u\left(t\right)\right)}{dt^2}+$$

$$+\rho\left(t\right)A^2\left(\varphi\left(t\right)u\left(t\right)\right)+A_1\frac{d\left(\varphi\left(t\right)u\left(t\right)\right)}{dt}+A_2\left(\varphi\left(t\right)u\left(t\right)\right)=$$

$$=\varphi\left(t\right)P\left(\frac{d}{dt}\right)u\left(t\right)-2\frac{d\varphi\left(t\right)}{dt}\frac{du\left(t\right)}{dt}-u\left(t\right)\frac{d^2\varphi\left(t\right)}{dt^2}+$$

$$+\frac{d\varphi\left(t\right)}{dt}A_1u(t).$$

As we know

$$P\left(\frac{d}{dt}\right)u\left(t\right)=0.$$

Then

$$P\left(\frac{d}{dt}\right)v\left(t\right)=-2\frac{d\varphi\left(t\right)}{dt}\frac{du\left(t\right)}{dt}-$$

$$-u\left(t\right)\frac{d^2\varphi\left(t\right)}{dt^2}+\frac{d\varphi\left(t\right)}{dt}A_1u(t).$$

From here we get that

$$\left\|P\left(\frac{d}{dt}\right)v\right\|_{L_2(R_+;H)}=$$

$$=\left\|-2\frac{d\varphi}{dt}\frac{du}{dt}-u\frac{d^2\varphi}{dt^2}+\frac{d\varphi}{dt}A_1u\right\|_{L_2(R_+;H)}.$$

Since from

$$\max_t\left|\varphi^{(s)}(t)\right|\le const\ \ ,\ s=1,2,$$

the following inequalities are hold:

$$\left\|-2\frac{d\varphi}{dt}\frac{du}{dt}-u\frac{d^2\varphi}{dt^2}+\frac{d\varphi}{dt}A_1u\right\|_{L_2((a,\ b);H)}\le$$

$$\le\left\|2\frac{d\varphi}{dt}\frac{du}{dt}\right\|_{L_2((a,\ b);H)}+\left\|u\frac{d^2\varphi}{dt^2}\right\|_{L_2((a,\ b);H)}+\left\|\frac{d\varphi}{dt}A_1u\right\|_{L_2((a,\ b);H)}\le$$

$$\le 2\max_{t\in(a,\ b)}\left|\varphi'(t)\right|\left\|\frac{du}{dt}\right\|_{L_2((a,\ b);H)}+\max_{t\in(a,\ b)}\left|\varphi''\left(t\right)\right|\left\|A^{-1}Au\right\|_{L_2((a,\ b);H)}+$$

$$+\max_{t\in(a,\ b)}\left|\varphi'\left(t\right)\right|\left\|A_1A^{-1}Au\right\|_{L_2((a,\ b);H)}\le const\left\|\frac{du}{dt}\right\|_{L_2((a,\ b);H)}+$$

$$+const \left\|A^{-1}\right\| \|Au\|_{L_2((a,\ b);H)} + const \left\|A_1A^{-1}\right\| \|Au\|_{L_2((a,\ b);H)}.$$

Applying the theorem on intermediate derivatives to the last inequality [6] it is obtained that

$$\left\| -2\frac{d\varphi}{dt}\frac{du}{dt} - u\frac{d^2\varphi}{dt^2} + \frac{d\varphi}{dt}A_1u \right\|_{L_2((a,\ b);H)} \leq$$

$$\leq const \|u\|_{W_2^1((a,\ b);H)} + const \left\|A^{-1}\right\| \|u\|_{W_2^1((a,\ b);H)} +$$

$$+const \left\|A_1A^{-1}\right\| \|u\|_{W_2^1((a,\ b);H)} \leq const\|u\|_{W_2^1((a,\ b);H)}. \tag{2.3}$$

Thus, from inequalities (2.2) and (2.3) we get:

$$\|u\|_{W_2^2((a',b');H)} \leq$$

$$\leq const \left\| P\left(\frac{d}{dt}\right) v \right\|_{L_2(R_+;H)} \leq const \|u\|_{W_2^1((a,\ b);H)}.$$

Hence, we obtain that the following inequality is valid

$$\|u\|_{W_2^1((a,\ b);H)} \geq const\|u\|_{W_2^2((a',b');H)}. \tag{2.4}$$

Using inequality (2.4), let's show that space $\mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ is an internally compact space.

We must prove that the set W_M is a compact set in the norm of space $W_2^1\left((a',b');H\right)$, here $a < a' < b' < b$.

Let's denote with $\widetilde{W_M}$ intersection of $W_M \cap Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$,

$$\widetilde{W_M} = W_M \cap Ker\left(P\left(\frac{d}{dt}\right), R_+\right).$$

Let us show that the set $\widetilde{W_M}$ is a compact set with respect to the norm of space $W_2^1\left((a',b');H\right)$. If

$$u(t) \in \widetilde{W_M} \subset\ Ker\left(P\left(\frac{d}{dt}\right), R_+\right)$$

then the following inequality hold:

$$\|u\|_{W_2^2((a',b');H)} \leq const\|u\|_{W_2^1((a,b);H)}.$$

This means that the set $\widetilde{W_M}$ is a bounded set with respect to the norm $\|u\|_{W_2^2((a',b');H)}$. Since A^{-1} operator is a completely continuous operator then the following statement is compact:

$$W_2^2\left(\left(a',b'\right);H\right) \subset W_2^1\left(\left(a',b'\right);H\right)$$

(see [1], [7]). That is why the set $\widetilde{W_M}$ is a compact set in the space $W_2^1\left((a',b');H\right)$.

Consequently, the space $\mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ is an internally compact space. Hence, since the space $\mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ is an invariant space with respect to sliding, it follows from P.D. Lax's theorem (see [5]) that there is such a number that $\varkappa_0 > 0$, for any $u(t) \in \mathcal{L}\left(P\left(\frac{d}{dt}\right), R_+\right)$ the following evaluation is valid:

$$\int_0^{+\infty} e^{-2\varkappa_0 t}\left(\left\|\frac{du(t)}{dt}\right\|_H^2 + \|Cu(t)\|_H^2\right) dt < +\infty.$$

The theorem is proved.

Let us noted that over the last 15 years, interest in the issues of internal compactness has increased, and several such studies have conducted in various aspects (for example, see [2], [3], [4]).

References

[1] Agmon S., Nirenberg L. Properties of solutions of ordinary differential equations in Banach space. Comm. Pure Appl. Math., 1963, vol. 16, pp. 121-239.

[2] Baranov, A. D. Interior-compact subspaces and differentiation in model subspaces. Journal of Mathematical Sciences, 2006, vol. 139, no. 2, pp. 6369-6373.

[3] Chodounský D., Murtinová E. Internal normality and internal compactness. Topology Appl., 2008, vol. 155, no. 4, pp. 201-206.

[4] Gasanov R.G. On intrinsic compactness properties of generalized solutions of a fourth order operator-differential equation on the segment. Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., 2006, vol. 26, no. 7, pp. 63-70.

[5] Lax P.D. A Phragmén-Lindelöf theorem in harmonic analysis and its application to some questions in the theory of elliptic equations. Comm. Pure Appl. Math., 1957, vol. 10, pp. 361-389.

[6] Lions J.-L., Magenes E. Non-homogeneous boundary value problems and applications. Mir, Moscow, 1971, 371 pp. (in Russian).

[7] Yakubov S.Ya. Linear differential-operator equations and their applications. Elm, Baku, 1985, 220 pp. (in Russian).

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