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On basicity of double systems in Banach spaces

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Abstract. The bases of double systems with operator coefficients in Banach spaces are considered. A relation between the basicity of these systems and the solvability of the corresponding operator equations is established. A necessary condition for a basicity is obtained and a concrete example is presented.

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1. Introduction

Bases of the double systems are natural generalizations of the classical system of exponentials $\left\{e^{int}\right\}_{n\in Z}$ (Z is the set of all integers). These generalizations include also the following perturbation of system of exponents

$$\left\{e^{i(n+\alpha sign\, n)t}\right\}_{n\in Z}. \tag{1}$$

Paley-Wiener [11] and N.Levinson [7] were the first mathematicians (to be followed by many others) who have studied the basis properties of this perturbation in $L_p(-\pi,\pi)$ ($L_\infty = C[-\pi,\pi]$), $1 \le p \le +\infty$. The latest results on this topic are obtained in [5,8,14]. The most general case is considered in [1,2]. Note that for the study of basicity in [1,2,8] the methods of boundary value problems of the theory of analytic functions are used. Abstracts generalizations of these results are given in [3].

In this paper we present one necessary condition for the basicity of double system in Banach space. The obtained results are applied to specific cases. This allows to establish the accuracy of the estimate with respect to the measurable function $\alpha(t)$, that provides the basicity of the system of exponents

$$\left\{e^{i(nt+\alpha(t)sign\, n)}\right\}_{n\in Z}, \tag{2}$$

in $L_p(-\pi,\pi)$, $1 < p < +\infty$. It should be noted that the systems of the form (2) appear when solving some problems of mathematical physics by Fourier method. More details about these problems can be found in [9,10,12,15].

2. Needful concepts and facts

We state some ideas from the theory of bases and common facts. Assume that X is some Banach space with a basis $\{x_n\}_{n\in N} \subset X$, $\{x_n^*\}_{n\in N} \subset X^*$ is a system biorthogonal to basis, and X^* is a space conjugated to X. By $\{P_n\}_{n\in N} \subset L(X)$ we denote the family of projector S

$$P_m x = \sum_{n=1}^{m} x_n^*(x)\, x_n\,,\ \forall m \in N,$$

where $L(X)$ is an algebra of bounded operators from X to X. It is known that the family $\{P_n\}_{n\in N}$ is bounded in $L(X)$, i.e. $\exists M > 0 : \|P_n\| \leq M\,,\ \forall n \in N$.

Let $\{x_n^+; x_n^-\}_{n\in N} \subset X$ be some double system. We'll call this system a basis in X, if for $\forall x \in X, \exists!\, \{\lambda_n^\pm\}_{n\in N} \subset C : x = \sum_{n=1}^{\infty} \lambda_n^+ x_n^+ + \sum_{n=1}^{\infty} \lambda_n^- x_n^-$.

We'll denote the closure of the linear span $\{x_n^\pm\}_{n\in N}$ in X by $X^\pm$. In other words, this definition means that the system $\{x_n^\pm\}_{n\in N}$ forms a basis for $X^\pm$, the spaces X^+ and X^- are complementable in X and the direct expansion

$$X = X^+ \dot{+} X^- \tag{3}$$

holds.

3. Main results

Let X be a Banach space with the basis $\{x_n^+; x_n^-\}_{n\in N}$ and let $T^\pm \in L(X)$ be some automorphisms. Assume that the system $\{T^+x_n^+; T^-x_n^-\}_{n\in N}$ also forms a basis for X. Thus, for $\forall y \in X$ the equation

$$T^+x^+ + T^-x^- = y\ , \tag{4}$$

is solvable in $X^+ \times X^-$, i.e. $\exists\,(x^+; x^-) \in X^+ \times X^-$, which satisfies relation (4). Let $X_0 \subset X$ be some manifold. Assume that there exist automorphisms $A^\pm$; $B^\pm \in L(X)$ and an operator $S : X_0 \to X$ such that for $\forall y \in X_0$ the solution of equation (4) is expressed by the formula

$$x^\pm = A^\pm y + B^\pm S y. \tag{5}$$

Let's prove that if the system $\{T^+x_n^+; T^-x_n^-\}_{n\in N}$ forms a basis for X, the operator S is bounded in X_0. Show that the equation (4) has a unique solution. Consider the homogeneous equation

$$T^+x^+ + T^-x^- = 0.$$

Expand $x^\pm$ with respect to the basis $\{x_n^\pm\}_{n\in N}$: $x^\pm = \sum_{n=1}^{\infty} \lambda_n^\pm x_n^\pm$. We have

$$\sum_{n=1}^{\infty} \lambda_n^+ T^+ x_n^+ + \sum_{n=1}^{\infty} \lambda_n^- T^- x_n^- = 0.$$

