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$p(x)$ -Admissible Sublinear Singular Operators in the Local "Complementary" Generalized Variable Exponent Morrey Spaces

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Abstract. In this paper we consider local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ with variable exponent $p(x)$ and a general function $\omega(r)$ defining a Morrey-type norm. We prove the boundedness of the $p(x)$ -admissible sublinear singular operators local "complementary" generalized Morrey spaces in case of unbounded sets $\Omega \subset \mathbb{R}^n$.

Key Words and Phrases: Maximal operator, singular integral operators, commutators, local "complementary" generalized Morrey space, BMO space.

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1. Introduction

The variable exponent analysis is a popular topic which continues to attract many researchers, both in view of possible applications and also because of difficulties in investigation and existing challenging problems. There is an evident increase of investigations, last two decades related to both the theory of variable exponent function spaces and operator theory in these spaces. The study of variable exponent function spaces has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [12], [41], [47]). Various results on non-weighted and weighted boundedness in Lebesgue spaces with variable exponents $p(x)$ have been proved for maximal, singular and fractional type operators, we refer to surveying papers [14] and [43].

In 1938 C. Morrey [36] studied Morrey spaces for the first time in connection to its applications in partial differential equations. Until recently, a rapid growth has been seen in the study of Morrey type spaces because of its applications in major fields of engineering and sciences. Function spaces with non-standard growth has seen a major focus in recent times because of its wide range of applications in the area of image processing, the study of thermorheological fluids and modeling of electrorheological fluids. It would be next to impossible to give a complete account of the literature which is available to this subject. Let us quote at least the efforts of D.R. Adams, V. Burenkov, F. Chieranza, G. Di Fazio, M. Frasca, A. Gogatishvili, V.S. Guliyev, J.J. Hasanov, A. Meskhi, T. Mizuhara, Y. Mizuta, E. Nakai, T. Shimomura, J. Peetre, M.A. Ragusa, S.G. Samko that resulted in a long series of papers (see [2, 5, 6, 7, 9, 17, 15, 21, 30, 32, 34, 35, 37, 39, 40, 41]).

Variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$, were introduced and studied in [3] in the Euclidean setting. In [3] the boundedness of the maximal operator was proved in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\mathbb{R}^n)$ under the log-condition on $p(\cdot)$, and for potential operators a Sobolev type $\mathcal{L}^{p(\cdot),\lambda(\cdot)} \to \mathcal{L}^{q(\cdot),\lambda(\cdot)}$ theorem was proved under the same log-condition in the case of bounded sets. Hästö in [27]

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used his new local-to-global approach to extend the result of [3] on the maximal operator to the case of the whole space $\mathbb{R}^n$.

The generalized variable exponent Morrey spaces were introduced and studied in [23] in the case of bounded sets. In [23] (in the case of unbounded sets [26]) the boundedness of the maximal operator, potential operators and singular integral operators in variable exponent Morrey spaces under the certain conditions were proved.

In [30] the boundedness of the maximal operator and the singular integral operator in variable exponent Morrey spaces $\mathcal{L}^{p(\cdot),\lambda(\cdot)}$ in the general setting of metric measure spaces was proved. In the case of constant p and λ , the results on the boundedness of potential operators and classical Calderón-Zygmund singular operators go back to [1] and [39] respectively, while the boundedness of the maximal operator in the Euclidean setting was proved in [9]; for further results in the case of constant p and λ see for instance [6]– [8].

In [23] the boundedness of the classical integral operators in the generalized variable exponent Morrey spaces $\mathcal{M}^{p(\cdot),\varphi}(\Omega)$ over an open bounded set $\Omega \subset \mathbb{R}^n$ was studied. Generalized Morrey spaces in the case of constant p is defined with norm

$$\|f\|_{\mathcal{M}^{p,\varphi}} := \sup_{x,\, r>0} \frac{r^{-\frac{n}{p}}}{\varphi(r)} \|f\|_{L^p(B(x,r))}, \tag{1.1}$$

where $B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$. Under some assumptions on φ , generalized Morrey spaces were studied in [16], [32], [34], [37], [38]. Results of [23] were extended in [24] to the case of the generalized Morrey spaces $\mathcal{M}^{p(\cdot),\theta(\cdot),\omega(\cdot)}(\Omega)$ (where the L^∞ -norm in r in the definition of the Morrey space is replaced by the Lebesgue L^θ -norm), we refer to [6] for such spaces in the case of constant exponents.

In [19] (see, also [20]) local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)$ with constant p , the space of all functions $f \in L^p(\mathbb{R}^n \backslash B(x_0,r))$, $r > 0$ with finite norm

$$\|f\|_{{}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} \frac{r^{\frac{n}{p'}}}{\omega(r)} \|f\|_{L^p(\mathbb{R}^n \backslash B(x_0,r))}$$

were introduced and studied.

We denote by [19] the local "complementary" Morrey spaces ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$ with constant p , the space of all functions $f \in L^p(\mathbb{R}^n \backslash B(x_0,r))$, $r > 0$ with finite norm

$$\|f\|_{{}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)} = \sup_{r>0} r^{\frac{\lambda}{p'}} \|f\|_{L^p(\mathbb{R}^n \backslash B(x_0,r))}, \quad x_0 \in \mathbb{R}^n,$$

where $p' = \frac{p}{p-1}$, $1 \le p < \infty$ and $0 \le \lambda < n$. Note that ${}^{\complement}\mathcal{L}^{p,0}_{\{x_0\}}(\mathbb{R}^n) = L^p(\mathbb{R}^n)$.

