Journal of Contemporary Applied Mathematics
V. 11, No 2, 2021, December
ISSN 2222-5498

Problems for the First-Order Differential equations with Discrete Additive and Discrete Multiplicative Derivatives

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Abstract. The paper is devoted to the study of the solution of the Cauchy and boundary value problems for the ordinary differential equation of the first order with discrete additive and discrete multiplicative derivatives. Based on the definition of discrete derivatives, the considered problem is reduced to the different difference equations, for the solution of which the analytical expression is obtained. The analytical formulas for the solution are proved by the method of mathematical induction. Formulation of the problem and the obtained results are new and relevant when solving not only linear ones, even for nonlinear difference equations. Note that the considered equation and the boundary conditions, are nonlinear.

Key Words and Phrases: Discrete additive derivative; discrete multiplicative derivative; Cauchy problem; boundary value problem; solution of the boundary value problem; nonlinear difference equation.

2010 Mathematics Subject Classifications: Primary 34B09; 34B15.

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1. Introduction

It is known that both for the ordinary differential equation [1], [2] and for the partial differential equations [3, 4], various problems for discrete additive derivatives [5] and the so-called difference equations [6] have been well studied.

Multiplicative derivatives in the continuous case have appeared recently. In [7] the definition of the multiplicative derivative and the multiplicative integral and their main properties are given.

We started to study the discretely multiplicative derivative in [8]. Further, different problems were considered for the differential equation with discrete additive-multiplicative and discrete multiplicative-additive derivatives [9], [10].

Then we gave the definition of the discrete powerative derivative and formulation of the various problems for three types of discrete derivative [11, 12].

2. Problem statement and main results

Consider the first-order discrete differential equation with various discrete derivatives

$$y_n^{[1]} + a y_n^{(1)} - a^2 y_n^2 = 0, \quad n \geq 0 \ , \tag{1}$$

where a is a given real constant, y_n is a sought real valued sequence.

Definition 2.1. *Discrete additive derivative of the real valued sequence y_n is defined as*

$$y_n^{(1)} = y_{n+1} - y_n \tag{2}$$

Definition 2.2. *Discrete multiplicative derivative of the real valued sequence y_n is defined by the relation*

$$y_n^{[1]} = \frac{y_{n+1}}{y_n} \ . \tag{3}$$

Note that the definition of the discrete additive integral and the discrete multiplicative integral also belong to us. We assume that y_n is not a constant sequence, i.e.

$$y_n \neq c. \tag{4}$$

Taking into account the definitions of the discrete additive derivative (2) and discrete multiplicative derivative (3), first-order equation (1) can be represented in the following form of a nonlinear difference equation

$$\frac{y_{n+1}}{y_n} + a\left(y_{n+1} - y_n\right) - a^2 y_n^2 = 0, \quad n \geq 0 \ .$$

Multiplying the last relation by y_n we get

$$y_{n+1} + a y_n \left(y_{n+1} - y_n\right) - a^2 y_n^3 = 0, \ n \ \geq 0 \ .$$

After some simple transformations we obtain

$$y_{n+1} + a y_n y_{n+1} - a y_n^2 - a^2 y_n^3 = 0, \quad n \geq 0 \ \ ,$$

$$\left(1 + a y_n\right) y_{n+1} - a y_n^2 \left(1 + a y_n\right) = 0, \ n \geq 0 \ , \tag{5}$$

$$\left(1 + a y_n\right)\left(y_{n+1} - a y_n^2\right) = 0, \ n \geq 0 \ .$$

Considering that y_n is not a constant sequence, we can assume that

$$1 + a y_n \neq 0, \quad n \geq 0 \tag{6}$$

Then from (5) we obtain the following equation

$$y_{n+1} - ay_n^2 = 0, \quad n \geq 0$$

or

$$y_{n+1} = ay_n^2, \quad n \geq 0. \tag{7}$$

Now we solve difference equation (7). From (7) at $n = 0$ we have

$$y_1 = ay_0^2, \tag{8}$$

at $n = 1$,

$$y_2 = ay_1^2.$$

Considering (8) we find

$$y_2 = a(ay_0^2)^2 = \ a^3 y_0^4. \tag{9}$$

At $n = 2$ from (7) we get

$$y_3 = ay_2^3.$$

Considering (9) we obtain

$$y_3 = a(a^3 y_0^4)^2 = a^7 y_0^8. \tag{10}$$

Continuing this process we finally get

$$y_n = a^{2^n - 1} y_0^{2^n}, \quad n \geq 0. \tag{11}$$

As one can see, relations (8)-(10) are obtained from (11) at $n = 1,\ n = 2$ and $n = 3$.
Now let us prove formula (11) by the method of mathematical induction.
Suppose that (11) is valid up to the index k inclusively, i.e.