It directly follows from the basicity of the system $\{T^+x_n^+;T^-x_n^-\}_{n\in N}$ that $\lambda_n^\pm=0$, $\forall n\in N$. Thus, every $y\in X$ is corresponded by a unique pair $(x^+;x^-)\in X^+\times X^-$. Assume $Tx=T^+x^++T^-x^-$, $\forall x=x^++x^-\in X$, where $x^\pm\in X^\pm$. By Banach theorem, it follows from the above reasonings that the operator T is invertible in $L(X)$, i.e. $T^{-1}\in L(X)$. It is easy to see that we can define the inverse operator T^{-1} as follows:

$$T^- = \begin{pmatrix} (T^+)^{-1}P^+ & O \\ O & (T^-)^{-1}P^- \end{pmatrix},$$

where P^+ and P^- are the projectors on X^+ and X^-, respectively, generated by expansion (3). This operator acts on the element $x=(x^+;x^-)\in X^+\times X^-$ according to the matrix rule. Thus, $x^\pm$ is expressed by the formula $x^\pm=(T^\pm)^{-1}P^\pm y$. Having taken $y\in X_0$, from the uniqueness of the solution and from (5) we get

$$\left[(T^\pm)^{-1}P^\pm\right]\Big/_{X_0}=A^\pm+B^\pm S,$$

where $T/_{X_0}$ is the contraction of the operator T on X_0. Consequently,

$$S=(B^\pm)^{-1}\left[\left[(T^\pm)^{-1}P^\pm\right]\Big/_{X_0}-A^\pm\right],$$

and, as a result, it becomes clear that S is bounded on X_0.

So the following theorem is true.

Theorem 1. *Let the Banach space X has direct expansion (3), the system $\{x_n^\pm\}_{n\in N}$ forms a basis for $X^\pm$, $T^\pm;A^\pm;B^\pm\in L(X)$ be automorphisms and $S:X_0\to X$ be some operator for which formula (5) is valid with respect to the solution of equation (4), with $X_0\subset X$ being some set. Assume that the system $\{T^+x_n^+;T^-x_n^-\}_{n\in N}$ forms a basis for X. Then the operator S is bounded on X_0.*

This theorem implies the following

Corollary 1. *Let all the conditions of Theorem 1 be fulfilled and let X_0 be dense everywhere in X. Assume that the system $\{T^+x_n^+;T^-x_n^-\}_{n\in N}$ forms a basis for X. Then the operator S can be boundedly continued to the whole of X.*

Consider the specific case. Let $\Gamma\subset C$ be some rectifiable Jordan curve on a complex plane. Consider Cauchy type integral

$$[Kf](z)\equiv\frac{1}{2\pi}\int_\Gamma\frac{f(t)dt}{t-z},z\notin\Gamma$$

and the corresponding singular integral

$$Sf\equiv\frac{1}{2\pi}\int_\Gamma\frac{f(\tau)}{\tau-t}d\tau,$$

where $f\in L_1(\Gamma)$ is a function summable on Γ. Consider the weight $\rho(t)$ of the form

$$\rho(t)\equiv\prod_{k=1}^m|t-t_k|^{\beta_k},$$

where $\{t_k\}_1^m \subset \Gamma$, $t_i \neq t_j$ is $i \neq j$. Let $t = t(s)$, $0 \leq s \leq l$, be a parametric equation of the curve Γ with respect to the length of the arc $\overset{\cup}{at}$, where a and l are the origin and the length of Γ, respectively. Γ is said to be Radon curve if $t = t(s)$ is a function with bounded variation on $[0, l]$. Denote by $L_{p,\rho}()$ the Lebesque weight class of functions on Γ with the norm $\|\cdot\|_{p,\rho}$: $\|f\|_{p,\rho} \equiv \left(\int_\Gamma |f(t)|^p \rho^p(t)\, |dt|\right)^{1/p}$.

We'll need the following result (see e.g. [6]).

Theorem [6]. *Let Γ be either a Lyapunov or Radon curve without cusps. The operator S acts boundedly from $L_{p,\rho}(\Gamma)$ to $L_{p,\rho}(\Gamma)$ if and only if the following inequalities are fulfilled*

$$-\frac{1}{p} < \beta_k < \frac{1}{q}, k = \overline{1, m},$$

where $1 < p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$.

4. Application

Consider the system (2), where $\alpha \in L_\infty(-\pi, \pi)$ is some measurable real-valued function. As proved in [13], in case $\|\alpha\|_{L_\infty} < \frac{\pi}{4}$ the system (2) forms a Riesz basis for $L_2(-\pi, \pi)$. Of course, there arises the question: how necessary this condition is? Consider the special case $\alpha(t) \equiv \alpha\, t$, where $\alpha \in R$ is some real parameter. In this case we have $|\alpha| < \frac{1}{4}$. As it follows from the results of [8,14], this condition is necessary for the basicity of the system (2) for $L_2(-\pi, \pi)$. The same result may be derived from Corollary 1. In fact, let $|\alpha| = \frac{1}{4}$ and let the system (2) form a basis for $L_2(-\pi, \pi)$. Denote by $\{h_n\}_{n \in Z}$ a system biorthogonal to it, i.e.