In this paper we consider local "complementary" generalized Morrey spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ with variable exponent $p(\cdot)$, see Definition 3. In the case of constant exponent p such spaces, defined by the condition

$$\sup_{r>0} \frac{r^{\frac{n}{p'}}}{\omega(r)} \|f\|_{L^p(\Omega \backslash B(x_0,r))} < \infty, \quad x_0 \in \Omega \tag{1.2}$$

were introduced and studied in [19], [20] (see also [22]).

According to the definition of ${}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)$, we recover the space ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega)$ under the choice $\omega(r) = r^{\frac{n-\lambda}{p'}}$:

$${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\Omega) = {}^{\complement}\mathcal{M}^{p,\omega}_{\{x_0\}}(\Omega)\Big|_{\omega(r)=r^{\frac{n-\lambda}{p'}}}.$$

In contrast to the Morrey space, where one measures the regularity of a function f near the point x_0 (in the case of local Morrey spaces) and near all points $x \in \Omega$ (in the case of global Morrey spaces), the norm (1.2) is aimed to measure a "bad" behaviour of f near the point x_0 in terms of the possible grow of $\|f\|_{L^p(\Omega\backslash B(x_0,r))}$ as $r \to 0$. Correspondingly, one admits $\varphi(0) = 0$ in (1.1) and $\omega(0) = \infty$ in (1.2).

In the spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ over unbounded sets $\Omega \subset \mathbb{R}^n$ we consider the following operators:
1) Hardy-Littlewood maximal operator

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y)|dy,$$

where $\widetilde{B}(x,r) = B(x,r) \cap \Omega$.
2) Calderón-Zygmund singular operators

$$Af(x) = \int_{\Omega} K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard" singular kernel, that is, a function continuous on $\{(x,y) \in \Omega \times \Omega : x \neq y\}$ and satisfying the estimates

$$|K(x,y)| \le C|x-y|^{-n} \text{ for all } x \neq y,$$

$$|K(x,y) - K(x,z)| \le C\frac{|y-z|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if } |x-y| > 2|y-z|,$$

$$|K(x,y) - K(\xi,y)| \le C\frac{|x-\xi|^{\sigma}}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \quad \text{if } |x-y| > 2|x-\xi|.$$

Let

$$A^*f(x) = \sup_{\varepsilon>0} |A_{\varepsilon}f(x)|$$

be the maximal singular operator, where $A_{\varepsilon}f(x)$ is the usual truncation

$$A_{\varepsilon}f(x) = \int_{|x-y|\ge\varepsilon} K(x,y)f(y)dy.$$

We find the condition on the function $\omega(r)$ for the boundedness of the $p(x)$ -admissible singular operator T in the local "complementary" generalized Morrey space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ with variable $p(x)$ under the log-condition on $p(\cdot)$.

Let T be a sublinear operator, that is, $|T(f+g)| \le |Tf| + |Tg|$. ($p(x)$ -admissible singular operators). Let sublinear operator T will be called $p(x)$ -admissible singular operators, if:

1) T satisfies the size condition of the form

$$\chi_{B(x,r)}(z) \left|T\Big(f\chi_{\mathbb{R}^n\backslash B(x,2r)}\Big)(z)\right| \le C\chi_{B(x,r)}(z) \int_{\mathbb{R}^n\backslash B(x,2r)} \frac{|f(y)|}{|y-z|^n}dy \tag{1.3}$$

for $x \in \mathbb{R}^n$ and $r > 0$;

2) T is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$.

($p(x)$ -admissible commutator singular operators). Let sublinear operator T will be called $p(x)$ -admissible singular operators, if:

1) T satisfies the size condition of the form

$$\chi_{B(x,r)}(z)\left|[b,T]\Big(f\chi_{\mathbb{R}^n\setminus B(x,2r)}\Big)(z)\right|\le C\chi_{B(x,r)}(z)\int_{\mathbb{R}^n\setminus B(x,2r)}\frac{|b(y)-b(z)|\,|f(y)|}{|y-z|^n}dy \qquad (1.4)$$

for $x\in\mathbb{R}^n$ and $r>0$;

2) $[b,T]$ is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$.

We use the following notation: $\mathbb{R}^n$ is the n -dimensional Euclidean space, $\Omega\subset\mathbb{R}^n$ is an open set, $\chi_E(x)$ is the characteristic function of a set $E\subseteq\mathbb{R}^n$, by c , C, c_1, c_2 etc, we denote various absolute positive constants, which may have different values even in the same line.