$$y_n = a^{2^n - 1} \cdot y_0^{2^n},\ n \geq k. \tag{12}$$

We should obtain (11) at $n = k + 1$.
At $n = k$ from (7) we get

$$y_{k+1} = ay_k^2 \tag{13}$$

At $n = k$ from (12) we get

$$y_k = a^{2^k - 1} y_0^{2^k},$$

Substituting this into (13) we find

$$y_{k+1} = a(a^{2^k - 1} y_0^{2^k})^2 = a^{1 + 2 \cdot 2^k - 2} y_0^{2 \cdot 2^k} = a^{2^{k+1} - 1} y_0^{2^{k+1}},$$

which coincides with (12) at $n = k + 1$.
Relation (11) is proved.
Thus, we obtained the following statement.

Theorem 2.3. *If a is a given real number, then under conditions* (16) *there exists a general solution to equation* (1) *that can be represented in the form* (11), *where y_0 is an arbitrary constant number.*

Indeed, taking into account (11), using relations (2) and (3), we calculate the discrete additive and discrete multiplicative derivative y_n i.e.

$$y_n^{(1)} = y_{n+1} - y_n = a^{2^{n+1}-1} y_0^{2^{n+1}} - a^{2^n-1} y_0^{2^n} = a^{2^{n+1}-1} y_0^{2^n} \left(a^{2^n} y_0^{2^n} - 1\right), \tag{14}$$

and

$$y_n^{[1]} = \frac{y_{n+1}}{y_n} = \frac{a^{2^{n+1}-1} y_0^{2^{n+1}}}{a^{2^n-1} y_0^{2^n}} = a^{2^{n+1}-1-2^n+1} y_0^{2^{n+1}-2^n} = a^{2^n} y_0^{2^n}, \tag{15}$$

Substituting (14) and (15) into the first order equation of (1), we get

$$y_n^{[1]} + a y_n^{(1)} - a^2 y_n^2 = a^{2^n} y_0^{2^n} + a \cdot a^{2^n-1} y_0^{2^n} \left(a^{2^n} y_0^{2^n} - 1\right) - a^2 \cdot \left(a^{2^n-1} y_0^{2^n}\right)^2 =$$

$$= a^{2^n} y_0^{2^n} + a^{2^n} y_0^{2^n} \left(a^{2^n} y_0^{2^n} - 1\right) - \left(a^{2^n} y_0^{2^n}\right)^2 - a^{2^n} y_0^{2^n} +$$
$$+ a^{2^{n+1}} y_0^{2^{n+1}} - a^{2^n} y_0^{2^n} - a^{2^{n+1}} y_0^{2^{n+1}} = 0,$$

i.e. the first order equation (1) is satisfied.

Cauchy problem. Consider first-order equation (1) with the following initial condition

$$y_0 = x. \tag{16}$$

Then it is clear from (11) that the following statement is true.

Theorem 2.4. *Under the condition of Theorem 1, if x is a given real number, then there is a unique solution to Cauchy problem* (1), (16) *having the form*

$$y_n = a^{2^n-1} x^{2^n}. \tag{17}$$

It was shown that (11) satisfies equation (1), since (17) was obtained from (11) for $y_0 = x$. Then (17) also satisfies (1). If we substitute $n = 0$ in (17), then we have $y_0 = x$ which indeed is initial condition (16).

Boundary value problem: Suppose that equation (1) is satisfied for $n = \overline{0, N-1}$, i.e.

$$y_n^{[1]} + a y_n^{(1)} - a^2 y_n^2 = 0 , \quad n = \overline{0, N-1}, \tag{18}$$

Then we add the following boundary condition to (18)

$$y_0^{\propto} y_N^{\beta} = \gamma, \tag{19}$$

where $\propto, \beta$ and γ are given real numbers. Then, taking into account that the general solution of equation (18) is given in the form (11), considering the specified boundary condition (19), we have

$$y_0^{\propto}\left(a^{2^N-1} y_0^{2^N}\right)^{\beta} = \gamma,$$

or

$$a^{\beta(2^N-1)} y_0^{\propto+\beta\cdot 2^N} = \gamma,$$

or the same

$$y_0^{\propto+\beta\cdot 2^N} = \gamma \cdot a^{-\beta(2^N-1)}. \tag{20}$$

Thus y_0 is not uniquely determined from algebraic equation (20).