$$\int_{-\pi}^{\pi} e^{i(n\, t + \alpha(t)\, sign\, n)} \overline{h_k(t)}\, dt = \delta_{nk}\,, \;\; \forall n, k \in Z,$$

where $(\bar{\cdot})$ is a complex conjugation, δ_{nk} is the Kronecker symbol. Denote by $P^\pm : L_2 \to L_2$ the following projectors.

$$P^+ f = \sum_{n=0}^{\infty} f_n e^{i(n\, t + \alpha(t))}; \; P^- f = \sum_{n=1}^{\infty} f_{-n} e^{-i(n\, t + \alpha(t))}\,, \tag{6}$$

where $f_n = \int_{-\pi}^{\pi} f(t)\overline{h_n(t)} dt\,,\; n \in Z$. It is clear that the projectors $P^\pm$ are continuous, i.e. $\exists M > 0$:

$$\left\|P^\pm f\right\|_{L_2} \leq M \left\|f\right\|_{L_2} \forall f \in L_2(-\pi, \pi)\,,$$

where $\|\cdot\|$ is an ordinary norm in $L_2(-\pi, \pi)$. Let H_2^+ be a Hilbertian Hardy class of functions analytic interior to the unit circle, and let H_2^- be a similar class of functions analytic exterior to the unit circle and vanishing at infinity. It follows from the convergence of series (**??**) in $L_2(-\pi, \pi)$ that the functions

$$F^+(z) \equiv \sum_{n=0}^{\infty} f_n z^n, F^-(z) \equiv \sum_{n=1}^{\infty} f_{-n} z^{-n},$$

belong to H_2^+ and H_2^-, respectively. Consequently, the pair $(F^+; F^-) \in H_2^+ \times H_2^-$ is the solution of the following Riemann problem.

$$\begin{cases} e^{i\alpha(t)}F^+(\tau) + e^{-i\alpha(t)}F^-(\tau) = f(t), \\ F^-(\infty) = 0, \; \tau = e^{it}, \; t \in (-\pi, \pi). \end{cases} \tag{7}$$

Assume

$$Z_0^{\pm}(z) \equiv \exp\left\{ \mp \frac{i}{2\pi} \int_{-\pi}^{\pi} \alpha(t) \frac{e^{it}+z}{e^{it}-z} dt \right\}, \tag{8}$$

and let

$$Z(z) \equiv \begin{cases} Z_0^+(z), \, |z| < 1, \\ \left[Z_0^-(z)\right]^{-1}, \, |z| > 1. \end{cases}$$

It is known that the solution of problem (7) in the classes $H_2^+ \times H_2^-$ has the following form (see e.g. [4]):

$$F(z) \equiv \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i\alpha(t)} f(t)}{Z^+(e^{it})} \frac{dt}{1 - ze^{-it}},$$

where $Z^+\left(e^{it}\right)$ are non-tangential boundary values of the function $Z(z)$ on the unit circumference inside the unit circle. Applying the Sokhotskii-Plemelj formulae, we get

$$F^{\pm}(e^{it}) = \pm\frac{1}{2} e^{-i\alpha(t)} f(t) + S_0[f], \tag{9}$$

where

$$S_0[f] = \frac{Z^+(e^{it})}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i\alpha(\xi)} f(\xi)}{Z^+(e^{i\xi})} \frac{d\xi}{1 - e^{i(t-\xi)}}.$$

Identify $H_2^{\pm}$ with the subspaces of $L_2(-\pi, \pi)$. Denote by $T_0^{\pm}$ the operators of multiplication by $e^{\pm i\alpha(t)}$ in $H_2^{\pm}$, respectively, i.e. $T_0^{\pm}F^{\pm} = e^{\pm i\alpha(t)}F^{\pm}$, $\forall F^{\pm} \in H_2^{\pm}$. Then we can rewrite the boundary value problem (7) as follows

$$T_0^+ F^+ + T_0^- F^- = f \;, \;\; f \in L_2(-\pi, \pi).$$

Let $A_0^{\pm} f = \pm\frac{1}{2} e^{-i\alpha(t)} f$, $\forall \, f \in L_2(-\pi, \pi)$. It is clear that $A_0^{\pm}$ is an automorphism in $L_2(-\pi, \pi)$. From (9) we get $F^{\pm} = A_0^{\pm} f + S_0[f]$. Then it follows from Corollary 1 that S_0 acts boundedly in $L_2(-\pi, \pi)$. Using (8) and the results of [4], it is easy to get

$$\left|Z^+\left(e^{it}\right)\right| \mid |t| - \pi|^{2\alpha}, t \in (-\pi, \pi).$$

Combined with theorem of [6], the latter relation yields that S_0 is not bounded for $|\alpha| = \frac{1}{4}$. The obtained contradiction means that the system (2) doesn't form a basis for $L_2(-\pi, \pi)$ in this case.

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