2. Preliminaries on variable exponent Lebesgue and Morrey spaces

In this section we refer to the book [12] for variable exponent Lebesgue spaces and give some basic definitions and facts. Let $p(\cdot)$ be a measurable function on Ω with values in $[1,\infty)$. An open set Ω is assumed to be unbounded throughout the whole paper. We mainly suppose that

$$1<p_-\le p(x)\le p_+<\infty, \qquad (2.1)$$

where $p_-:=\operatorname*{ess\,inf}_{x\in\Omega}p(x),\quad p_+:=\operatorname*{ess\,sup}_{x\in\Omega}p(x).$ By $L^{p(\cdot)}(\Omega)$ we denote the space of all measurable functions $f(x)$ on Ω such that

$$I_{p(\cdot)}(f)=\int_\Omega|f(x)|^{p(x)}dx<\infty.$$

Equipped with the norm

$$\|f\|_{p(\cdot)}=\inf\left\{\eta>0:\ I_{p(\cdot)}\left(\frac{f}{\eta}\right)\le1\right\},$$

this is a Banach function space. By $p'(\cdot)=\frac{p(x)}{p(x)-1}$, $x\in\Omega,$ we denote the conjugate exponent.

For the basics on variable exponent Lebesgue spaces we refer to [31], [45].

$\mathcal{P}(\Omega)$ is the set of bounded measurable functions $p:\Omega\to[1,\infty)$;

$\mathcal{P}^{log}(\Omega)$ is the set of exponents $p\in\mathcal{P}(\Omega)$ satisfying the local log-condition

$$|p(x)-p(y)|\le\frac{A}{-\ln|x-y|},\quad |x-y|\le\frac12\ \ x,y\in\Omega, \qquad (2.2)$$

where $A=A(p)>0$ does not depend on x,y ;

$\mathcal{A}^{log}(\Omega)$ is the set of bounded exponents $p:\Omega\to\mathbb{R}$ satisfying the condition (2.2);

$\mathbb{P}^{log}(\Omega)$ is the set of exponents $p\in\mathcal{P}^{log}(\Omega)$ with $1<p_-\le p(x)\le p_+<\infty$;

for Ω which may be unbounded, by $\mathcal{P}_\infty(\Omega)$, $\mathcal{P}^{log}_\infty(\Omega)$, $\mathbb{P}^{log}_\infty(\Omega)$, $\mathcal{A}^{log}_\infty(\Omega)$ we denote the subsets of the above sets of exponents satisfying the decay condition (when Ω is unbounded)

$$|p(x)-p(\infty)|\le\frac{A_\infty}{\ln(2+|x|)},\quad x\in\mathbb{R}^n. \qquad (2.3)$$

where $p(\infty)=\lim_{x\to\infty}p(x)>1$.

The following theorem gives the boundedness of the maximal operator M in the variable exponent Lebesgue spaces $L^{p(\cdot)}(\Omega)$. ([10], [11]) Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set and $p \in \mathbb{P}^{log}_\infty(\Omega)$. Then the maximal operator M is bounded in $L^{p(\cdot)}(\Omega)$.

Singular operators within the framework of the spaces with variable exponents were studied in [13]. From Theorem 4.8 and Remark 4.6 of [13] and the known results on the boundedness of the maximal operator, we have the following statement, which is formulated below for our goals for a bounded Ω , but valid for an arbitrary open set Ω under the corresponding condition in $p(x)$ at infinity.

([13]) Let $\Omega \subset \mathbb{R}^n$ be an unbounded open set and $p \in \mathbb{P}^{log}_\infty(\Omega)$. Then the singular integral operator T is bounded in $L^{p(\cdot)}(\Omega)$.

We will also make use of the estimate provided by the following lemma (see [12], Corollary 4.5.9).

$$\|\chi_{\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \le C r^{\theta_p(x,r)}, \quad x \in \Omega,\ p \in \mathbb{P}^{log}_\infty(\Omega), \tag{2.4}$$

where $\theta_p(x,r) = \begin{cases} \frac{n}{p(x)}, & r \le 1, \\ \frac{n}{p(\infty)}, & r \ge 1 \end{cases}$.

The following lemma. [4] Let Ω be an unbounded open set, let $p \in \mathbb{P}^{log}_\infty(\Omega)$ satisfy the assumption $1 \le p_- \le p(x) \le p_+ < \infty$ and the function $\nu(x)$ satisfy the assumptions of Lemma **??** and additionally $\sup_{x\in\Omega}[n + \nu(x)p(\infty)] < 0$. Then

$$\||x-\cdot|^{\nu(x)}\chi_{\Omega\backslash\widetilde{B}(x,r)}(\cdot)\|_{p(\cdot)} \le C r^{\nu(x)+\theta_p(x,r)}, \quad x \in \Omega,\ r > 0, \tag{2.5}$$

where C does not depend on x and r .

Let $\lambda(x)$ be a measurable function on Ω with values in $[0,n]$. The variable Morrey space $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$ is defined as the set of integrable functions f on Ω with the finite norm

$$\|f\|_{\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)} = \sup_{x\in\Omega,\ t>0} t^{-\frac{\lambda(x)}{p(x)}} \|f\chi_{\widetilde{B}(x,t)}\|_{L^{p(\cdot)}(\Omega)}.$$

The following statements are known.

([3]) Let Ω be bounded, $p \in \mathcal{P}^{log}(\Omega)$ and let a measurable function λ satisfy the conditions

$$0 \le \lambda(x), \quad \sup_{x\in\Omega}\lambda(x) < n. \tag{2.6}$$

Then the maximal operator M is bounded in $\mathcal{L}^{p(\cdot),\lambda(\cdot)}(\Omega)$.