Indeed, since the two-term equation

$$X^m = A, \tag{21}$$

where $A > 0$ and $m \in N$ (N is the set of natural numbers) has m number of solutions of the form

$$X_k = \sqrt[m]{A} E^{i\frac{2k\pi}{m}}, \quad k = \overline{0, m-1}, \tag{22}$$

which for $k = m$ turns to

$$X_m = X_0,$$

and for $k = m+1$ turns to

$$X_{m+1} = X_1$$

and at $k \in Z$ the obtained solution is repeated. Thus, equation (21) has m different solutions, i.e. equation (20) has an infinite number of solutions in the form

$$y_{0k} = \sqrt[x+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}} E^{i\frac{2K\pi}{x+\beta\cdot 2^N}}, \; k \in Z. \tag{23}$$

Therefore, using (11) and (23) non-unique solution to boundary value problem (18), (19), is represented in the form

$$y_{nk} = a^{2^{n-1}} \left(\sqrt[\alpha+\beta\cdot 2^N]{\gamma + a^{-\beta(2^N-1)}} E^{\frac{2k\pi}{i\alpha+\beta\cdot 2^N}} \right)^{2n}, \; 0 \le n < N; \; k \in Z, \tag{24}$$

where $i = \sqrt{-1}$.

Indeed, taking into account the definition of the discrete additive derivative (2) we obtain from (24)

$$y_{nk}^{(1)} = y_{n+1k} - y_{nk} = a^{2^{n+1}-1}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^{n+1}} - \\ -a^{2^n-1}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^n}. \tag{25}$$

In the same way, taking into account the definitions of the discrete multiplicative derivative (3) from (24), we obtain

$$y_{nk}^{[1]} = \frac{y_{n+1k}}{y_{nk}} = \frac{a^{2^{n+1}-1}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^{n+1}}}{a^{2^n-1}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^n}} = \\ = a^{2^{n+1}-1-2^n+1}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^{n+1}-2^n} = \\ = a^{2^n}\cdot\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^n}. \tag{26}$$

Substituting (24) - (26) into equation (18), we have

$$a_{nk}^{[1]} + ay_{nk}^{(1)} - a^2y_{nk}^2 = \\ a^{2^n}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^n} + \\ +a\left[a^{2^{n+1}-1}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^{n+1}} - a^{2^n-1}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^n}\right] - \\ -a^2\cdot a^{2^{n+1}-2}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^{n+1}} = a^{2^n}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^n} + \\ +a^{2^{n+1}}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^{n+1}} - a^{2^n}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^n} - \\ -a^{2^{n+1}}\left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}}E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}}\right)^{2^{n+1}} = 0\ .$$

Thus (24) satisfies equation (18). To show how expression (24) satisfies boundary condition (19) we can write

$$y_{0k}^{\partial} y_{Nk}^{\beta} = \left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}} E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}} \right)^{\partial} \cdot \left(a^{2^N-1} \right)^{\beta} \cdot \left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}} E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}} \right)^{2^N\beta} =$$

$$= a^{\beta\left(2^N-1\right)} \cdot \left(\sqrt[\propto+\beta\cdot 2^N]{\gamma a^{-\beta(2^N-1)}} E^{i\frac{2k\pi}{\propto+\beta\cdot 2^N}} \right)^{\partial+\beta 2^N} =$$

$$= \ a^{\beta\left(2^N-1\right)} \cdot \gamma \cdot a^{-\beta\left(2^N-1\right)} \cdot E^{i2k\pi} = \gamma \cdot E^{2k\pi i} = \gamma.$$

Theorem 2.5. *Under the conditions of Theorem 1, if ∂, β and γ are given positive numbers, then there is not a unique solution to boundary value problem* (18), (19) *of form* (23).

Remark 2.6. *If, instead of boundary condition* (9), *to impose the following one*

$$y_n + xy_0 = \beta \ , \tag{27}$$

then obtained boundary value problem (1), (27) *also does not have a unique solution, because y_0 is not determined uniquely.*

If to assume that we seek only real valued solution for boundary value problem (18), (19) then it is easy to see that this solution is unique and may be presented in the form

$$y_n = a^{2^n-1} \left(\gamma a^{-\beta(2^N-1)} \right) \frac{2^n}{\gamma + \beta \cdot 2^N} \ , \ \overline{n = 0 \ , \ N-1}. \tag{28}$$

Theorem 2.7. *Under the conditions of Theorem 1, if ∂, β and γ are given positive numbers, then there is a unique real-valued solution to boundary value problem* (18), (19) *represented in the form* (28).

3. Conclusions

The paper considers the Cauchy problem and the boundary value problem for the nonlinear differential equation of the first order with discrete additive and discrete multiplicative derivatives. For the considered problems the explicit analytical expression for the solution is obtained. Note that the solution to the Cauchy problem for this nonlinear equation is unique, and the solution to the boundary value problem for the first-order nonlinear equation is not unique.

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Received 10 February 2021
Accepted 7 June 2021