Theorem 2 was extended to unbounded domains in [26] and [27]. Note that the boundedness of the maximal operator in Morrey spaces with variable $p(x)$ was studied in [30] in more general setting of quasimetric measure spaces.

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{\widetilde{B}(x,r)} |f(y) - f_{\widetilde{B}(x,r)}| dy,$$

where $f_{\widetilde{B}(x,t)} = |\widetilde{B}(x,t)|^{-1} \int_{\widetilde{B}(x,t)} f(z)dz$.

We define BMO space, as the set of locally integrable functions f with finite norm

$$\|f\|_* = \sup_{x\in\Omega} M^\sharp f(x) = \sup_{t>0,\ x\in\Omega} |B(x,t)|^{-1} \int_{\widetilde{B}(x,t)} |f(y) - f_{\widetilde{B}(x,t)}| dy.$$

We define the $BMO_{p(\cdot)}(\Omega)$ space as the set of all locally integrable functions f with finite norm

$$\|f\|_{BMO_{p(\cdot)}} = \sup_{x\in\Omega,\, r>0} \frac{\|(f(\cdot)-f_{\widetilde{B}(x,r)})\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}{\|\chi_{\widetilde{B}(x,r)}\|_{L^{p(\cdot)}(\Omega)}}.$$

([28]) Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$, then the norms $\|\cdot\|_{BMO_{p(\cdot)}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.

3. Local "complementary" generalized variable exponent Morrey spaces

Everywhere in the sequel the functions $\omega(r),\ \omega_1(r)$ and $\omega_2(r)$ used in the body of the paper, are non-negative measurable functions on $(0,\infty)$.

The local generalized Morrey spaces $\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ and global generalized Morrey spaces $\mathcal{M}^{p(\cdot),\omega}(\Omega)$ with variable exponent are defined (see [23]) by the norms

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}} = \sup_{r>0} \frac{r^{-\theta_p(x_0,r)}}{\omega(r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x_0,r))}$$

and

$$\|f\|_{\mathcal{M}^{p(\cdot),\omega}} = \sup_{x\in\Omega, r>0} \frac{r^{-\theta_p(x,r)}}{\omega(r)} \|f\|_{L^{p(\cdot)}(\widetilde{B}(x,r))},$$

where $x_0 \in \Omega$ and $1 \le p_- \le p(x) \le p_+ < \infty$ for all $x \in \Omega.$

We find it convenient to introduce the variable exponent version of the local "complementary" space as follows (compare with the condition (1.2)). Let $x_0 \in \Omega$, $1 \le p_- \le p(x) \le p_+ < \infty$. The local "complementary" variable exponent Morrey space ${}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}_{\{x_0\}}(\Omega)$ is defined by the norm

$$\|f\|_{{}^{\complement}\mathcal{L}^{p(\cdot),\lambda(\cdot)}_{\{x_0\}}(\Omega)} = \sup_{t>0} t^{\frac{\lambda(x)}{p'(x)}} \|f\chi_{\Omega\setminus\widetilde{B}(x_0,t)}\|_{L^{p(\cdot)}(\Omega)}.$$

Let $x_0 \in \Omega$, $1 \le p_- \le p(x) \le p_+ < \infty$. The local "complementary" generalized variable exponent Morrey spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ and global "complementary" generalized variable exponent Morrey spaces $\mathcal{M}^*_{p(\cdot),\omega}(\Omega)$ are defined by the norms

$$\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}} = \sup_{r>0} \frac{r^{\theta_{p'}(x_0,r)}}{\omega(r)} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,r))}$$

and

$$\|f\|_{\mathcal{M}^*_{p(\cdot),\omega}} = \sup_{x\in\Omega, r>0} \frac{r^{\theta_{p'}(x,r)}}{\omega(r)} \|f\|_{L_{p(\cdot)}(\Omega\setminus\widetilde{B}(x,r))},$$

respectively.

Everywhere in the sequel we assume that

$$\sup_{r>0} \frac{r^{\theta_{p'}(x_0,r)}}{\omega(r)} < \infty, \tag{3.1}$$

which makes the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ non-trivial, since it contains $L^{p(\cdot)}(\Omega)$ in this case.

Also if $\inf\limits_{r>0} \frac{r^{\theta_{p'}(x_0,r)}}{\omega(r)} > 0$, then ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega) = L^{p(\cdot)}(\Omega)$. Therefore, to guarantee that the "complementary" space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ is strictly larger than $L^p(\Omega)$, one should be interested in the cases where

$$\lim_{r\to 0} \frac{r^{\theta_{p'}(x_0,r)}}{\omega(r)} = 0. \tag{3.2}$$

Clearly, the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ may contain functions with a non-integrable singularity at the point x_0 , if no additional assumptions are introduced. To study the operators in ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$, we need its embedding into $L^1(\Omega)$. Dini condition on ω is sufficient for such an embedding.

Now we give the following lemma which we need while proving our main results.

[4] Let $p \in \mathbb{P}^{log}_{\infty}(\Omega)$ and $f \in L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))$ for every $s \in (0,\infty)$, $\gamma \in \mathbb{R}$. Then

$$\int_{\widetilde{B}(x_0,t)} |y-x_0|^{\gamma}|f(y)|dy \le C\int_0^t s^{\gamma+\theta_{p'}(x_0,r)-1}\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds \tag{3.3}$$

for every $t \in (0,\infty)$, where C does not depend on f, t and x_0 .

Note that the statements on the boundedness of the maximal and singular operators in the "complementary" Morrey spaces known for the case of the constant exponent p , obtained in [19], read as follows. Note that the theorems below do not assume any monotonicity type conditions on the functions ω, ω_1 and ω_2 .

([19], Theorem 1.4.6) Let $1 < p < \infty$, $x_0 \in \mathbb{R}^n$ and $\omega_1(r)$ and $\omega_2(r)$ be positive measurable functions satisfying the condition

$$\int_0^r \omega_1(t)\frac{dt}{t} \le c\,\omega_2(r)$$

with $c > 0$ not depending on $r > 0$. Then the operators M and T are bounded from ${}^{\complement}\mathcal{M}^{p,\omega_1}_{\{x_0\}}(\mathbb{R}^n)$ to ${}^{\complement}\mathcal{M}^{p,\omega_2}_{\{x_0\}}(\mathbb{R}^n)$. ([19]) Let $1 < p < \infty$, $x_0 \in \mathbb{R}^n$ and $0 \le \lambda < n$. Then the operators M and T are bounded in the space ${}^{\complement}\mathcal{L}^{p,\lambda}_{\{x_0\}}(\mathbb{R}^n)$.

The introduction of global "complementary" Morrey-type spaces has no big sense, neither in case of constant exponents, nor in the case of variable exponents. In the case of constant exponents this was noted in [7], pp 19-20; in this case the global space defined by the norm

$$\sup_{x\in\Omega, r>0} \frac{r^{\frac{n}{p}}}{\omega(r)}\|f\|_{L^p(\Omega\backslash B(x,r))}$$

reduces to $L^p(\Omega)$ under the assumption (3.1). In the case of variable exponents there happens the same. In general, to make it clear, note that for instance under the assumption (3.2) if we admit that $\sup\limits_{r>0} \frac{r^{\frac{n}{p(\cdot)}}}{\omega(r)}\|f\|_{L^p(\Omega\backslash B(x,r))}$ for two different points $x = x_0$ and $x = x_1$, $x_0 \ne x_1$, this would immediately imply that $f \in L^{p(\cdot)}$ in a neighbourhood of both the points x_0 and x_1 .

4. $p(x)$ -admissible singular operators in the spaces ${}^{\complement}\mathcal{M}^{p(\cdot),\omega}_{\{x_0\}}(\Omega)$ and $\mathcal{M}^*_{p(\cdot),\omega}(\Omega)$

Let $L^{\infty}(\mathbb{R}_+, v)$ be the weighted L^{∞} -space with the norm

$$\|g\|_{L^{\infty}(\mathbb{R}_+,v)} = \operatorname*{ess\,sup}_{t>0} v(t)g(t).$$

In the sequel $\mathfrak{M}(\mathbb{R}_+)$, $\mathfrak{M}^+(\mathbb{R}_+)$ and $\mathfrak{M}^+(\mathbb{R}_+;\uparrow)$ stand for the set of Lebesgue-measurable functions on $\mathbb{R}_+$, and its subspaces of nonnegative and nonnegative non-decreasing functions, respectively. We also denote

$$\mathbb{A} = \left\{ \varphi \in \mathfrak{M}^+(\mathbb{R}_+;\uparrow) : \lim_{t\to 0+} \varphi(t) = 0 \right\}.$$

Let u be a continuous and non-negative function on $\mathbb{R}_+$. We define the supremal operator $\overline{S}_u$ by

$$(\overline{S}_u g)(t) := \|u\, g\|_{L_1(0,t)}, \;\; t \in (0,\infty).$$

([5]) Suppose that v_1 and v_2 are nonnegative measurable functions such that $0 < \|v_1\|_{L_\infty(0,t)} < \infty$ for every $t > 0$. Let u be a continuous nonnegative function on $\mathbb{R}_+$. Then the operator $\overline{S}_u$ is bounded from $L_\infty(\mathbb{R}_+, v_1)$ to $L_\infty(\mathbb{R}_+, v_2)$ on the cone $\mathbb{A}$ if and only if

$$\left\| v_2 \overline{S}_u \left(\|v_1\|^{-1}_{L_\infty(0,\cdot)} \right) \right\|_{L_\infty(\mathbb{R}_+)} < \infty.$$

We will use the following statement on the boundedness of the weighted Hardy operators

$$H_w g(t) := \int_0^t g(s)w(s)ds, \;\; H^*_w g(t) := \int_t^\infty g(s)w(s)ds, \quad 0 < t < \infty,$$

where w is a weight.

The following theorem was proved in [21].

([21]) Let v_1 , v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality

$$\sup_{t>0} v_2(t) H^*_w g(t) \le C \sup_{t>0} v_1(t) g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)ds}{\operatorname*{ess\,sup}_{s<\tau<\infty} v_1(\tau)} < \infty.$$

([21]) Let v_1 , v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality

$$\sup_{t>0} v_2(t) H_w g(t) \le C \sup_{t>0} v_1(t) g(t) \tag{4.1}$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_0^t \frac{w(s)ds}{\sup_{0<\tau<s} v_1(\tau)} < \infty.$$

Moreover, the value $C = B$ is the best constant for (4.1).

Let Ω be an open unbounded set, $p \in \mathbb{P}^{log}_\infty(\Omega)$ and $f \in L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))$ for every $t \in (0,\infty)$. If the integral

$$\int_0^\infty r^{\theta_{p'}(x_0,r)-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))} dr$$

is convergent, then

$$\|Tf\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \le C t^{-\theta_{p'}(x_0,t)} \int_0^t r^{\theta_{p'}(x_0,r)-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))} dr, \tag{4.2}$$

where C does not depend on f , x_0 and $t \in (0,\infty)$.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{\Omega\backslash\widetilde{B}(x_0,t)}(y) \quad f_2(y) = f(y)\chi_{\widetilde{B}(x_0,t)}(y) \tag{4.3}$$

and have

$$\|Tf\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} \le \|Tf_1\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} + \|Tf_2\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))}.$$

Taking into account that $f_1 \in L^{p(\cdot)}(\Omega)$, by Definition 1 we have

$$\|Tf_1\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} \le \|Tf_1\|_{L^{p(\cdot)}(\Omega)} \le C\|f_1\|_{L^{p(\cdot)}(\Omega)} = C\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))}.$$

By the monotonicity of the norm $\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))}$ with respect to t we have

$$\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \le Ct^{-\theta_{p'}(x_0,t)} \int_0^t s^{\theta_{p'}(x_0,s)-1}\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds \tag{4.4}$$

and then

$$\|Tf_1\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,t))} \le Ct^{-\theta_{p'}(x_0,t)} \int_0^t r^{\theta_{p'}(x_0,r)-1}\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,r))}dr. \tag{4.5}$$

To estimate $\|Tf_2\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))}$, note that $\frac{1}{2}|x_0-z| \le |z-y| \le \frac{3}{2}|x_0-z|$ for $z \in \Omega\backslash\widetilde{B}(x_0,2t)$ and $y \in \widetilde{B}(x_0,t)$, so that

$$\begin{aligned}\|Tf_2\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} &\le C\left\|\int_{\widetilde{B}(x_0,t)} |z-y|^{-n}f(y)dy\right\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} \\ &\le C\int_{\widetilde{B}(x_0,t)} |f(y)|dy\||x_0-z|^{-n}\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))}.\end{aligned}$$

Therefore, with the aid of the estimate (2) and inequality (3.3), we get

$$\|Tf_2\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,2t))} \le Ct^{-\theta_{p'}(x_0,t)} \int_0^t s^{\theta_{p'}(x_0,s)-1}\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds,$$

which together with (4.5) yields (4.2). □

Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $x_0 \in \Omega$, $p \in \mathbb{P}^{log}_\infty(\Omega)$ and the functions $\omega_1(t)$, $\omega_2(t)$ satisfy the condition

$$\int_0^t \frac{\inf\limits_{0<r<s} \omega_1(r)r^{-\theta_{p'}(x_0,r)}}{s^{1-\theta_{p'}(x_0,s)}}ds \le C\omega_2(t), \tag{4.6}$$

where C does not depend on t . Then the $p(x)$ -admissible singular operators T is bounded from the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)$ to the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)$.

Proof. For $f \in {}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)$ we have

$$\|Tf\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)} = \sup_{t>0} \frac{t^{\theta_{p'}(x_0,t)}}{\omega_2(t)}\|Tf\chi_{\Omega\backslash\widetilde{B}(x_0,t)}\|_{L^{p(\cdot)}(\Omega)}, \tag{4.7}$$

we estimate $\|Tf\chi_{\Omega\backslash\widetilde{B}(x_0,t)}\|_{L^{p(\cdot)}(\Omega)}$ by means of Theorem 4 and Theorem 4 we obtain

$$\|Tf\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)} \le C\sup_{t>0}\omega_2^{-1}(t)\int_0^t r^{\theta_{p'}(x,r)-1}\|f\|_{L_{p(\cdot)}(\Omega\backslash\widetilde{B}(x,r))}dr$$

$$\leq C \sup_{t>0} \frac{t^{\theta_{p'}(x_0,t)}}{\omega_1(t)} \|f\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,t))} = C\|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)}.$$

□

Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$ and $\omega_1(t)$ and $\omega_2(t)$ fulfill condition (4.6). Then the singular integral operator T and T^* are bounded from the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)$ to the space ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)$.

The commutator generated by M and a suitable function b is formally defined by

$$[b, M]f = bM(f) - M(bf).$$

Given a measurable function b the maximal commutator is defined by

$$M_b(f)(x) := \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy,$$

for all $x \in \mathbb{R}^n$.

This operator plays an important role in the study of commutators of singular integral operators with BMO symbols (see, for instance [18], [33], [44]). The maximal operator M_b has been studied intensively and there exist plenty of results about it. Pu Zhang and Jianglong Wu [46] proved the following statement.

([46]) Let $b \in L^{loc}_1(\mathbb{R}^n)$ and $p \in \mathbb{P}^{log}_{\infty}(\mathbb{R}^n)$, then M_b is bounded in $L^{p(\cdot)}(\mathbb{R}^n)$ if and only if $b \in BMO(\mathbb{R}^n)$.

Operators M_b and $[b, M]$ essentially differ from each other. For example, M_b is a positive and sublinear operator, but $[b, M]$ is neither positive nor sublinear. However, if b satisfies some additional conditions, then operator M_b controls $[b, M]$.

Let Ω be an open unbounded set, $p \in \mathbb{P}^{log}_{\infty}(\Omega)$, $b \in BMO(\Omega)$ and $f \in L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0, t))$ for every $t \in (0, \infty)$. If the integral

$$\int_0^{\infty} s^{\theta_{p'}(x_0,s)-1} \|f\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,s))} ds$$

is convergent, then

$$\|[b, T]f\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,t))} \leq Ct^{-\theta_p(x_0,t)} \|b\|_* \int_0^t s^{\theta_{p'}(x_0,s)-1} \left(1 + \ln \frac{t}{s}\right) \|f\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,s))} ds, \tag{4.8}$$

where C does not depend on f , x_0 and $t \in (0, \infty)$.

Proof. We split the function f in the form $f_1 + f_2$ as in (4.3) and have

$$\|[b, T]f\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,t))} \leq \|[b, T]f_1\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,t))} + \|[b, T]f_2\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,t))}.$$

Taking into account that $f_1 \in L^{p(\cdot)}(\Omega)$, from Definition 1 we have

$$\|[b, T]f_1\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,t))} \leq \|[b, T]f_1\|_{L^{p(\cdot)}(\Omega)}$$

$$\leq C\|b\|_*\|f_1\|_{L^{p(\cdot)}(\Omega)} = C\|b\|_*\|f\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,t))}.$$

Then in the view of (4.4)

$$\|[b, T]f_1\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,t))} \tag{4.9}$$

$$\leq Ct^{-\theta_p(x_0,t)}\, \|b\|_* \int_0^t s^{\theta_{p'}(x_0,s)-1} \left(1+\ln\frac{t}{s}\right) \|f\|_{L^{p(\cdot)}(\Omega\backslash \widetilde{B}(x_0,s))} ds.$$

To estimate $[b,T]f_2(z)$, we observe that for $z\in\Omega\backslash\widetilde{B}(x_0,t)$ we have

$$\begin{aligned}
&|[b,T]f_2(z)| \leq \int_\Omega |b(y)-b(z)||z-y|^{-n}|f_2(y)|dy \\
&\leq \int_{\widetilde{B}(x_0,t)} |b(y)-b(z)||x_0-z|^{-n}|f(y)|dy \\
&= |x_0-z|^{-n}\int_{\widetilde{B}(x_0,t)} |b(y)-b(z)||f(y)|dy \\
&\leq 2^n\beta|x_0-z|^{-n}\int_{\widetilde{B}(x_0,t)} |b(y)-b(z)||x_0-y|^{-\beta}|f(y)| \left(\int_0^{|x_0-y|} s^{\beta-1}ds\right) dy \\
&= 2^n\beta\,|x_0-z|^{-n}\int_0^t s^{\beta-1}\left(\int_{\{y\in\Omega: s<|x_0-y|<t\}} |b(y)-b(z)||x_0-y|^{-\beta}|f(y)|dy\right) ds \\
&\leq 2^n\beta\,|x_0-z|^{-n}\int_0^t s^{-1}\left(\int_{\{y\in\Omega: s<|x_0-y|<t\}} |b(y)-b(z)||f(y)|dy\right) ds \\
&\leq 2^n\beta\,|x_0-z|^{-n}\int_0^t s^{-1}\left(\int_{\{y\in\Omega: s<|x_0-y|<t\}} |b(y)-b_{\widetilde{B}(x_0,s)}||f(y)|dy\right) ds \\
&+ 2^n\beta\,|x_0-z|^{-n}\int_0^t s^{-1}\,|b(z)-b_{\widetilde{B}(x_0,s)}|\left(\int_{\{y\in\Omega: s<|x_0-y|<t\}} |f(y)|dy\right) ds \\
&= J_1+J_2.
\end{aligned}$$

By applying Hölder inequality we get

$$\begin{aligned}
J_1 &\leq C\int_0^t s^{-n-1}\,\|b(\cdot)-b_{\widetilde{B}(x_0,t)}\|_{L^{p'(\cdot)}(\widetilde{B}(x_0,t))}\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds \\
&+Ct^{-n}\int_0^t s^{\theta_{p'}(x_0,s)-1}\,|b_{\widetilde{B}(x_0,t)}-b_{\widetilde{B}(x_0,s)}|\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds \\
&\leq C\|b\|_* t^{-n}\int_0^t s^{\theta_{p'}(x_0,s)-1}\,\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds \\
&+C\|b\|_* t^{-n}\int_0^t s^{\theta_{p'}(x_0,s)-1}\ln\frac{t}{s}\,\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds \\
&\leq C\|b\|_* t^{-n}\int_0^t \left(1+\ln\frac{t}{s}\right) s^{\theta_{p'}(x_0,s)-1}\,\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds. \qquad (4.10)
\end{aligned}$$

To estimate J_2 , we have

$$\begin{aligned}
J_2 &= Ct^{-n}\int_0^t s^{-1}\,|b(z)-b_{\widetilde{B}(x_0,s)}|\left(\int_{\{y\in\Omega: s<|x_0-y|<t\}} |f(y)|dy\right) ds \\
&\leq C\,t^{-n}\,|B(x_0,t)|^{-1}\int_{\widetilde{B}(x_0,t)} |b(z)-b(y)|dy\int_0^t s^{\theta_{p'}(x_0,s)-1}\,\|f\|_{L^{p(\cdot)}(\Omega\backslash\widetilde{B}(x_0,s))}ds
\end{aligned}$$

$$+ Ct^{-n} \int_0^t s^{\theta_{p'}(x_0,s)-1} |b_{\widetilde{B}(x_0,t)} - b_{\widetilde{B}(x_0,s)}| \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds$$

$$\leq C\, t^{-n}\, M_b \chi_{B(x_0,t)}(z) \int_0^t s^{\theta_{p'}(x_0,s)-1} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds$$

$$+ C\|b\|_*\, t^{-n} \int_0^t s^{\theta_{p'}(x_0,s)-1} \ln\frac{t}{s}\, \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds,$$

where C does not depend on x_0 and t. Then from the Theorem 4 and (4.10) we obtain

$$\|[b,T]f_2\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} \leq \|I_1\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} + \|I_2\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))}$$

$$\leq C\, \|b\|_* t^{-n}\, \|\chi_{\Omega\setminus\widetilde{B}(x_0,t))}\|_{L^{p(\cdot)}(\Omega)} \int_0^t \left(1+\ln\frac{t}{s}\right) s^{\theta_{p'}(x_0,s)-1} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds$$

$$+C\, t^{-n}\, \|M_b\chi_{B(x_0,t)}\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} \int_0^t s^{\theta_{p'}(x_0,s)-1} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds$$

$$+C\, \|b\|_* t^{-n}\, \|\chi_{\Omega\setminus\widetilde{B}(x_0,t))}\|_{L^{p(\cdot)}(\Omega)} \int_0^t s^{\theta_{p'}(x_0,s)-1} \ln\frac{t}{s}\, \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds$$

$$\leq Ct^{-\theta_{p'}(x_0,t)}\, \|b\|_* \int_0^t s^{\theta_{p'}(x_0,s)-1} \left(1+\ln\frac{t}{s}\right) \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds$$

$$+C\|b\|_* \|\chi_{B(x_0,t)}\|_{L^{p(\cdot)}(\Omega)} \int_0^t s^{\theta_{p'}(x_0,s)-1} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds$$

$$+C\, \|b\|_* \|\chi_{\Omega\setminus\widetilde{B}(x_0,t))}\|_{L^{p(\cdot)}(\Omega)} \int_0^t s^{\theta_{p'}(x_0,s)-1} \ln\frac{t}{s}\, \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds$$

$$\leq Ct^{-\theta_p(x_0,t)}\, \|b\|_* \int_0^t s^{\theta_{p'}(x_0,s)-1} \left(1+\ln\frac{t}{s}\right) \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds \tag{4.11}$$

which together with (4.9) and (4.11) yields (4.8). □

Let $\Omega \subset \mathbb{R}^n$ be an open unbounded set, $x_0 \in \Omega$, $p \in \mathbb{P}^{log}_\infty(\Omega)$, $b \in BMO(\Omega)$ and $\omega_1(t)$ and $\omega_2(t)$ satisfy the condition

$$\int_0^t \left(1+\ln\frac{s}{t}\right) \frac{\inf\limits_{0<r<s} \omega_1(x,r) r^{-\theta_{p'}(x_0,r)}}{s^{1-\theta_{p'}(x_0,s)}} ds \leq C\omega_2(x,t), \tag{4.12}$$

where C does not depend on t. Then the operator $[b,T]$ is bounded from ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)$ to ${}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)$, where $b \in BMO(\Omega)$.

Proof. Let $f \in {}^{\complement}\mathcal{M}^{\{x_0\}}_{p(\cdot),\omega_1}(\Omega)$. We follow the procedure already used in the proof of Theorem 4: in the norm

$$\|[b,T]f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)} = \sup_{t>0} \frac{t^{\theta_{p'}(x_0,t)}}{\omega_2(t)} \|[b,T]f\chi_{\Omega\setminus\widetilde{B}(x_0,t)}\|_{L^{p(\cdot)}(\Omega)}, \tag{4.13}$$

we estimate $\|[b,T]f\chi_{\Omega\setminus\widetilde{B}(x_0,t)}\|_{L^{p(\cdot)}(\Omega)}$ by means of Theorem 4 and Theorem 4 we obtain

$$\|[b,T]f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_2}_{\{x_0\}}(\Omega)}$$

$$\leq C\|b\|_* \sup_{t>0} \frac{1}{\omega_2(t)} \int_0^t s^{\theta_{p'}(x_0,s)-1} \left(1+\ln\frac{t}{s}\right) \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,s))} ds$$

$$\leq C\|b\|_* \sup_{t>0} \frac{t^{\theta_{p'}(x_0,t)}}{\omega_1(t)} \|f\|_{L^{p(\cdot)}(\Omega\setminus\widetilde{B}(x_0,t))} = C\|b\|_* \|f\|_{{}^{\complement}\mathcal{M}^{p(\cdot),\omega_1}_{\{x_0\}}(\Omega)}.$$

□

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